Monomization of Power Ideals and Generalized Parking Functions

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Abstract

A power ideal is an ideal in a polynomial ring generated by powers of homogeneous linear forms. Power ideals arise in many areas of mathematics, including the study of zonotopes, approximation theory, and fat point ideals; in particular, their applications in approximation theory are relevant to work on splines and pertinent to mathematical modeling, industrial design, and computer graphics. For this reason, understanding the structure of power ideals, especially their Hilbert series, is an important problem.

Unfortunately, due to the computational complexity of power ideals, this is a difficult problem. Only a few cases of this problem have been solved; efficient ways to compute the Hilbert series of a power ideal are known only for power ideals of certain forms $\mathcal{J}_G$, $\mathcal{J}_{G,-1}$, and $\mathcal{J}_{G,1}$, where G is a graph.

In this paper, we find an efficient way to compute the Hilbert series of a class of power ideals $\mathcal{J}_{G,\Delta}$ determined by a graph G and an abstract simplicial complex Δ on $\{1,\ldots,n\}$. This result generalizes and interpolates between the known cases of $\mathcal{J}_G$ and $\mathcal{J}_{G,1}$, which can be recovered by setting $\Delta = \{\emptyset\}$ and $\Delta = \mathcal{P}(\{1,\ldots,n\})$, respectively. We also find a combinatorial interpretation for the Hilbert series of the algebra $\mathcal{B}_{G,\Delta} = \mathbb{K}[x_1,\ldots,x_m]/\mathcal{J}_{G,\Delta}$ in terms of parking functions and forests of G, generalizing previous work on G-parking functions.

1 Introduction

Power ideals, ideals generated by powers of linear forms, have many diverse applications, including approximation theory [5] [6], linear diophantine equations and splines [7], fat point ideals [10] [12], zonotopal algebras [13], Cox rings [21], and the geometry of the flag manifold [18]. The Hilbert series of the quotient of the polynomial ring by a power ideal is instrumental to such applications; consequently, an important problem regarding power ideals is the computation of such a Hilbert series [1]. In particular, Gröber Basis techniques for computing a power ideal's Hilbert Series are computationally inefficient [8], and an efficient means of computing the Hilbert series of a general power ideal is not known.

Among the many applications of power ideals, of particular importance is their connection to approximation theory. Through their inverse systems, power ideals play a role in work on box splines [5] [6], which are relevant to computer graphics and mathematical modeling.

The problem of computing a power ideal's Hilbert Series, or equivalently, the Hilbert Series of its quotient algebra, has been solved for three classes of power ideals related to graphs. The quotient algebras of these power ideals belong to three classes of *zonotopal algebras* described by Holtz and Ron in [13]. Such algebras can be associated with a zonotope and its dual hyperplane arrangement; for Type A hyperplane arrangements, they can be understood in terms of graphs. An efficient way to compute the Hilbert series of the *central algebra*, one of these three classes, was given by Postnikov and Shapiro in [17]. In [8], Desjardins generalized this methodology to the remaining two classes, the *internal* and *external algebras*. In this paper, we extend these results to a class of algebras that interpolate between the central and external zonotopal algebras; in doing so, we solve the problem of computing a power ideal's Hilbert series for a large class of power ideals.

Let $\mathbb{K}$ be a field with characteristic 0. We work in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. One way to compute the Hilbert series of the quotient of $\mathbb{K}[x_1, \ldots, x_n]$ by a power ideal is *monomization*, a technique developed by Postnikov and Shapiro in [17]: we associate a power ideal $\mathcal{J}$ in $\mathbb{K}[x_1, \ldots, x_n]$ with an ideal $\mathcal{I}$ generated by monomials such that the set of monomials not in $\mathcal{I}$, known as the *standard monomial basis* of the algebra $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$, is a basis of the algebra $\mathcal{B} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}$. Monomization allows us to compute the Hilbert series of $\mathcal{B}$ and $\mathcal{J}$ by computing the Hilbert series of $\mathcal{A}$ and $\mathcal{I}$, which are computationally much simpler.

Monomial ideals, ideals generated by monomials, themselves have many interesting properties and applications. They are related to G-parking functions, which have many interpretations in combinatorics and statistical physics, notably in chip-firing games [2] and the abelian sandpile model [9] [15] introduced by Dhar. For example, for a class of digraphs G including symmetric digraphs, the G-parking functions biject to the *recurrent states* of the abelian sandpile model [11]. Therefore, monomization also serves to draw connections between the theory of power ideals and monomial ideals.

We briefly describe the known monomizations of the central, internal, and external zonotopal algebras $\mathcal{B}_G$, $\mathcal{B}_{G,-1}$, and $\mathcal{B}_{G,1}$. Let G be a graph[1] on the vertices $\{0, 1, \ldots, n\}$. For nonempty $I \subseteq \{1, \ldots, n\}$ and $i \in I$, let $d_I(i)$ denote the number of edges from i to vertices outside I, and let $D_I = \sum_{i \in I} d_I(i)$ be the total

[1]In this paper, we will use the terms "directed graph" and "digraph" to refer to directed graphs and "undirected graph" and "graph" to refer to undirected graphs. We allow all directed and undirected graphs to have multiple edges, but not loops.

number of edges from vertices in I to vertices outside I. Let

$$m_I = \prod_{i \in I} x_i^{d_I(i)} \qquad \text{and} \qquad p_I = \left(\sum_{i \in I} x_i \right)^{D_I},$$

and let the ideals $\mathcal{I}_G = \langle m_I \rangle$ and $\mathcal{J}_G = \langle p_I \rangle$ be generated by all such m_I and p_I. Define the algebras $\mathcal{A}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_G$ and $\mathcal{B}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_G$. Let $|G|$ denote the number of edges of G.

Theorem 1.1. *[17] The standard monomial basis of $\mathcal{A}_G$ is a basis of $\mathcal{B}_G$. The algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ have dimension equal to the number of spanning trees of G. They have equal Hilbert series, and the dimension of the graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ equals the number of spanning trees of G with external activity $|G|-n-k$.*

Thus $\mathcal{I}_G$ is a monomization of $\mathcal{J}_G$. For all nonempty $I = \{i_1 < \cdots < i_r\} \subseteq \{1, \ldots, n\}$, let

$$m_I^+ = x_{i_1} \prod_{i \in I} x_i^{d_I(i)} \qquad \text{and} \qquad p_I^+ = \left(\sum_{i \in I} x_i \right)^{D_I+1}.$$

Define the ideals $\mathcal{I}_{G,1} = \langle m_I^+ \rangle$ and $\mathcal{J}_{G,1} = \langle p_I^+ \rangle$ and the algebras $\mathcal{A}_{G,1} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{G,1}$ and $\mathcal{B}_{G,1} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{G,1}$.

Theorem 1.2. *[8] The standard monomial basis of $\mathcal{A}_{G,1}$ is a basis of $\mathcal{B}_{G,1}$. The algebras $\mathcal{A}_{G,1}$ and $\mathcal{B}_{G,1}$ have dimension equal to the number of forests of G. They have the same Hilbert series, and the dimension of the graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ equals the number of forests F of G with external activity $|G| - |F| - k$.*

Suppose G has the property that every pair of vertices has at least one edge between them. For all nonempty $I = \{i_1 < \cdots < i_r\} \subseteq \{1, \ldots, n\}$, let

$$m_I^- = x_{i_1}^{d_I(i_1)-1} \prod_{i \neq i_1, i \in I} x_i^{d_I(i)} \qquad \text{and} \qquad p_I^- = \left(\sum_{i \in I} x_i \right)^{D_I-1}.$$

Define the ideals $\mathcal{I}_{G,-1} = \langle m_I^- \rangle$ and $\mathcal{J}_{G,-1} = \langle p_I^- \rangle$ and the algebras $\mathcal{A}_{G,-1} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{G,-1}$ and $\mathcal{B}_{G,-1} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{G,-1}$.

Theorem 1.3. *[8] The standard monomial basis of $\mathcal{A}_{G,-1}$ is a basis of $\mathcal{B}_{G,-1}$. The algebras $\mathcal{A}_{G,-1}$ and $\mathcal{B}_{G,-1}$ have dimension equal to the number of spanning trees of G with internal activity 0. They have the same Hilbert series, and the dimension of the graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ equals the number of spanning trees of G with internal activity 0 and external activity $|G| - n - k$.*

Thus, $\mathcal{I}_{G,1}$ and $\mathcal{I}_{G,-1}$ are monomizations of $\mathcal{J}_{G,1}$ and $\mathcal{J}_{G,-1}$.

