Cylindric Young Tableaux and their Properties

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Introduction

- Young tableaux
- Cylindric tableaux
- Schur polynomials

- A *partition* λ of a nonnegative integer n is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$.
- For example, a partition of 10 is $(5, 2, 2, 1)$.

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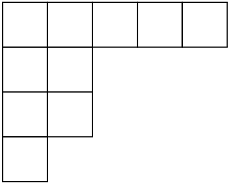
- For example, a partition of 10 is $(5, 2, 2, 1)$.
- Partitions can be represented with boxes (Young diagrams):



Young Tableaux

- We can fill in Young diagrams boxes with numbers.
- If entries strictly increase from top to bottom and weakly increase from left to right, we have a *semistandard Young tableau* (henceforth, *tableau*).

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3	6			
7	7			
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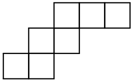
- If a tableau T is the Young diagram of a partition λ with its boxes filled, we say that λ is the *shape* of T.
- In the example above, the shape of the tableau is $(5, 2, 2, 1)$.

Skew Young Diagrams and Skew Tableaux

- Given two partitions λ and μ, with μ inside λ, the *skew Young diagram* λ/μ consists of the boxes inside the Young diagram of λ but outside the Young diagram of μ.
- Example:
  - $\lambda = (5, 3, 2)$
  - $\mu = (2, 1)$
  - $\lambda/\mu = (5, 3, 2)/(2, 1)$
  - Young diagram of λ/μ:



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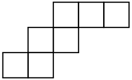
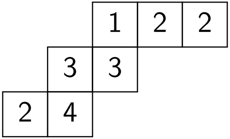
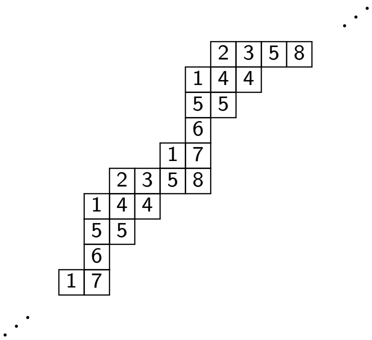


- A *skew tableau* is a skew Young diagram with its boxes filled according to the same rules as regular tableaux.
- Example:



- A *cylindric tableau* is an "infinite" skew tableau where every row repeats if you go k rows down but move m steps to the left, for some fixed k and m.
- Corresponding entries are considered the *same entry*, because we can think of them as corresponding to the same place on a cylinder.



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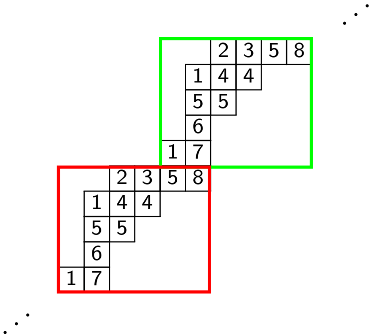
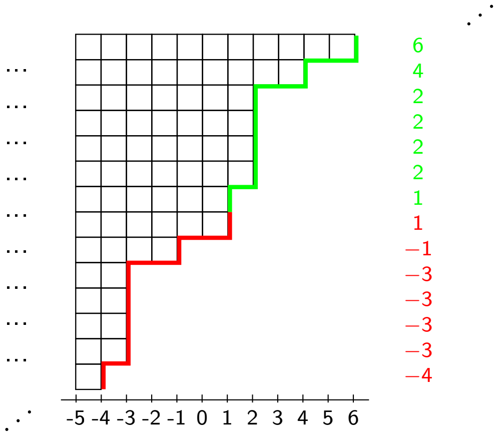


Cylindric Partitions

- A cylindric partition is a "periodic", weakly decreasing sequence of integers.
- It can be represented as a Young diagram that extends infinitely far left.
- A cylindric tableau is bounded by two cylindric partitions.
- Corresponding boxes in a partition are actually the same box.



