Generalizations of the Joints Problem

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Abstract

In this paper we explore generalizations of the joints problem introduced by B. Chazelle et al. A joint is formed when three noncoplanar lines intersect in $\mathbb{R}^3$, and other authors have proved an $O(n^{\frac{3}{2}})$ bound on the number of joints formed by n lines. We narrow the constant in this bound to between $\frac{\sqrt{2}}{3}$ and $\frac{4}{3}$, and explore the problem when the dimension of the space, the dimension of the intersecting hyperplanes, and the dimension of their mutual intersection is changed. We also consider cases where the intersecting hyperplanes do not all have the same dimension, focusing on the simplest nontrivial case in $\mathbb{R}^4$. This case is used to reconsider the original joints problem with an eye towards extending current results to higher dimensions. We prove an analogue of the joints theorem for this case and use it to give a new proof of the theorem in $\mathbb{R}^3$.

1 Introduction

Recently, a new and powerful method of solving incidence geometry problems has been developed by Guth and Katz. This method, known as the polynomial method, has enabled progress to be made on many outstanding problems in the field, including the joints problem posed in 1992 by B Chazelle et al. [8] The joints problem asks this: Given n lines in $\mathbb{R}^3$, how many joints can be made? A *joint* is defined as an intersection of three lines such that their tangent vectors form a basis for $\mathbb{R}^3$. Note that to exclude trivial cases we will only count $k > 3$ noncoplanar lines intersecting at a common point as a single joint.

The joints problem is motivated by a number of related problems in computer science; at their center is the *hidden surface removal problem*, which is the computer graphics problem of computing the view of a scene from some viewpoint. Given a set of objects with polygonal boundary, it is easier to render them when there exists an ordering of the polygons by distance; that is, there are no cycles, or chains of n polygons such that P_i covers P_{i+1} and P_n covers P_1. The joints problem, in addition to its purely mathematical value, is related to the study of understanding algorithmic and combinatorial properties of lines, rods, and cycles in $\mathbb{R}^3$ [8].

In [8], the authors conjectured an upper bound of $O(n^{\frac{3}{2}})$ joints given n lines. The following result is recent, and the proof relies on ideas from the polynomial method:

Theorem 1. *Any n lines in space determine at most $10n^{\frac{3}{2}}$ joints.*

This result is due to Guth [7], with the first proof that the bound is $O(n^{\frac{3}{2}})$ given by Guth and Katz ([3]). This first paper introduced a number of groundbreaking algebraic techniques that are at the center of further research on what is known as the Kayeka conjecture. A simpler proof using similar techniques was then given by Elekes et al. ([4]). This proof is concerned only with determining the exponent $n^{\frac{3}{2}}$. In the original 1992 paper presenting the problem, the best bound was $O(n^{\frac{7}{4}})$ [8]. One of the goals of this project is to investigate the constant factor of 10: can it be lowered? In this paper, we prove a result that lowers this constant factor substantially, giving a new upper bound of $\frac{4}{3}n^{\frac{3}{2}}$ joints that can be made with n lines.

The other goal of the project is to investigate generalizations of the joints problem — playing with the dimensions of the intersecting objects, the dimension of the space, the dimension of the intersection, and the number of the objects that must intersect to make a joint. This paper establishes a lower bound for all variations of the joints problem where the objects that intersect to make joints have a common dimension.

We will also look at a new way to approach the generalized joints problem by using a further generalization — giving the intersecting objects differing dimensions. We prove an upper bound on a new generalization of the problem to $\mathbb{R}^4$ where joints are made by the intersection of two lines and a plane, and use it to give a new proof of the original joints theorem on incidence in $\mathbb{R}^3$.

1.1 Acknowledgements

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2 Methods and Techniques

One major result used in previous important work on the joints problem is the following fundamental theorem:

Theorem 2. *Given k points in $\mathbb{R}^m$, there exists a nonzero polynomial of degree $d \leq (m!k)^{\frac{1}{m}}$ that vanishes on each of the points.*

This theorem, presented with derivation by Guth ([6]) forms the basis of a new technique in incidence geometry; the theorem itself follows by considering the vector space of polynomials in m variables over $\mathbb{R}$ of degree $\leq d$ and applying linear algebra to the (linear) evaluation map $E(P) = (E(x_1), \ldots, E_(x_k))$.