The goal of this paper is to extend this monomization theory to a larger class of power ideals. A family of subsets Δ of $\{1, \ldots, n\}$ is an (abstract) *simplicial complex* on $\{1, \ldots, n\}$ if $I \in \Delta$ and $J \subseteq I$ implies $J \in \Delta$. For graphs G on $\{0, \ldots, n\}$ and simplicial complexes Δ on $\{1, \ldots, n\}$, define

$$m_{I,\Delta} = \begin{cases} m_I^+ & I \in \Delta \\ m_I & I \notin \Delta \end{cases} \qquad \text{and} \qquad p_{I,\Delta} = \begin{cases} p_I^+ & I \in \Delta \\ p_I & I \notin \Delta \end{cases}.$$

We define the ideals $\mathcal{I}_{G,\Delta} = \langle m_{I,\Delta} \rangle$ and $\mathcal{J}_{G,\Delta} = \langle p_{I,\Delta} \rangle$ and the algebras $\mathcal{A}_{G,\Delta} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{G,\Delta}$. Our main result is the following.

Theorem 1.4. *The standard monomial basis of $\mathcal{A}_{G,\Delta}$ is a basis of $\mathcal{B}_{G,\Delta}$.*

This theorem generalizes Theorems 1.1 and 1.2 and interpolates between these two results. Indeed, setting $\Delta = \{\emptyset\}$ and $\Delta = \mathcal{P}(\{1, \ldots, n\})$ in Theorem 1.4 recovers Theorems 1.1 and 1.2, respectively.

Our methodology is as follows. We define the (G, Δ)-parking functions, a generalization of the G-parking functions, which correspond to the elements of the standard monomial basis of $\mathcal{I}_{G,\Delta}$. We establish and prove a bijection between the (G, Δ)-parking functions and a set of forests of G with a property determined by Δ, proving that $\dim \mathcal{A}_{G,\Delta}$ equals the number of such forests. We define $\mathcal{C}_{G,\Delta}$, a subalgebra of $\mathcal{B}_{G,\Delta}$, and prove that $\mathcal{C}_{G,\Delta}$ has dimension equal to the number of these forests. Using results on monotone monomial ideals we show that $\operatorname{Hilb} \mathcal{A}_{G,\Delta} \geq \operatorname{Hilb} \mathcal{B}_{G,\Delta} \geq \operatorname{Hilb} \mathcal{C}_{G,\Delta}$; because $\dim \mathcal{A}_{G,\Delta} = \dim \mathcal{C}_{G,\Delta}$, these Hilbert series are equal. The theory of monotone monomial ideals then implies that $\mathcal{A}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta}$ share a monomial basis. Much of this work is motivated by computer experiments from the software **Macaulay2**.

The remainder of this paper is organized as follows. In Section 2, we review parking functions and monotone monomial ideals. In Section 3, we define the (G, Δ)-parking functions and formulate our two main results, Theorems 3.1 and 3.2; Theorem 3.1 establishes a bijection between the (G, Δ)-parking functions and a class of forests of G, while Theorem 3.2 refines Theorem 1.4 in the context of (G, Δ)-parking functions and forests of G. In Section 4, we illustrate Theorem 3.2 with examples. In Sections 5 and 6, we prove Theorems 3.1 and 3.2; while these proofs are inspired by the work of Chebikin-Pylyavskyy [4] and Postnikov-Shapiro [17], respectively, the combinatorial details of these proofs are different, due to their added generality. In Section 7 we discuss the implications of this work on questions regarding ρ-parking functions and ρ-algebras. Finally, in Section 8 we summarize our results and examine possibilities for future investigation.

2 Preliminaries

2.1 Parking Functions

The *parking functions* are sequences of n nonnegative integers $(b_1, \ldots, b_n)$ whose decreasing rearrangements are termwise less than $(n, n-1, \ldots, 1)$. These are interesting in and of themselves; they count, for example, the number of spanning trees of the complete graph on $n+1$ vertices [14] and the number of regions of the Shi hyperplane arrangement [19]. For more properties of parking functions, see e.g. [14], [19], and [20].

The G-parking functions [17] are a broad generalization of the classical parking functions. Let G be a directed graph on the vertices $\{0, 1, \ldots, n\}$. For a nonempty $I \subseteq \{1, \ldots, n\}$ and a vertex $i \in I$, let $d_I(i)$ denote the number of directed edges from i to vertices outside I. In the case that G is an undirected graph, we may treat G as a symmetric directed graph, replacing each undirected edge with a pair of opposing directed edges; in this case the definition of $d_I(i)$ coincides with the definition given in Section 1. A *G-parking function* is a sequence of nonnegative integers $(b_1, \ldots, b_n)$ with the property that for each nonempty $I \subseteq \{1, \ldots, n\}$, there exists a vertex $i \in I$ such that $b_i < d_I(i)$.

Remark 2.1. The K_{n+1}-parking functions are the parking functions of size n.

An *oriented subtree* of a digraph G is a subgraph $T \subseteq G$ such that for every vertex $i \in T$, there exists a unique directed path in T from i to 0. An *oriented spanning tree* of G is an oriented subtree of G that includes every vertex of G. If G is an undirected graph, the oriented spanning trees of G correspond to the ordinary spanning trees of G.

Theorem 2.2. *[17] The number of G-parking functions equals the number of oriented spanning trees of G.*

Many bijective proofs of Theorem 2.2 exist in the literature; see e.g. [3] and [4]. We will generalize the bijection of Chebikin-Pylyavskyy [4] in Section 5.

We may define the algebra $\mathcal{A}_G$ for digraphs the same way as we defined it for graphs in Section 1. The sequence $(b_1, \ldots, b_n)$ is a G-parking function if and only if the monomial $\prod_{i \in I} x_i^{b_i}$ is an element of the standard monomial basis of $\mathcal{I}_G$. Therefore, the number of G-parking functions and oriented spanning trees of G equals $\dim \mathcal{A}_G$. This idea and its generalizations will be useful in the proof of our result.

2.2 Monotone Monomial Ideals and their Deformations

Consider a set of monomials $\{m_I\}$ in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, one for each nonempty subset $I \subseteq \{1, \ldots, n\}$. Such a set is a *monotone monomial family* [17] if m_I is a monomial in the variables x_i, $i \in I$, for all I, and $J \subseteq I$ and $i \in J$ implies $\deg_{x_i} m_J \geq \deg_{x_i} m_I$ for all I, J. A *monotone monomial ideal* is the ideal generated in $\mathbb{K}[x_1, \ldots, x_n]$ by a monotone monomial family. If we let $I = \{i_1, \ldots, i_r\}$, then a homogeneous polynomial p_I in the variables $x_{i_1}, \ldots, x_{i_r}$ is an *I-deformation of m_I* if $\deg(p_I) = \deg(m_I)$ and

$$\mathbb{K}[x_{i_1}, \ldots, x_{i_r}] = \langle R_{m_I} \rangle \oplus (p_I),$$

where $\langle R_{m_I} \rangle$ denotes the linear span of monomials not divisible by m_I and (p_I) denotes the ideal in $\mathbb{K}[x_{i_1}, \ldots, x_{i_r}]$ generated by p_I. Furthermore, if p_I is an I-deformation of m_I for all nonempty $I \subseteq \{1, \ldots, n\}$ and $\mathcal{I} = \langle m_I \rangle$ is a monotone monomial ideal, then we say that the ideal $\mathcal{J} = \langle p_I \rangle$ is a *deformation* of $\mathcal{I}$.

Lemma 2.3. *[17] Suppose $I = \{i_1, \ldots, i_r\}$ and m_I is a monomial in $x_{i_1}, \ldots, x_{i_r}$. If $\alpha_{i_1}, \ldots, \alpha_{i_r}$ are nonzero elements of $\mathbb{K}$, then*

$$(\alpha_{i_1} x_{i_1} + \cdots + \alpha_{i_r} x_{i_r})^{\deg(m_I)}$$

is an I-deformation of m_I.

Remark 2.4. Observe that $\mathcal{I}_G$, $\mathcal{I}_{G,1}$, $\mathcal{I}_{G,-1}$, and $\mathcal{I}_{G,\Delta}$ are monotone monomial ideals, and $\mathcal{J}_G$, $\mathcal{J}_{G,1}$, $\mathcal{J}_{G,-1}$, and $\mathcal{J}_{G,\Delta}$ are, respectively, deformations of these ideals.

The following property of monotone monomial ideals will be important to proving monomization.

Theorem 2.5. *[17] Let $\mathcal{I}$ be a monotone monomial ideal in $\mathbb{K}[x_1, \ldots, x_n]$, and let $\mathcal{J}$ be a deformation of $\mathcal{I}$. Define the algebras $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$ and $\mathcal{B} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}$. The standard monomial basis of $\mathcal{A}$ spans $\mathcal{B}$. Consequently, the Hilbert series* $\operatorname{Hilb}\mathcal{I}$, $\operatorname{Hilb}\mathcal{J}$, $\operatorname{Hilb}\mathcal{A}$, *and* $\operatorname{Hilb}\mathcal{B}$ *obey the termwise inequalities* $\operatorname{Hilb}\mathcal{I} \leq \operatorname{Hilb}\mathcal{J}$ *and* $\operatorname{Hilb}\mathcal{A} \geq \operatorname{Hilb}\mathcal{B}$.