Schur Polynomials

- Let T be a tableau with entries from $\{1, 2, \ldots, n\}$.
- If T has μ_k k's for $1 \leq k \leq n$, then the *content* of T is the tuple $(\mu_1, \mu_2, \ldots, \mu_n)$.

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- The *Schur polynomial* of a partition λ in n variables, denoted $s_\lambda(x_1, x_2, \dots, x_n)$, is obtained by:
  - taking, for each tableau T of shape λ, the monomial $x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n}$, where $(\mu_1, \mu_2, \dots, \mu_n)$ is the content of T,
  - adding these monomials together.
- Example:
  - $\lambda = (2, 1)$
  - $n = 3$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

  - $s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$

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  - Notice: s_λ is symmetric!

Theorem

For regular, skew, and cylindric tableaux, Schur polynomials are

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- It suffices to show that the number of tableaux with content $(k_1, k_2, \ldots, k_i, k_{i+1}, \ldots, k_n)$ is the same as the number of tableaux with content $(k_1, k_2, \ldots, k_{i+1}, k_i, \ldots, k_n)$ for any $1 \leq i < n$.

Proof of Schur Polynomial Symmetry (2)

- We will create a bijection (Bender-Knuth involution). Here's an example:
  - Let $i = 2$ and T be the following tableau:

1	1	1	1	1	1	1	1	2	2	2	2	3	3
1	2	2	2	2	2	2	3	3	3	3	5	6	
3	3	3	4	5	5	5							

  - Leave the white and blue boxes alone.
  - Reverse the number of green and red boxes in each row:

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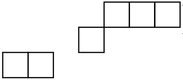
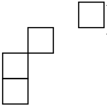
- This is a bijection, since re-applying the transformation gives back T.
- This proof also works for skew and cylindric tableaux.

Horizontal and Vertical Strips: Definition

- A *horizontal i-strip* is a set of i boxes, none of which are in the same column. (Example: )
- A *vertical i-strip* is a set of i boxes, none of which are in the same row. (Example: )

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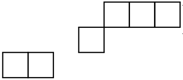
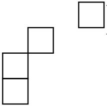
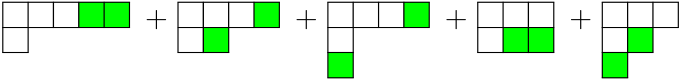
- A *horizontal i-strip* is a set of i boxes, none of which are in the same column. (Example: )
- A *vertical i-strip* is a set of i boxes, none of which are in the same row. (Example: )
- $h_i(\lambda)$ is the formal sum of all partitions you can get after **attaching** a **horizontal** i-strip to λ.
- $e_i(\lambda)$ is the formal sum of all partitions you can get after **attaching** a **vertical** i-strip to λ.
- $h_i^*(\lambda)$ is the formal sum of all partitions you can get after **removing** a **horizontal** i-strip from λ.
- $e_i^*(\lambda)$ is the formal sum of all partitions you can get after **removing** a **vertical** i-strip from λ.

Horizontal and Vertical Strips: Example

- $\lambda = (3, 1)$
- $h_2(\lambda) =$ 

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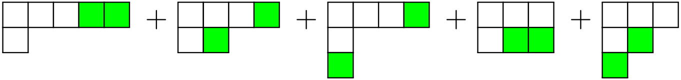
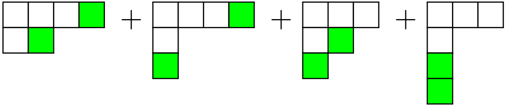
- $h_2(\lambda) =$ 
- $e_2(\lambda) =$ 

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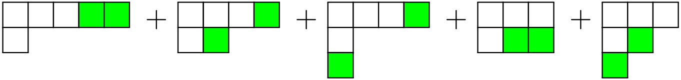
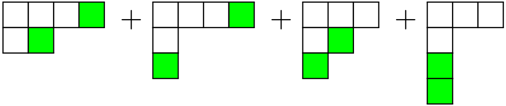
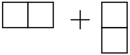
- $h_2(\lambda) =$ 
- $e_2(\lambda) =$ 
- $h_2^*(\lambda) =$ 