We will also make use of some properties of reguli. A regulus is defined as a degree two algebraic surface that is ruled by two families of lines L_1 and L_2 such that any pair of lines from different families is coplanar and any pair from the same family is noncoplanar. It is easy to observe that a regulus and a plane in $\mathbb{R}^4$ that intersect in more than two noncollinear points must lie in a common three-dimensional subspace. In addition, a line cannot intersect a given regulus more than twice without being contained in it. Finally (this follows almost by definition): if we have two families of lines $\mathscr{L}_1$ and $\mathscr{L}_2$ in $\mathbb{R}^m$ such that each line in $\mathscr{L}_1$ intersects each line in $\mathscr{L}_2$, then the lines lie on a common regulus (or plane, since planes are degenerate reguli). Reguli were used extensively in the original $O(n^{\frac{7}{4}})$ bound given by Chazelle et al; for our purposes they will help us examine a problem related to the joints problem concerning planes and lines in $\mathbb{R}^4$.

3 Joints in $\mathbb{R}^3$

We now turn to the first part of the project: investigating the joints theorem in three dimensions.

As a warm-up, it is fruitful to consider the simpler case of joints in $\mathbb{R}^2$. Define a *joint* to be the intersection of any two lines in the plane. It is clear that the asymptotically maximal construction with n lines has $O(n^2)$ joints; in fact, we can compute it explicitly as $\binom{n}{2} \approx \frac{n^2}{2}$ by taking n lines in general position.

The problem in three dimensions is more difficult. A trivial upper bound of $O(n^2)$ can be computed by bounding the number of intersections (since certainly the number of joints is at most the number of intersections), but it is not obvious how to obtain a bound of $O(n^{2-\epsilon})$ for any $\epsilon > 0$. Since the problem was presented in 1992 along with the initial upper bound of $O(n^{\frac{7}{4}})$, the bound has gradually lowered in [9] and [10] until the recent result of $O(n^{\frac{3}{2}})$ in [3]. It has been shown (and we will reproduce this later) that there are constructions with n lines and $\geq kn^{\frac{3}{2}}$ joints for some fixed constant k.

This raises an interesting and currently open question: Let $f(n)$ be the maximal number of joints it is possible to create with n lines. What is $\lim_{n\to\infty} \frac{f(n)}{n^{\frac{3}{2}}}$?

Since this question has never been seriously attacked, it is not surprising that the previous bounds are not as tight as they could have been even using the techniques already in play. Our work will be based off a later proof of the theorem by Guth which gives a bound of $10n^{\frac{3}{2}}$ [7].

Guth's own work in this proof implies an upper bound constant (we will omit the $n^{\frac{3}{2}}$ for brevity) of $3\sqrt{3}$, already significantly better than 10. It is possible to reduce this still further to $\sqrt{6}$ using a slight optimization to Guth's proof, replacing his approximation of $(n!)^{\frac{1}{n}} \approx n$ with the more efficient $(3!)^{\frac{1}{3}} < 3$.

However, getting below $\sqrt{6}$ requires a new approach, which we apply below. The key idea is approximating an integer sequence by a differentiable function.

The proof relies on the following lemma:

Lemma 1. *Given J joints made by L lines, there exists some line with $\leq (6J)^{\frac{1}{3}}$ joints on it.*

Proof. We proceed by contradiction. Suppose each line has more than $(6J)^{\frac{1}{3}}$ joints on it. We consider the minimal nonzero polynomial that vanishes on all the joints. By the polynomial method dimension-counting theorem, it has degree $d \leq (6J)^{\frac{1}{3}}$. We know that any polynomial of degree $\leq d$ that vanishes on more than d points on a line must be the zero polynomial, so the minimal polynomial vanishes on each line as well. This means that each of its partial derivatives vanish on every line. At each joint, therefore, the function has three directional derivatives that are each equal to zero; by our spanning condition the directions are linearly independent. Hence the gradient of the polynomial is equal to zero on all the joints. Thus any single partial derivative of the polynomial must vanish at all the joints. But the polynomial was already minimal, and its partial derivatives have a smaller degree, a contradiction.

□

We now proceed to the theorem itself.