3 (G,Δ)-Parking Functions

We generalize the notion of a G-parking function. Let G be a digraph on $\{0,1,\ldots,n\}$ and Δ be a simplicial complex on $\{1,\ldots,n\}$. If G is an undirected graph, we may treat it as a symmetric directed graph. We define the ideal $\mathcal{I}_{G,\Delta}$ and the algebra $\mathcal{A}_{G,\Delta}$ for digraphs G as we did for graphs in section 1. The sequence of nonnegative integers $(b_1,\ldots,b_n)$ is a (G,Δ)-*parking function* if and only if $\prod_i x_i^{b_i}$ is nonvanishing in $\mathcal{A}_{G,\Delta}$. Observe that the $(G,\{\emptyset\})$-parking functions are the G-parking functions. The monomials not in $\mathcal{I}_{G,\Delta}$ comprise the standard monomial basis of $\mathcal{A}_{G,\Delta}$; so, the number of (G,Δ)-parking functions equals $\dim \mathcal{A}_{G,\Delta}$.

Define an *oriented forest* of a directed graph as a collection of vertices, some of which are designated roots, and directed edges among these vertices, such that from each vertex there is a unique path to a root. Define a *proper forest* as an oriented forest in which each vertex is rooted at a vertex smaller than or equal to itself. Define a Δ-*proper forest* as a proper forest in which the set of vertices not rooted at 0 is an element of Δ. For undirected graphs, define a Δ-forest as an acyclic edge set for which the set of vertices not connected to 0 is an element of Δ. If a graph is undirected, its Δ-proper forests correspond to its Δ-forests.

Theorem 3.1. *For any digraph G and any simplicial complex Δ on $\{1,\ldots,n\}$, the (G,Δ)-parking functions biject to the $(n+1)$-vertex Δ-proper forests of G.*

We will present this bijection in Section 5. This theorem implies that $\dim \mathcal{A}_{G,\Delta}$ equals the number of Δ-proper forests of G. Observe that when $\Delta = \{\emptyset\}$, we recover Theorem 2.2 from Theorem 3.1.

We have the following refinement of Theorem 1.4. We will prove this theorem in Sections 5 and 6.

Theorem 3.2. *For all undirected graphs G and all simplicial complexes Δ on $\{1,\ldots,n\}$, the monomials $\prod_i x_i^{b_i}$, as $(b_1,\ldots,b_n)$ ranges over all (G,Δ)-parking functions, form a basis of $\mathcal{B}_{G,\Delta}$, and*

$$\dim \mathcal{A}_{G,\Delta} = \dim \mathcal{B}_{G,\Delta} = N_{G,\Delta},$$

where $N_{G,\Delta}$ is the number of Δ-forests of G. Furthermore, the k^{th} graded components $\mathcal{A}^k_{G,\Delta}$ and $\mathcal{B}^k_{G,\Delta}$ have dimension equal to the number of Δ-forests F of G with external activity $|G|-|F|-k$.

4 Examples

To demonstrate Theorem 3.2 and the notions of (G,Δ)-parking functions and Δ-forests, we present examples of Theorem 3.2 for various values of Δ and the graph

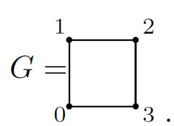

Figure 1: An example graph.

Example 4.1. Let $\Delta = \{\emptyset\}$. In this case, the (G,Δ)-parking functions are the G-parking functions and the Δ-proper forests of G are the spanning trees of G. The graph G has four spanning trees.

Figure 2: Spanning trees of G.

We have

$$\mathcal{I}_{G,\Delta} = \langle x_1^2, x_2^2, x_3^2, x_1x_2, x_1^2x_3^2, x_2x_3, x_1x_2^0x_3\rangle$$

$$\mathcal{J}_{G,\Delta} = \langle x_1^2, x_2^2, x_3^2, (x_1+x_2)^2, (x_1+x_3)^4, (x_2+x_3)^2, (x_1+x_2+x_3)^2\rangle.$$

The monomials not in $\mathcal{I}_{G,\Delta}$ are $1, x_1, x_2, x_3$. These are a basis for $\mathcal{A}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta}$ and give rise to four (G,Δ)-parking functions. The algebras $\mathcal{A}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta}$ both have dimension 4, the number of spanning trees of G, and both have Hilbert series $1+3t$.

Example 4.2. Let $\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}\}$. The Δ-forests of G are forests of G in which the set of vertices not connected to 0 is $\emptyset$, $\{1\}$, $\{2\}$, or $\{3\}$. In addition to the four spanning trees above, three more forests of G are Δ-forests for this Δ.

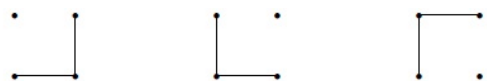

Figure 3: Δ-forests of G.

In this case,

$$\mathcal{I}_{G,\Delta} = \langle x_1^3, x_2^3, x_3^3, x_1x_2, x_1^2x_3^2, x_2x_3, x_1x_2^0x_3\rangle$$

$$\mathcal{J}_{G,\Delta} = \langle x_1^3, x_2^3, x_3^3, (x_1+x_2)^2, (x_1+x_3)^4, (x_2+x_3)^2, (x_1+x_2+x_3)^2\rangle.$$

The monomials not in $\mathcal{I}_{G,\Delta}$ are $1, x_1, x_2, x_3, x_1^2, x_2^2, x_3^2$, which correspond to seven (G,Δ)-parking functions. These monomials form a basis for $\mathcal{A}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta}$. The dimension of $\mathcal{A}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta}$ is 7, which equals the number of Δ-proper forests of G, and the common Hilbert Series of $\mathcal{A}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta}$ is $1+3t+3t^2$.

Example 4.3. Let $G = \mathcal{P}(\{1,2,3\})$. In this case, any forest of G is a Δ-forest. In addition to the seven forests listed above, there are eight more.

Figure 4: Forests of G.

We have

$$\mathcal{I}_{G,\Delta} = \langle x_1^3, x_2^3, x_3^3, x_1^2x_2, x_1^3x_3^2, x_2^2x_3, x_1^2x_2^0x_3\rangle$$

$$\mathcal{J}_{G,\Delta} = \langle x_1^3, x_2^3, x_3^3, (x_1+x_2)^3, (x_1+x_3)^5, (x_2+x_3)^3, (x_1+x_2+x_3)^3\rangle.$$

The monomials not in $\mathcal{I}_{G,\Delta}$, are $1, x_1, x_2, x_3, x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3, x_1x_2^2, x_1x_3^2, x_2x_3^2, x_1x_2x_3, x_1x_2x_3^2$, which correspond to 15 (G,Δ)-parking functions. These form a basis of $\mathcal{A}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta}$. The dimension of $\mathcal{A}_{G,\Delta}$ and $\mathcal{B}_{G,\Delta}$ is 15, the number of forests of G, and their Hilbert series is $1+3t+6t^2+4t^3+t^4$.

5 A Bijection from (G, Δ)-Parking Functions to Δ-Proper Forests

In this section we give a bijection from the (G, Δ)-parking functions to the $(n+1)$-vertex Δ-proper forests of the digraph G. Let $\mathcal{P}_{G,\Delta}$ and $\mathcal{F}_{G,\Delta}$ denote the sets of (G, Δ)-parking functions and $(n+1)$-vertex Δ-proper forests of G, respectively.

Say that an oriented forest F is a *subforest* of an oriented forest F' if the vertices, edges, and roots of F are subsets of, respectively, the vertices, edges, and roots of F'. Observe that because Δ is a simplicial complex, any subforest of a Δ-proper forest is a Δ-proper forest.

Furthermore, for any oriented forest F and any vertex $i \in F$, let $r_F(i)$ and $e_F(i)$ denote, respectively, the vertex at which i is rooted in F and the edge coming out of i in F, if it exists.

For every Δ-proper forest $F \subseteq G$, we assign a total order $\pi(F)$ to the vertices of F. Let $i >_{\pi(F)} j$ denote that i is larger than j in this order. A set of such orders $\Pi(G, \Delta)$ is a *proper set of forest orders* if the following conditions hold:

1. For all F, if $e_F(i) = (i, j)$, then $i >_{\pi(F)} j$.

2. For all F, if vertices $i, j \in F$ satisfy $r_F(i) > r_F(j)$, then $i >_{\pi(F)} j$.

3. For all F, if F' is a subforest of F, then the orders $\pi(F)$ and $\pi(F')$ are consistent.

One example of a proper set of forest orders is the breadth-first search order: let $h_F(i)$ denote the length of the unique path in F from i to a root; for all F and all $i, j \in F$, let $i >_{\pi(F)} j$ if: $r_F(i) > r_F(j)$, or $r_F(i) = r_F(j)$ and $h_F(i) > h_F(j)$, or $r_F(i) = r_F(j)$ and $h_F(i) = h_F(j)$ and $i > j$.

