Pattern Avoidance and Compact Dot Representations

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- We define a permutation θ to contain another permutation π if and only if there exists a subsequence of θ order-isomorphic to π.
- For example, the permutation 31452 contains the permutation 132 (as the subsequences 152 and 142) but avoids the permutation 321.
- The study of permutation avoidance has been a very important area of study - first introduced with Donald Knuth's work on stack-sorting permutations.

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- The Stanley-Wilf conjecture, proven by Marcus and Tardos in 2004, stated that $\lim_{n\to\infty} S_n(\pi)^{\frac{1}{n}}$ is finite for all permutations π.
- Current work mostly concerns the value of the above limit; a recent paper of Fox showed that the limit is typically exponential in k, disproving the conjecture of many that it was polynomial in k.

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- Relatively new and interesting area of study first suggested by Richard Arratia - many parallel conjectures as to the length of the smallest superpattern (whose value we denote $L(k)$)

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- Upper bound improved by Alison Miller in 2006 to $\frac{k(k+1)}{2}$ using an explicit construction.
- Currently conjectured asymptotic bounds include $\frac{k^2}{2}$ [Eriksson et. al.], $\frac{k^2}{4}$ [Alon], and $\frac{k^2}{e^2}$ [Arratia] - lots of work to be done.

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- An **inverse descent** in π occurs wherever there is a descent in the inverse permutation of π.

Tilted Square

The n-**tilted square** is the n^2 permutation obtained by ordering the indices $1, 2, \ldots, n^2$ into n sequences of length n each decreasing by n and putting the sequences in ascending order [Eriksson et. al]

Compact Dot Representation

- The compact dot representation of a permutation is the set of points representing the permutation in the grid of the tilted square farthest to the west and south.

Lemma

Lemma [Eriksson et. al]: The compact dot representation of a permutation is achieved by taking each symbol π_i *in* π *and placing a dot in the site with as many columns to the left of it as there are ascents before it and as many rows below it as there are inverse descents under it.*

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- Our motivation is to find a line under which we can find the compact dot representations of all patterns of a given length.
- The method could be used to push the line down further, as long as we maintain a sub-quadratic number of dots above - could potentially improve the current estimate for a superpattern if our approach is generalized.

The Result

Theorem

The line under which we may bound all k-permutations is roughly $a(\pi) + d(\pi^{-1}) \leq 2k - 2\sqrt{k}$.

In addition, we find a permutation which reaches this line in the compact dot representation, showing the bound is sharp.

Further Directions

- As mentioned earlier, our method could be extended to potentially find a better estimate for $L(k)$.

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- It would be interesting to consider the compact dot representation in its own right - its behavior under certain transformations and potential application to other problems.

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Thanks for listening!