Theorem 1. *n lines in $\mathbb{R}^3$ determine at most $\left(\frac{4}{3}+\epsilon\right) n^{\frac{3}{2}}$ joints.*

Proof. We follow the general approach outlined in [7], but modify it slightly to take advantage of a new method of sharpening the constant factor:

Let L_0 be a configuration of lines and joints with n lines and $j_0 = J$ joints. Let $L_0, L_1, \ldots, L_y$ be a sequence of configurations of lines and joints. The $i + 1^{\text{th}}$ configuration is obtained by removing a line with $\leq (6j_i)^{\frac{1}{3}}$ joints from the i^{th} configuration, where j_i is the number of joints in the i^{th} configuration; the existence of such a line is guaranteed by the lemma above. Furthermore we will make sure that $j_y = 0$ by letting $y = n$.

We can now sketch the argument given in [7], with some modifications for clarity:

Each of the quantities $j_i - j_{i+1}$ is less than $(6j_i)^{\frac{1}{3}}$; in particular, since the sequence j_i is decreasing, they are each less than $(6j_0)^{\frac{1}{3}}$. The sum $\sum_{i=0}^{y-1} j_i - j_{i+1}$ is equal to $j_0 - 0$, and it is bounded above by $y(6j_0)^{\frac{1}{3}}$. Hence $j_0 \leq y(6j_0)^{\frac{1}{3}}$; simplifying gives $J \leq \sqrt{6} n^{\frac{3}{2}}$ when we let $y = n$.

However, we can make this bound better. Notice that in an asymptotically maximal construction each $j_i - j_{i+1}$ will be close to $(6j_i)^{\frac{1}{3}}$.

Let us define $g(x)$ as follows: $g(0) = 0$, and $g(x+1) - g(x) = (6g(x+1))^{\frac{1}{3}}$.

Claim: $g(i)$ is an asymptotic upper bound for j_{y-i}. This is clear because $g(0) = j_{y-0}$, and at each "step" $g(i+1) - g(i) \geq j_{y-i} - j_{y-(i+1)}$ by the above. Thus $j_0 \leq g(y)$.

We can further approximate (asymptotically as always) g by noting that $g(x+1) - g(x) \approx g'(x+1)$. This only makes "sense" if we interpolate g by a differentiable function; this second function f is the solution to the differential equation $f'(x) = (6f(x))^{\frac{1}{3}}$ with $f(0) = 0$, and g approaches f asymptotically.

Solving the differential equation:

$$\begin{aligned} f'(x) &\leq (6f(x))^{\frac{1}{3}} \\ \int \frac{1}{(6f(x))^{\frac{1}{3}}} \, \mathrm{d}f(x) &\leq \int \mathrm{d}x \\ \frac{3(6^{-\frac{1}{3}})}{2} f(x)^{\frac{2}{3}} &\leq x \\ f(x) &\leq \left(\frac{4}{3}+\epsilon\right) x^{\frac{3}{2}} \end{aligned}$$

where $\epsilon > 0$ is arbitrarily small to account for our approximations, and plugging in $x = y = n$ gives us

$$\left(\frac{4}{3}+\epsilon\right) n^{\frac{3}{2}} \geq f(y) \approx g(y) \geq J.$$

□

As for a lower bound, it can be verified that an $s \times s \times s$ grid of lines contains s^3 joints and $3s^2$ lines, giving a simple construction with $J = \left(\frac{n}{3}\right)^{\frac{3}{2}}$ joints when we solve for the number of joints in terms of the number of lines [3]. Up to a constant factor, this implies that our $O(n^{\frac{3}{2}})$ bound was tight.

The best-known lower bound of $\frac{\sqrt{2}}{3}n^{\frac{3}{2}}$ is determined by considering k planes in general position. Any two of them intersect in a line, and any three intersect in a joint. Setting $n = \binom{k}{2} \approx \frac{k^2}{2}$ and solving for $\binom{k}{3} \approx \frac{k^3}{6}$ in terms of n gives the desired result. The approximations used are tight asymptotically. The example of planes in general position was introduced with the original problem in [8]; the explicit lower bound (including constant) has been computed here for the first time.

Notice that when n is not of the form $\binom{k}{2}$ we need to modify the construction. Let $\binom{c}{2}$ be the greatest binomial coefficient less than n; then notice that because

$$\lim_{c \to \infty} \frac{\binom{c+1}{3}}{\binom{c}{3}} = 1$$

holds, it is asymptotically tight (in terms of the lower bound) to use any construction containing a conjecturally optimal construction with c planes that is itself contained in a conjecturally optimal construction with $c+1$ planes (as such a construction will have between $\binom{c}{3}$ and $\binom{c+1}{3}$ joints.)