Fix a proper set of forest orders $\Pi(G, \Delta)$. If G has multiple edges, fix a total order on each set of multiple edges.

For each Δ-proper forest $F \subseteq G$ and each vertex $i \in G$, we define a total order on the edges from i to vertices in F. If $e = (i, j_1)$ and $e' = (i, j_2)$ are edges from i to vertices in F, let $e >_{\pi(F)} e'$ if $j_1 >_{\pi(F)} j_2$, or if $j_1 = j_2$ and e is larger than e' in the fixed order of multiple edges.

Define the function $\Theta_{\Pi,G,\Delta} : \mathcal{F}_{G,\Delta} \to \mathcal{P}_{G,\Delta}$ as follows: for $F \in \mathcal{F}_{G,\Delta}$, let $\Theta_{\Pi,G,\Delta}(F) = (b_1, \ldots, b_n)$, where b_i is the number of edges e from i such that $e <_{\pi(F)} e_F(i)$, if $e_F(i)$ exists, and the number of edges from i to vertices j such that $j <_{\pi(F)} i$, otherwise.

Proposition 5.1. *$\Theta_{\Pi,G,\Delta}$ is a bijection between $\mathcal{P}_{G,\Delta}$ and $\mathcal{F}_{G,\Delta}$.*

Remark 5.2. This bijection preserves Chebikin and Pylyavskyy's bijection [4] between G-parking functions and oriented spanning trees of G.

We construct a function $\Phi_{\Pi,G,\Delta} : \mathcal{P}_{G,\Delta} \to \mathcal{F}_{G,\Delta}$, which we claim is the inverse of $\Theta_{\Pi,G,\Delta}$: let $P \in \mathcal{P}_{G,\Delta}$. Let the oriented forest F_0 consist of the vertex 0. We construct oriented forests $F_1, \ldots, F_n = \Phi_{\Pi,G,\Delta}(P)$ by the following algorithm, run for $m = 0, \ldots, n-1$. Let U_m consist of the set of vertices $i \notin F_m$ with more than b_i outgoing edges to vertices in F_m. We consider two cases.

If $|U_m| > 0$: for each $i \in U_m$, let e_i denote the $(b_i + 1)^{\text{st}}$ smallest edge from i to F_m in the order $\pi(F_m)$. Let f_m be the oriented forest consisting of F_m, all vertices $i \in U$, and all edges e_i, for $i \in U$. Let v_{m+1} be the smallest vertex in U_m in the order $\pi(f_m)$. Construct F_{m+1} by adding v_{m+1} and $e_{v_{m+1}}$ to F_m.

If $|U_m| = 0$: Let v_{m+1} be the numerically smallest vertex not in F_m. Construct F_{m+1} by adding v_{m+1} to F_m without adding an edge.

Example 5.3. Let

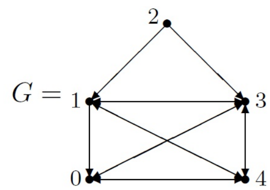

Figure 5: An Example Digraph.

and $\Delta = \mathcal{P}(\{1,2,3,4\})$. Let our proper set of forest orders $\Pi(G,\Delta)$ be the breadth-first search order. Consider the (G,Δ)-parking function $P = (3,1,0,0)$. The algorithm for constructing $\Phi_{\Pi,G,\Delta}(P)$ constructs the following oriented forests.

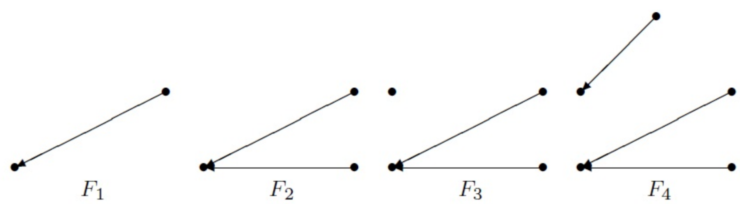

Figure 6: Process of constructing $\Phi_{\Pi,G,\Delta}(P)$.

We have $U_0 = \{3,4\}$, so f_0 consists of the vertices 0, 3, 4, and the edges $e_3 = (3,0)$ and $e_4 = (4,0)$. We have $3 <_{\pi(f_0)} 4$, so we construct F_1 by adding the vertex 3 and the edge $(3,0)$ to F_0.

Then, $U_1 = \{4\}$ and $e_4 = (4,0)$, so we construct F_2 by adding the vertex 4 and the edge $(4,0)$ to F_1.

Next, U_2 is empty. Thus, we construct F_3 by adding 1, the smallest vertex outside of F_2, to F_2.

Lastly, $U_3 = \{2\}$ and $e_2 = (2,1)$, so we construct $F_4 = \Phi_{\Pi,G,\Delta}(P)$ by adding the vertex 2 and the edge $(2,1)$ to F_3. Observe that $0 <_{\pi(F_4)} 3 <_{\pi(F_4)} 4 <_{\pi(F_4)} 1 <_{\pi(F_4)} 2$. Thus $\Theta_{\Pi,G,\Delta}(F_4) = (3,1,0,0)$, as expected.

We now show that $\Theta_{\Pi,G,\Delta}$ and $\Phi_{\Pi,G,\Delta}$ map the sets $\mathcal{F}_{G,\Delta}$ and $\mathcal{P}_{G,\Delta}$ to each other.

Lemma 5.4. *If $F \in \mathcal{F}_{G,\Delta}$, then $\Theta_{\Pi,G,\Delta}(F) \in \mathcal{P}_{G,\Delta}$.*

Proof. Let $\Theta_{\Pi,G,\Delta}(F) = (b_1,\ldots,b_n)$. Consider any nonempty $I \subseteq \{1,\ldots,n\}$, and let j be the minimal element of I in the order $\pi(F)$. We consider two cases.

Case 1: $e_F(j)$ exists. By definition of $\Theta_{\Pi,G,\Delta}$, there are b_j edges smaller than $e_F(j)$ in $\pi(F)$. Because $\Pi(G,\Delta)$ is a proper set of forest orders and j is minimal in the order $\pi(F)$, $e_F(j)$ and the b_j edges smaller than it in $\pi(F)$ must go to vertices outside I. Therefore $d_I(j) \geq b_j + 1$, and $\deg_{x_j} m_{I,\Delta} \geq d_I(j) > b_j = \deg_{x_j} \prod_i x_i^{b_i}$. So, $m_{I,\Delta}$ does not divide $\prod_i x_i^{b_i}$.

Case 2: $e_F(j)$ does not exist. Then j must be a root of F. Because $\Pi(G,\Delta)$ is a proper set of forest orders and j is minimal in $\pi(F)$, we must have $r_F(i) \geq r_F(j) = j$ for all $i \in I$. Because F is a proper forest, all vertices $i \in I$ must satisfy $i \geq r_F(i) \geq j$; thus j is the numerically smallest vertex in I. Moreover, because $r_F(i) \geq r_F(j) = j > 0$ for all $i \in I$, all elements of I are not rooted at 0; hence $I \in \Delta$. This implies $\deg_{x_j} m_{I,\Delta} = d_I(j) + 1$. By definition of $\Theta_{\Pi,G,\Delta}$, there are b_j edges from j to vertices smaller than j in $\pi(F)$. By minimality of j in $\pi(F)$, all of these edges must go to vertices outside I. Therefore $d_I(j) \geq b_j$. So, $\deg_{x_j} m_{I,\Delta} = d_I(j) + 1 \geq b_j + 1 > b_j = \deg_{x_j} \prod_i x_i^{b_i}$. Thus $m_{I,\Delta}$ does not divide $\prod_i x_i^{b_i}$.

Therefore $\prod_i x_i^{b_i}$ is not divisible by any $m_{I,\Delta}$ and does not vanish in $\mathcal{A}_{G,\Delta}$. □

Lemma 5.5. *Let $P \in \mathcal{P}_{G,\Delta}$. In the algorithm for constructing $\Phi_{\Pi,G,\Delta}(P)$, if $|U_m| = 0$, then the set of vertices $\{0, 1, \ldots, n\} \setminus F_m \in \Delta$.*

Proof. Let $P = (b_1, \ldots, b_n)$ and $\{0, 1, \ldots, n\} \setminus F_m = I$. Suppose for sake of contradiction that $I \notin \Delta$. Then $m_{I,\Delta} = \prod_{i \in I} x_i^{d_I(i)}$. Because $P \in \mathcal{P}_{G,\Delta}$, $m_{I,\Delta}$ does not divide $\prod_i x_i^{b_i}$. Thus, there exists $i \in I$ such that $d_I(i) > b_i$. But $d_I(i)$ is the number of edges from i to F_m, so there exists $i \in I$ with more than b_i edges to F_m. This contradicts $|U_m| = 0$. □

Lemma 5.6. *If $P \in \mathcal{P}_{G,\Delta}$, then $\Phi_{\Pi,G,\Delta}(P) \in \mathcal{F}_{G,\Delta}$.*

Proof. Let $\Phi_{\Pi,G,\Delta}(P) = F$. It is clear that each of the F_m is an oriented forest. Because each F_m has one more vertex than the previous, $F = F_n$ has $n + 1$ vertices. Also, F is a proper forest because the roots of F are precisely the vertices that were added to some F_m where $|U_m| = 0$, and each such vertex was the numerically smallest vertex not in that F_m when it was added.