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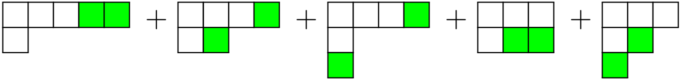
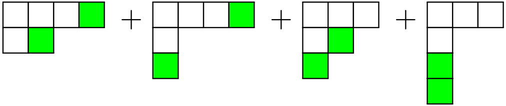
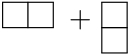
- $h_2(\lambda) =$ 
- $e_2(\lambda) =$ 
- $h_2^*(\lambda) =$ 
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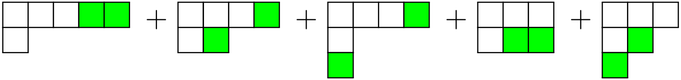
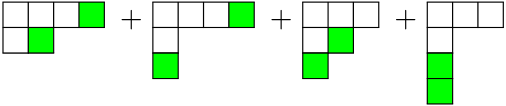
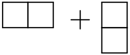
- $h_2(\lambda) =$ 
- $e_2(\lambda) =$ 
- $h_2^*(\lambda) =$ 
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Theorem

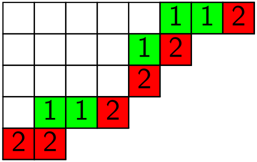
h and e commute with each other and with themselves.

- $h_j(h_i(\lambda)) = h_i(h_j(\lambda))$
- $e_j(e_i(\lambda)) = e_i(e_j(\lambda))$
- $h_j(e_i(\lambda)) = e_i(h_j(\lambda))$

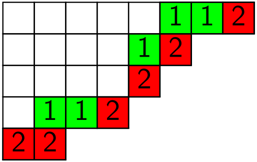
Similarly, h^* and e^* commute with each other and with themselves.

Proof that h Commutes with Itself (1)

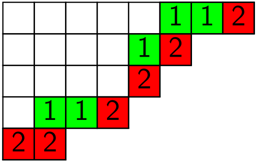
- Consider $h_j(h_i(\lambda))$ for any j, i, and λ.
- Let μ be λ with the horizontal i-strip added.
- Let ν be μ with the horizontal j-strip added.
- Consider the Young diagram of ν/λ.
  - Fill the boxes of μ/λ with 1's.
  - Fill the boxes of ν/μ with 2's.
- Example:
  - $\lambda = (5, 4, 4, 1)$
  - $i = 5$
  - $j = 6$
  - One summand of $h_j(h_i(\lambda))$:



Proof that h Commutes with Itself (2)



- Since we can do this for every pair of horizontal strips that is added, the number of times ν is in $h_j(h_i(\lambda))$ is the number of skew tableaux of shape ν/λ with i 1's and j 2's.



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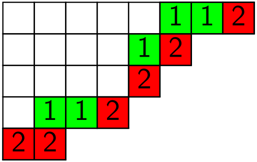

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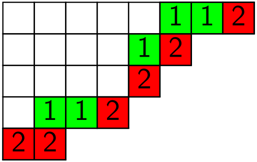

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- Therefore, $h_j(h_i(\lambda)) = h_i(h_j(\lambda))$.
- This proof also works for cylindric partitions.

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- The fact that there are nice properties of cylindric tableaux that don't exist for regular tableaux is encouraging.

Goals

- Goal 1: extend notions applicable to regular tableaux to cylindric tableaux.
  - Cylindric tableau product (different equivalent methods for regular tableau products yield different results for cylindric tableaux)
  - Robinson-Schensted-Knuth Correspondence (bijection between matrices and pairs of tableaux)
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  - Commutativity of h, e, h^*, and e^*
- Goal 3: find applications of cylindric tableaux to other parts of math.
  - Regular tableaux have a variety of applications in combinatorics and abstract algebra.
  - Very few, if any, applications are known for cylindric tableaux.

Acknowledgements

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