However, there is a large difference between $\frac{\sqrt{2}}{3}$ and $\frac{4}{3}$. The following observation is of note:

$$\frac{4}{3}\left(\frac{n}{2}\right)^{\frac{3}{2}} = \frac{\sqrt{2}}{3}n^{\frac{3}{2}}.$$

This motivates the following:

Conjecture 1. *Suppose we have a set S of n lines $\{\ell_i\}$ in $\mathbb{R}^3$. Given any such set S, let $f(S)$ be the number of joints formed by lines in S. Also, let $g(\ell_i, S)$ be the number of joints formed by the line ℓ_i and two members of S. Then there exists a sequence $\{a_i,\ 1 \leq i \leq k\}$ with $k \leq \frac{n}{2}$ satisfying the following property for all $0 \leq x \leq k-1$:*

$$g(\ell_{a_{x+1}}, S/\{\ell_{a_i},\ i \leq x\}) \leq \left(6f(S/\{\ell_{a_i},\ i \leq x\})\right)^{\frac{1}{3}}.$$

In addition each joint intersects ℓ_{a_i} for some i.

This conjecture implies that the number of joints satisfies $J \leq \frac{\sqrt{2}}{3}n^{\frac{3}{2}}$, as it allows us to take $y = \frac{n}{2}$ in the above proof; since we have a construction that gives this lower bound asymptotically, its proof would completely solve the joints problem in three dimensions. Conceptually, this conjecture can be stated as follows: We can take away half of the lines, with each line containing a small number of the total joints, such that when we are done the remaining lines do not define any joints. This implies that there must not have been many joints to begin with.

4 Looking at m dimensions

The same techniques that Guth and Katz use in their original paper can be used to bound the number of joints in m dimensions, where a joint is formed by the intersection of m lines whose tangent vectors are linearly independent. As in three dimensions, although the bound of $O(n^{\frac{m}{m-1}})$ derived below has been proved by Kaplan and Quilodrán in [1] and [2] respectively, the constant factor given here is the best known.

We will reproduce the calculation for the lower bound:

Take a planes in general position in $\mathbb{R}^m$. Any $m-1$ intersect in a line, and any m intersect in a point. Taking the points to be joints, we have $\binom{a}{m-1} \approx \frac{a^{m-1}}{(m-1)!} = n$ lines that intersect in $\binom{a}{m} \approx \frac{a^m}{m!} = J$ joints. Hence we have an asymptotic lower bound, by construction, of

$$J = \frac{n^{\frac{m}{m-1}}(m-1)!^{\frac{1}{m-1}}}{m},$$

where the construction is modified as above for a number of lines n not of the form $n = \binom{a}{m-1}$.

For an upper bound, we can adapt a proof of the lemma above to show that there must be some line with at most $(m!J)^{\frac{1}{m}}$ joints on it. The rest of the argument generalizes as well; the approximating function $f(x)$ is defined by the differential equation $f'(x) = (m!f(x))^{\frac{1}{m}},\ f(0) = 0$ with solution

$$f(x) = x^{\frac{m}{m-1}}(m!)^{\frac{1}{m-1}}\left(\frac{m-1}{m}\right)^{\frac{m}{m-1}},$$

which gives an asymptotic upper bound of $J \leq f(y)$ (up to smaller terms). When we let $y = n$ as before, we have the following theorem:

Theorem 2. *Suppose n lines in $\mathbb{R}^m$ determine J joints. Then:*

$$J \leq \left((m!)^{\frac{1}{m-1}}\left(\frac{m-1}{m}\right)^{\frac{m}{m-1}} + \epsilon\right) n^{\frac{m}{m-1}},$$

for any $\epsilon > 0$ when n is sufficiently large.

Something interesting happens if we can let $y = \frac{n}{m-1}$, which is a natural way to generalize letting $y = \frac{n}{2}$ before.

The new upper bound becomes:

$$J \leq \left((m!)^{\frac{1}{m-1}} \left(\frac{m-1}{m} \right)^{\frac{m}{m-1}} + \epsilon \right) \left(\frac{n}{m-1} \right)^{\frac{m}{m-1}}$$
$$\leq \left(\frac{(m-1)!^{\frac{1}{m-1}}}{m} + \epsilon \right) n^{\frac{m}{m-1}},$$

which is precisely the lower bound we obtained above. This motivates a generalization of the earlier conjecture to $\mathbb{R}^m$, the proof of which would resolve this particular generalization of the joints problem to m dimensions.