If every vertex in F is rooted at 0, F is clearly Δ-proper. Else, let F_m be such that every vertex in F_m is rooted at 0 and m is maximal. Then $|U_m| = 0$; by Lemma 5.5, the set of vertices $\{0, 1, \ldots, n\} \setminus F_m \in \Delta$. Hence, the set of vertices of F that are not rooted at 0 is an element of Δ. Therefore, F is an $(n+1)$-vertex Δ-proper forest. □

Next we show that $\Theta_{\Pi,G,\Delta}$ and $\Phi_{\Pi,G,\Delta}$ are inverses.

Lemma 5.7. *Let $P \in \mathcal{P}_{G,\Delta}$ and $F = \Phi_{\Pi,G,\Delta}(P)$. For $m = 1, \ldots, n$, let v_m be the vertex in F_m but not in F_{m-1}. Then $0 <_{\pi(F)} v_1 <_{\pi(F)} v_2 <_{\pi(F)} \cdots <_{\pi(F)} v_n$.*

Proof. Because $\Pi(G, \Delta)$ is a proper set of forest orders and 0 is the smallest root of F, 0 is minimal in $\pi(F)$. Therefore $0 <_{\pi(F)} v_1$. We inductively prove that $0 <_{\pi(F)} v_1 <_{\pi(F)} \cdots <_{\pi(F)} v_m$. Suppose $0 <_{\pi(F)} v_1 <_{\pi(F)} \cdots <_{\pi(F)} v_m$; we show that $v_m <_{\pi(F)} v_{m+1}$. We consider three cases.

Case 1: $|U_m| = 0$. Then v_{m+1} must be a root of F. As F_m is a subforest of F, $r_F(v_m) = r_{F_m}(v_m) \in F_m$. If $r_F(v_m)$=0, then $r_F(v_m) < v_{m+1} = r_F(v_{m+1})$, so $v_m <_{\pi(F)} v_{m+1}$ because $\Pi(G, \Delta)$ is a proper set of forest orders. Else, let F_i be such that $r_F(v_m) \in F_i$ and i is minimal. Then $r_F(v_m)$ must be the smallest vertex not in F_{i-1}. But, v_{m+1} is not in F_{i-1}, so $r_F(v_m) < v_{m+1} = r_F(v_{m+1})$. Because $\Pi(G, \Delta)$ is a proper set of forest orders, $v_m <_{\pi(F)} v_{m+1}$.

Case 2: $|U_m| > 0$, $v_{m+1} \notin U_{m-1}$. Then v_{m+1} has at most $b_{v_{m+1}}$ edges to F_{m-1} but at least $b_{v_{m+1}} + 1$ edges to F_m, so G has at least one edge from v_{m+1} to v_m. Moreover, by the inductive hypothesis, v_m is the maximal vertex in F_m in the order $\pi(F)$. As F_m is a subforest of F, v_m is also the maximal vertex in F_m in the order $\pi(F_m)$. Thus the $(b_{v_{m+1}} + 1)^{\text{st}}$ smallest edge from v_{m+1} to F_m in the order $\pi(F_m)$ is from v_{m+1} to v_m, and F includes the edge (v_{m+1}, v_m). Because $\Pi(G, \Delta)$ is a proper set of forest orders, $v_m <_{\pi(F)} v_{m+1}$.

Case 3: $|U_m| > 0$, $v_{m+1} \in U_{m-1}$. Because $v_{m+1} \in U_{m-1}$, $|U_{m-1}| > 0$. So, $v_m \in U_{m-1}$. Let e_{v_m} and $e_{v_{m+1}}$ denote, respectively, the $(b_{v_m} + 1)^{\text{st}}$ smallest edge from v_m to F_{m-1} in the order $\pi(F_{m-1})$ and the

$(b_{v_m+1}+1)^{\text{st}}$ smallest edge from v_{m+1} to F_{m-1} in the order $\pi(F_{m-1})$. By the inductive hypothesis, v_m is the largest vertex in F_m in $\pi(F)$; thus it is the largest vertex in F_m in $\pi(F_m)$. So, in $\pi(F_m)$, all edges from v_{m+1} to v_m are larger than edges from v_{m+1} to vertices in F_{m-1}; because there are at least $b_{v_{m+1}}+1$ edges from v_{m+1} to F_{m-1} and the orders $\pi(F_{m-1})$ and $\pi(F_m)$ are consistent, the $(b_{v_{m+1}}+1)^{\text{st}}$ smallest edge from v_{m+1} to F_{m-1} in $\pi(F_{m-1})$ is also the $(b_{v_{m+1}}+1)^{\text{st}}$ smallest edge from v_{m+1} to F_m in $\pi(F_m)$. Hence F_{m+1} is formed by adding $e_{v_{m+1}}$ and v_{m+1} to F_m. Let f consist of F_{m-1}, v_m, v_{m+1}, e_{v_m} and $e_{v_{m+1}}$. Then f is a subforest of both f_{m-1} and F, so $\pi(f)$ is consistent with both $\pi(f_{m-1})$ and $\pi(F)$. We have $v_m <_{\pi(f_{m-1})} v_{m+1}$ because v_m is the smallest vertex in U_{m-1} in the order $\pi(f_{m-1})$. Hence $v_m <_{\pi(f)} v_{m+1}$, and $v_m <_{\pi(F)} v_{m+1}$. □

Lemma 5.8. *Let $P \in \mathcal{P}_{G,\Delta}$ and $F = \Phi_{\Pi,G,\Delta}(P)$. If $|U_{m-1}| = 0$, then v_m has exactly b_{v_m} edges to vertices in F_{m-1}.*

Proof. Suppose $|U_{m-1}| = 0$. Then v_m is the numerically smallest vertex not in F_{m-1}. Let I be the set of vertices $\{0, 1, \ldots, n\} \setminus F_{m-1}$. By Lemma 5.5, $I \in \Delta$. Thus $m_{I,\Delta} = x_{v_m} \prod_{i \in I} x_i^{d_I(i)}$. Because $|U_{m-1}| = 0$, each vertex $i \in I$ has at most b_i edges to vertices in F_{m-1}. Hence $d_I(i) \le b_i$ for all $i \in I$. In particular, $d_I(v_m) \le b_{v_m}$. But, $\prod_i x_i^{b_i}$ is not divisible by $m_{I,\Delta}$. This is only possible if $\deg_{x_{v_m}} m_{I,\Delta} = d_I(v_m)+1 > b_{v_m}$, which requires that $d_I(v_m) = b_{v_m}$. Therefore v_m has exactly b_{v_m} edges to F_{m-1}. □

Lemma 5.9. *Let $P \in \mathcal{P}_{G,\Delta}$. Then $\Theta_{\Pi,G,\Delta}(\Phi_{\Pi,G,\Delta}(P)) = P$.*

Proof. Let $P = (b_1, \ldots, b_n)$, $F = \Phi_{\Pi,G,\Delta}(P)$, and $\Theta_{\Pi,G,\Delta}(F) = P' = (b'_1, \ldots, b'_n)$. As before, let $F_1, \ldots, F_n = F$ be the oriented forests made in the construction of $\Phi_{\Pi,G,\Delta}(P)$, and let v_m $(1 \le m \le n)$ be the vertex in F_m but not F_{m-1}. For each m, we consider two cases.

Case 1: $e_F(v_m)$ exists. The edge $e_F(v_m)$ must go to a vertex in F_{m-1}. By Lemma 5.7, all edges e from v_m such that $e <_{\pi(F)} e_F(v_m)$ must go to vertices in F_{m-1}. By construction, there are b_{v_m} edges e from v_m to F_{m-1} such that $e <_{\pi(F_{m-1})} e_F(v_m)$. Because the orders $\pi(F_{m-1})$ and $\pi(F)$ are consistent, there are b_{v_m} edges e from v_m such that $e <_{\pi(F)} e_F(v_m)$. Thus $b'_{v_m} = b_{v_m}$.

Case 2: $e_F(v_m)$ does not exist. Then $|U_{m-1}| = 0$. By Lemma 5.8, v_m has exactly b_{v_m} edges to vertices in F_{m-1}. By Lemma 5.7, these are the edges from v_m to vertices j such that $j <_{\pi(F)} v_m$. Therefore $b'_{v_m} = b_{v_m}$. It follows that $b'_{v_m} = b_{v_m}$ for all m. Therefore $P' = P$. □

Lemma 5.10. *Let $F \in \mathcal{F}_{G,\Delta}$. Then $\Phi_{\Pi,G,\Delta}(\Theta_{\Pi,G,\Delta}(F)) = F$.*

Proof. Let $P = (b_1, \ldots, b_n) = \Theta_{\Pi,G,\Delta}(F)$ and $F' = \Phi_{\Pi,G,\Delta}(P)$. Let $F_1, \ldots, F_n = F'$ be the oriented forests made in the construction of $\Phi_{\Pi,G,\Delta}(P)$, and let v_m $(1 \le m \le n)$ be the vertex in F_m but not F_{m-1}. We prove by induction on m that F_m is a subforest of F whose vertices are the $m+1$ smallest vertices of F in $\pi(F)$. 0 is the smallest vertex in the order $\pi(F)$, so the claim is true for $m = 0$.

Assume that F_{m-1} is a subforest of F whose vertices are the m smallest vertices of F in $\pi(F)$. Let v'_m be the $(m+1)^{\text{st}}$ smallest vertex of F in $\pi(F)$. We consider two cases.

Case 1: $e_F(v'_m)$ exists. Let $e_F(v'_m) = (v'_m, v)$. Because $\Pi(G, \Delta)$ is a proper set of forest orders, $v <_{\pi(F)} v'_m$. Thus $v \in F_{m-1}$. Because F_{m-1} consists of the m smallest vertices of F in $\pi(F)$, if an edge e from v'_m satisfies $e <_{\pi(F)} e_F(v'_m)$, then e is to a vertex in F_{m-1}. By definition of $\Theta_{\Pi,G,\Delta}$, there are $b_{v'_m}$ edges from v'_m such that $e <_{\pi(F)} e_F(v'_m)$. These edges and the edge $e_F(v'_m)$ all go from v'_m to vertices in F_{m-1}; hence $v'_m \in U_{m-1}$.

For each $i \in U_{m-1}$, let e_i be the $(b_i+1)^{\text{st}}$ smallest edge from i to F_{m-1} in the order $\pi(F_{m-1})$. Because F_{m-1} consists of the m smallest vertices of F in $\pi(F)$, all edges e coming out of $i \in U_{m-1}$ and satisfying $e <_{\pi(F)} e_i$ must go to a vertex in F_{m-1}. Moreover, because the orders $\pi(F_{m-1})$ and $\pi(F)$ are consistent, an edge from i satisfies $e <_{\pi(F_{m-1})} e_i$ if and only if it satisfies $e <_{\pi(F)} e_i$. Thus, for each $i \in U_{m-1}$ there are exactly b_i edges e from i satisfying $e <_{\pi(F)} e_i$. By choice of b_i, e_i is an edge in F. It follows that f_{m-1}, the oriented forest consisting of F_{m-1}, all $i \in U_{m-1}$, and all e_i for $i \in U_{m-1}$, is a subforest of F. So, the orders $\pi(f_{m-1})$ and $\pi(F)$ are consistent. Because $v'_m \in U_{m-1}$ and v'_m is the smallest vertex not in F_m in the order $\pi(F)$, v'_m is the smallest vertex in U_{m-1} in the order $\pi(f_{m-1})$. Therefore F_m consists of F_{m-1}, v'_m, and $e_F(v'_m)$ and is a subforest of F whose vertices are the $m+1$ smallest vertices of F in $\pi(F)$.

Case 2: $e_F(v'_m)$ does not exist. Then v'_m is a root of F. Because v'_m is the smallest vertex in $\pi(F)$ not in F_{m-1}, no edges in F go from a vertex outside F_{m-1} to a vertex in F_{m-1}. Because $\Pi(G,\Delta)$ is a proper set of forest orders, v'_m must be the numerically smallest root of F outside of F_{m-1}; moreover, because F is a proper forest, v'_m must be the numerically smallest vertex of F outside of F_{m-1}. By definition of $\Theta_{\Pi,G,\Delta}$, each $i \notin F_{m-1}$ has at most b_i edges to F_{m-1}; hence $|U_{m-1}| = 0$. Then F_m consists of F_{m-1} and v'_m; therefore F_m is a subforest of F whose vertices are the $m+1$ smallest vertices of F in $\pi(F)$.

This implies that $F_n = F'$ is a subforest of F whose vertices are the vertices of F. Thus $F' = F$. □

Proof of Proposition 5.1. Proposition 5.1 follows from Lemmas 5.4, 5.6, 5.9, and 5.10. □

Hence the (G,Δ)-parking functions biject to the Δ-proper forests of G, as claimed by Theorem 3.1.

6 Δ-Forest Algebras

Let G be a graph on $\{0,\dots,n\}$ and Δ be a simplicial complex on $\{1,\dots,n\}$. For each nonempty $I \subseteq \{1,\dots,n\}$, let H_I denote the set of edges between vertices in I and vertices in $\{0,1,\dots,n\} \setminus I$. Associate with each edge $e \in G$ a commutative variable ϕ_e, and let $\Phi_{G,\Delta}$ be the algebra over $\mathbb{K}$ generated by the ϕ_e, obeying $\phi_e^2 = 0$ for all edges $e \in G$ and $\prod_{e \in H_I} \phi_e = 0$ for all nonempty $I \notin \Delta$.

Define a set of edges $H \subseteq G$ to be Δ-*allowed* if $\prod_{e\in H} \phi_e$ does not vanish in $\Phi_{G,\Delta}$. Equivalently, H is Δ-allowed if the set of vertices not connected to 0 in $G \setminus H$ is an element of Δ. For $i = 1,\dots,n$, define

$$X_i = \sum_{\substack{e=(i,j)\in G \\ i<j}} \phi_e - \sum_{\substack{e=(i,j)\in G \\ i>j}} \phi_e,$$

and let $\mathcal{C}_{G,\Delta}$ be the subalgebra of $\Phi_{G,\Delta}$ generated by $X_1,\dots,X_n$.

Proposition 6.1. *For all graphs G and all Δ, $\dim \mathcal{C}_{G,\Delta} = N_{G,\Delta}$, where $N_{G,\Delta}$ is the number of Δ-forests of G. Moreover, the k^{th} graded component $\mathcal{C}^k_{G,\Delta}$ has dimension equal to the number of Δ-forests F of G with external activity $|G| - |F| - k$.*

Define $\mathcal{S}_{G,\Delta}$ as the subspace of $\mathbb{K}[y_1,\dots,y_n]$ linearly spanned by $\alpha_H = \prod_{e\in H}(\alpha_e)$ as H ranges over all Δ-allowed subgraphs of G, where $\alpha_e = y_i - y_j$ for $e = (i,j)$ with $0 < i < j$ and $\alpha_e = -y_j$ for $e = (0,j)$.

Lemma 6.2. *For any Δ-allowed edge set $H \subseteq G$ and sequence $a = (a_1,\dots,a_n)$ with sum $|H|$, the coefficient of $\prod_{e\in H}\phi_e$ in the expansion $\frac{1}{a_1!\cdots a_n!}X_1^{a_1}\cdots X_n^{a_n}$ equals the coefficient of $y_1^{a_1}\cdots y_n^{a_n}$ in the expansion α_H.*

Proof. For fixed H and a, define an (H, a)-*valid assignment* as an assignment of each edge of H to one of its endpoints such that each vertex $i \in \{1, \dots, n\}$ has a_i edges assigned to it. In each (H, a)-valid assignment, let the value of an edge be $+1$ if it is assigned to its smaller endpoint, and -1 if it is assigned to its larger endpoint. Define the value of an (H, a)-valid assignment to be the product of the values of its edges. Finally define $f(H, a)$ as the sum of the values of all (H, a)-valid assignments.

The coefficient of $\prod_{e \in H} \phi_e$ in the expansion $\frac{1}{a_1! \cdots a_n!} X_1^{a_1} \cdots X_n^{a_n}$ and the coefficient of $y_1^{a_1} \cdots y_n^{a_n}$ in the expansion α_H both count $f(H, a)$ — the first by choosing edges to assign to each vertex, and the second by choosing the vertex to which each edge is assigned. Therefore these coefficients are equal. □

Lemma 6.3. *For all G, Δ and all k, the k^{th} graded components $\mathcal{C}_{G,\Delta}^k$ and $\mathcal{S}_{G,\Delta}^k$ obey $\dim \mathcal{C}_{G,\Delta}^k = \dim \mathcal{S}_{G,\Delta}^k$.*

Proof. Define $b_{H,a} = f(H, a)$, and let the matrix $B = (b_{H,a})$, as H ranges over all Δ-allowed sets of k edges and $a = (a_1, \dots, a_n)$ ranges over all sequences of length n with sum k. Then, by Lemma 6.2, the dimensions of the k^{th} graded components of $\mathcal{C}_{G,\Delta}$ and $\mathcal{S}_{G,\Delta}$ both equal the rank of B. Therefore $\dim \mathcal{C}_{G,\Delta}^k = \dim \mathcal{S}_{G,\Delta}^k$. □

Fix an order on the edges of G. For all Δ-forests F in G, let F^+ be the graph consisting of F and all externally active edges.