Conjecture 2. *Suppose we have a set S of n lines $\{\ell_i\}$ in $\mathbb{R}^m$. Given any such set S, let $f(S)$ be the number of joints formed by lines in S. Also, let $g(\ell_i, S)$ be the number of joints formed by ℓ_i and two members of S. Then there exists a sequence $\{a_i,\ 1 \leq i \leq k\}$ with $k \leq \frac{n}{m-1}$ satisfying the following property for all $0 \leq x \leq k-1$:*

$$g(\ell_{a_{x+1}}, S/\{\ell_{a_i},\ i \leq x\}) \leq (m! f(S/\{\ell_{a_i},\ i \leq x\}))^{\frac{1}{m}}.$$

In addition each joint intersects ℓ_{a_i} for some i.

5 Generalizing the joints problem

With a conjecture that resolves the given problem, we now turn to expanding the scope of our investigation. The most obvious generalization of the problem has four parameters:

- The dimension a of the intersecting hyperplanes; this is 1 in the original problem.
- The dimension b of their intersection; this is 0 in the original problem.
- The number k of objects that intersect at a joint; this is 3 in the original problem.
- The dimension m of the space; this is also 3 in the original problem.

We will utilize the following formula for hyperplane intersection: In $\mathbb{R}^m$, a hyperplane of dimension a and a hyperplane of dimension b will, if they are in general position, intersect in a hyperplane of dimension $a+b-m$. (If $a+b<m$, of course, the hyperplanes will not intersect at all.)

In the original problem, we required that the tangent vectors of the intersecting objects formed a basis for $\mathbb{R}^3$. To ensure that this condition is met, we will suppose that $b+(a-b)k=m$. In addition necessarily $a>b$.

Given n hyperplanes of dimension a, a lower bound of $\frac{(k-1)!^{\frac{1}{k-1}}}{k} n^{\frac{k}{k-1}}$ "joints" can be proven by extending the 2-planes in general position argument. The appropriate hyperplanes to consider are of dimension $m+b-a$.

Suppose we have τ such hyperplanes.

Any $k-1$ of these hyperplanes intersect to determine a hyperplane of dimension a, and k such hyperplanes determine a hyperplane of dimension b. Hence there are $n=\binom{\tau}{k-1}\approx\frac{\tau^{k-1}}{(k-1)!}$ hyperplanes of dimension a and $J=\binom{\tau}{k}\approx\frac{\tau^k}{k!}$ of dimension b, giving an asymptotic construction with

$$J=\frac{(k-1)!^{\frac{1}{k-1}}}{k} n^{\frac{k}{k-1}}$$

as we asserted above.

Thus we have established the best known lower bound for this problem:

Theorem 3. *Define a* joint *as the b-dimensional intersection of k a-dimensional hyperplanes in $\mathbb{R}^m$ such that the tangent vectors of the k a-dimensional hyperplanes span $\mathbb{R}^m$. Necessarily $b+(a-b)k=m$. Then, for all $\epsilon>0$ and n sufficiently large, there exists configurations of n a-dimensional hyperplanes that determine*

$$\left(\frac{(k-1)!^{\frac{1}{k-1}}}{k}-\epsilon\right) n^{\frac{k}{k-1}}$$

joints.

It is of note that this lower bound is independent of a, b, and m. We conjecture that it is the tightest possible lower bound.

In this project, we explore a possible method of solution to this problem that uses the polynomial method in a different way, and use it to revisit the joints theorem in $\mathbb{R}^3$.

6 Inhomogeneous joints in $\mathbb{R}^4$

Suppose we have p planes and ℓ lines in $\mathbb{R}^4$, with $p+\ell=n$. Define a joint as an intersection of a plane and two lines in a point such that the tangent vectors of the lines and the plane form a basis for $\mathbb{R}^4$. Can we bound the number of joints as a function of n?

Let us look first at an easier problem — given p planes and ℓ lines in $\mathbb{R}^3$, how many joints can we make? A joint here will be any intersection of a line and a plane not containing that line. Furthermore we will let $p \approx \ell$ (in an asymptotic sense, $c_1 \leq \frac{p}{\ell} \leq c_2$ for some absolute positive constants c_1 and c_2; we can use the approximation $p = \ell$ since we will only worry about exponents for the remainder of this paper.)

Letting $p = \ell$, a clear upper bound is $J \leq p\ell$, since each plane can only intersect each line once.

Now let us revisit the two-dimensional joints problem: Suppose we have some configuration of ℓ lines and J joints in $\mathbb{R}^3$. As a warm-up, we will give a proof of the earlier $\frac{n^2}{2}$ upper bound using the result we just proved.

Separate the n lines randomly into two sets, A and B, of size $\frac{n}{2}$ each. With positive probability, half of the joints will be composed of two lines from different sets. Now we will change our configuration in $\mathbb{R}^2$ into one in $\mathbb{R}^3$ by making all lines in the set A into planes; their second tangent vector can be chosen arbitrarily as long as it does not lie in the same plane as the lines in set B. We will have $\frac{n}{2} = \ell$ lines and $\frac{n}{2} = p$ planes. Notice that we now have an upper bound of $p\ell$ by our above computation.

Since half of the original joints are joints under the new definition (the intersection of a line and a plane), we have the inequality $\frac{J}{2} \leq p\ell$ or $J \leq \frac{n^2}{2}$.

Although this reproves a trivial result, it motivates the examination of the problem in $\mathbb{R}^4$ as a way to explore the original joints problem in $\mathbb{R}^3$. Using the same process as above, we can "lift" constructions in $\mathbb{R}^3$ to create constructions in $\mathbb{R}^4$ with the same number of joints (up to a constant factor) and the same total number of planes/lines. Hence bounds on the problem in $\mathbb{R}^4$ lead to bounds in $\mathbb{R}^3$.

We can extend the expected value argument by supposing the lines (here we are considering a collection of n lines and J joints in $\mathbb{R}^3$) are partitioned randomly into three disjoint subsets A, B, and C, with $|A| = |B| = |C| = \frac{n}{3}$. We can compute that an expected $\frac{6}{27}J = \frac{2}{9}J$ of the original joints are now composed of three lines, one from each of A, B, and C.

Our partitioning also defines a new configuration in $\mathbb{R}^4$: Make all the lines in $|C|$ into planes with their second tangent vector not contained in the original three-dimensional subspace (it can be chosen arbitrarily given this condition). Our new set-up has (using the expected value) $\frac{n}{3} = p$ planes, $\frac{2n}{3} = \ell$ lines, and $\frac{2}{9}J$ joints.

Hence, since $\frac{9}{2}O(n^{\frac{3}{2}}) = O(n^{\frac{3}{2}})$, an upper bound of $Cn^{\frac{3}{2}}$ joints (for some constant C) in $\mathbb{R}^4$ allows us to transfer the upper bound given by the modified, inhomogeneous joints problem into an upper bound for the original problem in $\mathbb{R}^3$. For the remainder of our discussion of this problem, we will ignore the constant factor and focus just on the exponent, so our use of big O notation is valid.

Next, we will introduce the following theorem (proved using the polynomial method), presented in an equivalent form by Guth and Katz in [5] as Theorem 2.10:

Theorem 4. *For any $0 < \epsilon < \frac{1}{2}$, a collection of L lines in $\mathbb{R}^3$ $\mathscr{L}$ with at least $L^{\frac{3}{2}+\epsilon}$ intersections contains a plane or regulus with at least $C_1 L^{\frac{1}{2}+\epsilon}$ lines forming $C_2 L^{1+2\epsilon}$ intersection for absolute constants C_1, C_2.*

This theorem applies for $\mathscr{L} \in \mathbb{R}^4$ as well by projecting into a three-dimensional subspace.

Corollary 1. *For any $0 < \epsilon < \frac{1}{2}$, a collection of L lines in $\mathbb{R}^4$ $\mathscr{L}$ with at least $L^{\frac{3}{2}+\epsilon}$ intersections contains a plane or regulus with at least $C_1 L^{\frac{1}{2}+\epsilon}$ lines forming $C_2 L^{1+2\epsilon}$ intersection for absolute constants C_1, C_2.*

The proof of the corollary begins by noticing that a small modification of the proof in [5] yields the slightly stronger result:

Theorem 5. *Let $\mathscr{L}$ be a set of L lines in $\mathbb{R}^3$ with at least $L^{\frac{3}{2}+\epsilon}$ intersections (where $\epsilon > 0$). Then there exists a plane or a regulus that contains two families of lines $\mathscr{L}_1 \subset \mathscr{L}$ and $\mathscr{L}_2 \subset \mathscr{L}$ with $|\mathscr{L}_1|, |\mathscr{L}_1| \geq C L^{\frac{1}{2}+\epsilon}$ for some absolute constant C and each line in $\mathscr{L}_1$ intersects each line in $\mathscr{L}_2$.*

We will also use the following lemma:

Lemma 2. *Suppose $\mathscr{L}$ is a set of L lines in $\mathbb{R}^4$. Then there exists a three-dimensional subspace S such that each line $l \in \mathscr{L}$ can be projected to some $l' \in S$ and $l_1 \cap l_2 = \emptyset$ if and only if $l_1' \cap l_2' = \emptyset$.*

Proof. We note that $l_1' \cap l_2' = \emptyset$ implies $l_1 \cap l_2 = \emptyset$. Now suppose that l_1 and l_2 do not intersect, but their projections do. This means that the direction of projection must be parallel to a line connecting a point on l_1 and a point on l_2. The directions of this kind form a set of measure zero relative to the total number of projections, and since there are only a finite number of lines, there must be some subspace S and associated projection satisfying the desired property. □

Next, applying the lemma, we find a suitable projection that sends $\mathscr{L}$ to $\mathscr{L}'$. By our strengthened theorem, there exist subsets $\mathscr{L}_1'$ and $\mathscr{L}_2'$ of $\mathscr{L}$ of size $O(L^{\frac{1}{2}+\epsilon})$ that lie in a common plane or regulus such that each line in $\mathscr{L}_1'$ intersects each line in $\mathscr{L}_2'$. By the property of our projection, we have two subsets $\mathscr{L}_1$ and $\mathscr{L}_2$ of size $O(L^{\frac{1}{2}+\epsilon})$ where each line in one intersects each line in the other. Hence the lines all lie in a common three-dimensional subspace and lie in a common plane or regulus.

Our goal now is to use this corollary to prove that if we have p planes and ℓ lines in $\mathbb{R}^4$ with $2p = \ell$, then the number of joints we can make is $O((p+\ell)^{\frac{3}{2}})$.

We will start by applying the above theorem repeatedly to our collection of ℓ lines. Among them, we can suppose that the ℓ lines intersect $I = \ell^{\frac{3}{2}+k}$ times. Although not all these intersections need to be joints (recall that a joint is only formed when a *plane* intersects the mutual intersection of two lines), certainly there are at least as many line-line intersections as there are joints, and any optimal configuration (as we have seen by our method of transferring configurations in $\mathbb{R}^3$ to $\mathbb{R}^4$) can contain at least $C(p+\ell)^{\frac{3}{2}}$ joints for some absolute constant C.

We will apply the theorem, removing objects (planes and reguli) and the lines they contain from our configuration until the number of intersections I_f is less than $C\ell^{\frac{3}{2}}$ for some absolute constant C, at which point the number of joints remaining must necessarily be less than $C\ell^{\frac{3}{2}}$ (since there must be at least as many line-line intersections as joints).

Furthermore, when we have ℓ' lines left defining $I' = \ell'^{\frac{3}{2}+\epsilon}$ intersections, the theorem implies that there exists some plane or regulus containing at least $C_1\frac{I'}{\ell'}$ lines for some absolute constant C_1. We note that $C_1\frac{I'}{\ell'} \geq C_1\frac{C_2\ell^{\frac{3}{2}}}{\ell'} \geq C_1\frac{C_2\ell^{\frac{3}{2}}}{\ell} \geq C_3\ell^{\frac{1}{2}}$, for absolute constants C_2 and C_3. This uses the fact that $I' \geq C\ell^{\frac{3}{2}}$ for some constant C by previous reasoning (since when $I' = O(\ell^{\frac{3}{2}})$ we no longer have enough intersections to give a counterexample to the upper bound of $Cn^{\frac{3}{2}}$), and the additional fact that $\ell' \leq \ell$ (this is obvious).

We are now left with some number s of planes and reguli, each containing at least $C_3\ell^{\frac{1}{2}}$ lines; we have shown that the remaining lines have at most $O(\ell^{\frac{3}{2}})$ intersections and thus can be ignored in our upper bound analysis.