Lemma 6.4. *As F ranges over all Δ-forests of G, the $\alpha_{G \setminus F^+}$ linearly span $\mathcal{S}_{G,\Delta}$.*

Proof. Suppose for sake of contradiction that there exists a Δ-allowed edge set H such that α_H cannot be expressed as a linear combination of the $\alpha_{G \setminus F^+}$. Out of all such edge sets, let H be lexicographically maximal with respect to the order of G's edges. Observe that because H is Δ-allowed, all spanning forests of $G \setminus H$ are Δ-forests. We consider two cases.

Case 1: No edge $e \in H$ is an externally active edge of any spanning forest $F \subseteq G \setminus H$. We claim that $G \setminus H$ has a spanning forest F such that F^+ includes all edges of $G \setminus H$. We may construct such an F by starting with an arbitrary spanning forest f and repeatedly applying the following algorithm: if $f^+ = G \setminus H$, stop; otherwise, let $e \in G \setminus H$ be an edge not in f^+. Because e is not externally active with respect to f, there exists an edge e' in the cycle in $f \cup e$ that is smaller than e. Modify f by replacing e' with e. This algorithm must terminate because it replaces an edge in f by a larger edge at each step. So, there exists F such that $F^+ = G \setminus H$. Consequently $H = G \setminus F^+$, and $\alpha_H = \alpha_{G \setminus F^+}$ is a contradiction.

Case 2: There exists an edge $e \in H$ that is externally active in a spanning forest $F \subseteq G \setminus H$. Let $e, e_1, e_2, \dots, e_k$ be a cycle in G such that e is the minimal edge in this cycle and $e_1, \dots, e_k \in G \setminus H$. Then, $\alpha_e = -(\alpha_{e_1} + \cdots + \alpha_{e_n})$. Let $H_1, H_2, \dots, H_n$ be the Δ-allowed edge sets obtained from H by replacing e with $e_1, e_2, \dots, e_n$, respectively. These are lexicographically larger than H, so $\alpha_{H_1}, \alpha_{H_2}, \dots, \alpha_{H_n}$ are all expressible as linear combinations of the $\alpha_{G \setminus F^+}$. But now $\alpha_H = -(\alpha_{H_1} + \cdots + \alpha_{H_n})$ is a contradiction. □

Lemma 6.5. *As F ranges over all Δ-forests of G, the $\alpha_{G \setminus F^+}$ form a linear basis of $\mathcal{S}_{G,\Delta}$.*

Proof. By Lemma 6.4, it suffices to prove $\dim \mathcal{S}_{G,\Delta} = N_{G,\Delta}$, where $N_{G,\Delta}$ denotes the number of Δ-forests of G. We induct on the number of edges in G.

Say a Δ-forest F is a *minimal* Δ*-forest* if the forest produced by removing any edge $e \in F$ from F is not a Δ-forest. If G is a minimal Δ-forest, then $\dim \mathcal{S}_{G,\Delta} = 1 = N_{G,\Delta}$. If G is a forest that is not a Δ-forest, then $\dim \mathcal{S}_{G,\Delta} = 0 = N_{G,\Delta}$. This proves the induction's base case.

If G has at least one edge, choose an edge $e = (i, j)$ where $i < j$. For all $I \in \Delta$, define

$$f_e(I) = \begin{cases} I \setminus \{j\} & j \in I \\ I \setminus \{i\} & j \notin I, i \in I \\ I & \text{otherwise} \end{cases},$$

and let $\Delta_e = \{f_e(I) | I \in \Delta\}$. It is clear that Δ_e is also a simplicial complex.

Let $G - e$ be G with e removed; let G/e be G with e contracted and i and j both relabeled as i. The Δ-forests of G that do not include e are the Δ-forests of $G - e$, and the Δ-forests of G that include e biject to the Δ_e-forests of G/e by contraction of e. Thus $N_{G,\Delta} = N_{G-e,\Delta} + N_{G/e,\Delta_e}$. By the inductive hypothesis, $\dim \mathcal{S}_{G-e,\Delta} = N_{G-e,\Delta}$ and $\dim \mathcal{S}_{G/e,\Delta_e} = N_{G/e,\Delta_e}$.

Let $\mathcal{S}'_{G,\Delta}$ denote the span of the α_H, where H is Δ-allowed and $e \in H$. Let $\mathcal{S}''_{G,\Delta}$ denote the span of the α_H, where H is Δ-allowed and $e \notin H$. We have $\dim \mathcal{S}'_{G,\Delta} = \dim \mathcal{S}_{G-e,\Delta}$ because these spaces are isomorphic as vector spaces via multiplication by α_e. Let p be the vector space homomorphism that takes elements of $\mathcal{S}_{G,\Delta}$ modulo $y_i - y_j$. Then $p(\mathcal{S}''_{G,\Delta}) = \mathcal{S}_{G/e,\Delta_e}$. Thus $\dim \mathcal{S}''_{G,\Delta} = \dim \mathcal{S}_{G/e,\Delta_e} + \dim \ker(p)$. But, $\mathcal{S}'_{G,\Delta} \cap \mathcal{S}''_{G,\Delta} \subseteq \ker(p)$. Hence

$$\dim \mathcal{S}''_{G,\Delta} \geq \dim \mathcal{S}_{G/e,\Delta_e} + \dim(\mathcal{S}'_{G,\Delta} \cap \mathcal{S}''_{G,\Delta}).$$

Because $\mathcal{S}'_{G,\Delta}$ and $\mathcal{S}''_{G,\Delta}$ together span $\mathcal{S}_{G,\Delta}$, we have

$$\dim \mathcal{S}_{G,\Delta} = \dim \mathcal{S}'_{G,\Delta} + \dim \mathcal{S}''_{G,\Delta} - \dim(\mathcal{S}'_{G,\Delta} \cap \mathcal{S}''_{G,\Delta}).$$

Summing the last two relations yields

$$\dim \mathcal{S}_{G,\Delta} \geq \dim \mathcal{S}'_{G,\Delta} + \dim \mathcal{S}_{G/e,\Delta_e} = \dim \mathcal{S}_{G-e,\Delta} + \dim \mathcal{S}_{G/e,\Delta_e}.$$

By induction, the last quantity equals $N_{G-e,\Delta} + N_{G/e,\Delta_e} = N_{G,\Delta}$, so $\dim \mathcal{S}_{G,\Delta} \geq N_{G,\Delta}$. But Lemma 6.4 implies $\dim \mathcal{S}_{G,\Delta} \leq N_{G,\Delta}$. Thus $\dim \mathcal{S}_{G,\Delta} = N_{G,\Delta}$. ☐

Proof of Proposition 6.1. Proposition 6.1 follows from Lemmas 6.3 and 6.5. ☐

Lemma 6.6. *$\mathcal{C}_{G,\Delta}$ is a subalgebra of $\mathcal{B}_{G,\Delta}$.*

Proof. For all $I \in \Delta$,

$$\left(\sum_{i \in I} X_i\right)^{D_I + 1} = \left(\sum_{e \in H_I} \pm\phi_e\right)^{D_I + 1} = 0$$

because each term of the expansion $\left(\sum_{e \in H_I} \pm\phi_e\right)^{D_I+1}$ is divisible by the square of some ϕ_e. For all $I \notin \Delta$,

$$\left(\sum_{i \in I} X_i\right)^{D_I} = \left(\sum_{e \in H_I} \pm\phi_e\right)^{D_I} = 0$$

because the only square-free term of the expansion $\left(\sum_{e \in H_I} \pm\phi_e\right)^{D_I}$ is $\prod_{e \in H} \phi_e$, which is 0 because $I \notin \Delta$. ☐

We can now prove Theorem 3.2.