Note that $s = O(\ell^{\frac{1}{2}})$. We arrive at this conclusion by noting that each of the s objects contains at least $C_3\ell^{\frac{1}{2}}$ lines; in fact, our construction makes it clear that each of the s objects contains at least $C_3\ell^{\frac{1}{2}}$ lines that are not contained on any other plane or regulus (since, after each step, we "ignore" the lines already present in one of our removed objects). Thus we have the inequality

$$C_3\ell^{\frac{1}{2}}s \leq \ell,$$

and hence $s \leq C\ell^{\frac{1}{2}}$ for some absolute constant C.

Now, we will consider the joints in this problem. Recall that our four-dimensional joints are made by the intersection of a plane and two lines. There are two cases we need to consider: the line-line intersection can occur between two lines in the same plane or regulus, or between two lines in different surfaces.

If the two lines are contained in the same plane or regulus, then there is an easy upper bound on joints of this form. The plane being used to create the joint cannot intersect a given surface (plane or regulus) more than twice before they both lie in a common three-dimensional subspace and violate our spanning conditions; hence we have a limit of two joints per surface per plane, or $J \leq 2ps \leq 2C\ell^{\frac{1}{2}}p = 2C(\frac{2n}{3})^{\frac{1}{2}}\frac{n}{3} = O(n^{\frac{3}{2}})$.

Otherwise, the joint consists of a plane intersecting two lines from different objects (planes or reguli). Note that the number of joints of this kind is bounded above by the number of intersections between lines lying in different surfaces. If a given line lies in an object (plane or regulus) S_1 and intersects some other surface (plane or regulus) S_2 in more than two points, then it must be contained in S_2 as well. Hence any joint containing an intersection between our given line and another line in S_2 has already been counted, as it was bounded by our consideration of intersections that are completely contained in one surface (plane or regulus).

Hence each line, if it is to be contained in a joint of this second type, must intersect other surfaces in at most two points. Using the previous principle that the number of joints is bounded above by the number of intersections, we can see that the number of joints of this second kind satisfies $J \leq 2\ell s \leq 2C\ell^{\frac{1}{2}}\ell = 2C\left(\frac{2n}{3}\right)^{\frac{1}{2}}\frac{2n}{3} = O(n^{\frac{3}{2}})$.

Combined, the separate bounds proves the following theorem:

Theorem 6. *p planes and ℓ lines in $\mathbb{R}^4$ with $p = \frac{n}{3}$ and $\ell = \frac{2n}{3}$ combine to create at most $O(n^{\frac{3}{2}})$ joints.*

By extending a given construction in $\mathbb{R}^3$ to a configuration in $\mathbb{R}^4$ using our previous work, this theorem gives an entirely new proof of the joints theorem of Guth and Katz.

7 Further research

The joints problem is far from resolved, and two main directions for further research present themselves. The first of these is the problem of determining the asymptotic constant for the already solved cases of the joints theorem — that is, the case of m lines intersecting in a single point in $\mathbb{R}^m$. A proof of the "line-removal" conjecture above, along with a suitable generalization to m dimensions, would resolve this problem.

The second direction is dealing with the general problem. Fruitful results have already been obtained with the polynomial method, and it may be wise to attempt to apply polynomials to solve more cases of the theorem. However, polynomials are much more well-behaved on lines than they are on planes. In particular, a nonzero polynomial on a line can only vanish on a finite number of points, while a polynomial can vanish along a curve in a plane and not on the plane itself.

A new approach to this second direction, and one explored in this paper, is considering the inhomogeneous joints problem and attempting to bound the harder problem combinatorially. This still may incorporate the polynomial method, as the theorem due to Guth in [5] is proved using polynomials. Perhaps the next step is to consider a natural extension of the problem in $\mathbb{R}^4$ to $\mathbb{R}^7$, where joints are formed by the intersection of two planes and a 3-hyperplane. A bound here could be used to attack the problem in $\mathbb{R}^6$ where joints are formed by the mutual intersection of three planes.

In terms of the techniques developed in this paper, the most novel is the use of a differential equation to tighten the bound of the constant factor in the original joints theorem. This could have ramifications for other combinatorial or geometric problems, giving better constant-factor bounds on the behavior of sequences of the form $x_{i+1} = x_i - f(x_i)$, where $f(x)$ is an increasing function of x that can be well-approximated by a differentiable function. Of course such results are meaningless except asymptotically, since the approximation $f(x+1) - f(x)$ is roughly equal to $f'(x)$ in an asymptotic sense only for functions $f(x) = O(x^n)$.

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