Proof of Theorem 3.2. Recall that $\mathcal{J}_{G,\Delta}$ is a deformation of $\mathcal{I}_{G,\Delta}$. By Theorem 2.5 and Lemma 6.6, we have the termwise inequality of Hilbert series $\operatorname{Hilb}\mathcal{A}_{G,\Delta} \geq \operatorname{Hilb}\mathcal{B}_{G,\Delta} \geq \operatorname{Hilb}\mathcal{C}_{G,\Delta}$. But, by Theorem 3.1 and Proposition 6.1, $\dim \mathcal{A}_{G,\Delta} = N_{G,\Delta} = \dim \mathcal{C}_{G,\Delta}$. Thus we in fact have $\dim \mathcal{A}_{G,\Delta} = \dim \mathcal{B}_{G,\Delta} = \dim \mathcal{C}_{G,\Delta} = N_{G,\Delta}$ and $\operatorname{Hilb}\mathcal{A}_{G,\Delta} = \operatorname{Hilb}\mathcal{B}_{G,\Delta} = \operatorname{Hilb}\mathcal{C}_{G,\Delta}$. By Theorem 2.5, the standard monomial basis of $\mathcal{A}_{G,\Delta}$ is a basis of $\mathcal{B}_{G,\Delta}$. By Proposition 6.1, $\dim \mathcal{A}^k_{G,\Delta} = \dim \mathcal{B}^k_{G,\Delta} = \dim \mathcal{C}^k_{G,\Delta}$ equals the number of Δ-forests F of G with external activity $|G| - |F| - k$, and the theorem is proved. ☐

7 ρ-Algebras

In this section, we discuss ρ-algebras, a special class of power algebras and related monomial algebras. Let $\rho = (\rho_1, \ldots, \rho_n)$ be a nonincreasing sequence of positive integers. For all nonempty $I \subseteq \{1, \ldots, n\}$, let

$$m_{I,\rho} = \left(\prod_{i\in I} x_i\right)^{\rho_{|I|}} \qquad \text{and} \qquad p_{I,\rho} = \left(\sum_{i\in I} x_i\right)^{|I|\rho_{|I|}},$$

and $\mathcal{I}_\rho = \langle m_{I,\rho}\rangle$ and $\mathcal{J}_\rho = \langle p_{I,\rho}\rangle$ be ideals in $\mathbb{K}[x_1, \ldots, x_n]$ generated by all such $m_{I,\rho}$ and $p_{I,\rho}$, respectively. Define the algebras $\mathcal{A}_\rho = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_\rho$ and $\mathcal{B}_\rho = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_\rho$.

These algebras are related to the *ρ-parking functions*, a natural generalization of parking functions studied in [16] and [22]. A sequence $(b_1, \ldots, b_n)$ is a ρ-parking function if and only if its decreasing rearrangement is termwise less than ρ. Equivalently, $(b_1, \ldots, b_n)$ is a ρ-parking function if and only if $\prod_i x_i^{b_i}$ is nonvanishing in $\mathcal{A}_\rho$. Note that the $(n, n-1, \ldots, 1)$-parking functions are the original parking functions of size n.

Observe that $\mathcal{I}_\rho$ is a monotone monomial ideal and $\mathcal{J}_\rho$ is its deformation. While $\mathcal{I}_\rho$ is not always a monomization of $\mathcal{J}_\rho$, Theorem 2.5 implies that the monomials $\prod_i x_i^{b_i}$, as $(b_1, \ldots, b_n)$ ranges over all ρ-parking functions, span $\mathcal{B}_\rho$, and the Hilbert series of $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$ obey the termwise inequality $\operatorname{Hilb}\mathcal{A}_\rho \geq \operatorname{Hilb}\mathcal{B}_\rho$.

When $\rho = (l + (n-1)k, l + (n-2)k, \ldots, l)$, the ideals $\mathcal{I}_\rho$ and $\mathcal{J}_\rho$ are $\mathcal{I}_G$ and $\mathcal{J}_G$, where $G = K^{k,l}_{n+1}$, the graph with k edges between any two nonzero vertices and l edges between any nonzero vertex and 0. Hence Theorem 1.1 implies the following result.

Proposition 7.1. *When $\rho = (l + (n-1)k, l + (n-2)k, \ldots, l)$ is a decreasing arithmetic sequence, $\mathcal{I}_\rho$ is a monomization of $\mathcal{J}_\rho$.*

We claim that $\mathcal{I}_\rho$ is a monomization of $\mathcal{J}_\rho$ for another class of ρ as well.

Proposition 7.2. *When $\rho = (l + (n-1)k + 1, l + (n-2)k, \ldots, l)$ is a decreasing arithmetic sequence whose largest term is increased by 1, $\mathcal{I}_\rho$ is a monomization of $\mathcal{J}_\rho$.*

Observe that when $\rho = (l + (n-1)k + 1, l + (n-2)k, \ldots, l)$, $\mathcal{I}_\rho$ and $\mathcal{J}_\rho$ are $\mathcal{I}_{G,\Delta}$ and $\mathcal{J}_{G,\Delta}$, where $G = K^{k,l}_{n+1}$ and $\Delta = \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}$. Thus, Proposition 7.2 follows as a corollary to Theorem 3.2.

8 Conclusion

In this paper, we determined monomizations for a large class of power ideals, generalizing the monomizations of the power ideals $\mathcal{J}_G$ and $\mathcal{J}_{G,1}$ by Postnikov-Shapiro and Desjardins. Consequently, we found a way to compute the Hilbert series of a class of power algebras that interpolate between the central and external zonotopal algebras. We found a combinatorial interpretation of these monomizations in terms of (G, Δ)-parking functions and Δ-forests of G. In the process, we found a bijection between the (G, Δ)-parking functions and the Δ-proper forests of G, generalizing the known bijection between the G-parking functions and the spanning trees of G. We also specialized our monomization theory to ρ-algebras and found a new class of ρ-algebras that admit monomization. However, the general questions of when monomization is possible and how to find such a monomization remain unanswered; the following approaches may yield interesting and useful extensions of this paper's monomization theory and make progress toward an answer to these questions.

One approach for future investigation is to determine a way to compute the Hilbert series of power algebras that interpolate between the central and internal zonotopal algebras. Suppose a graph G on $\{0, 1, \ldots, n\}$ has at least one edge between any two vertices, and let Δ be a simplicial complex on $\{1, \ldots, n\}$. For all nonempty $I = \{i_1 < \cdots < i_r\} \subseteq \{1, \ldots, n\}$, let

$$m'_{I,\Delta} = \begin{cases} m_I^- & I \in \Delta \\ m_I & I \notin \Delta \end{cases} \qquad \text{and} \qquad p'_{I,\Delta} = \begin{cases} p_I^- & I \in \Delta \\ p_I & I \notin \Delta \end{cases}.$$

Let $\mathcal{I}'_{G,\Delta} = \langle m'_{I,\Delta} \rangle$ and $\mathcal{J}'_{G,\Delta} = \langle p'_{I,\Delta} \rangle$ be the ideals generated by all such $m'_{I,\Delta}$ and $p'_{I,\Delta}$, respectively. Then $\mathcal{I}'_{G,\Delta}$ is a monotone monomial ideal and $\mathcal{J}'_{G,\Delta}$ is its deformation. Define the algebras $\mathcal{A}'_{G,\Delta} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}'_{G,\Delta}$ and $\mathcal{B}'_{G,\Delta} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}'_{G,\Delta}$. Experiments by the computer program **Macaulay2** suggest the following conjecture.

Conjecture 8.1. *The standard monomial basis of $\mathcal{A}'_{G,\Delta}$ is a basis of $\mathcal{B}'_{G,\Delta}$.*

When $\Delta = \{\emptyset\}$ or $\Delta = \mathcal{P}(\{1, \ldots, n\})$, this conjecture reduces to Theorem 1.1 and Theorem 1.3, respectively. This conjecture would interpolate between these two results, further generalizing our monomization theory. The author has proven Conjecture 8.1 when the elements of Δ each contain at most one element. In the context of ρ-algebras, setting $G = K_{n+1}^{k,l}$ and $\Delta = \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}$ yields the following result.

Corollary 8.2. *When $\rho = (l + (n-1)k - 1, l + (n-2)k, \ldots, l)$ is a decreasing arithmetic sequence whose largest term is decreased by 1, $\mathcal{I}_\rho$ is a monomization of $\mathcal{J}_\rho$.*

Another approach is to characterize all ρ for which $\mathcal{I}_\rho$ is a monomization of $\mathcal{J}_\rho$. Say that ρ is *almost linear* if the differences $\rho_i - \rho_{i+1}$ differ from each other by at most one. Though Proposition 7.1, Proposition 7.2, and Corollary 8.2 describe a large class of ρ for which monomization holds, there are other cases as well; for instance, monomization holds for $\rho = (5, 5, 3)$ and $\rho = (8, 6, 5, 3)$. However, no strictly decreasing ρ is known such that ρ admits monomization and ρ is not almost linear [17]. Almost-linearity is also not a sufficient condition for monomization, as $\operatorname{Hilb} \mathcal{A}_\rho = \operatorname{Hilb} \mathcal{B}_\rho$ does not hold for the almost linear $\rho = (6, 4, 2, 1)$ and $\rho = (8, 6, 5, 3, 1)$. Moreover, for $n \geq 5$, the author does not know of a strictly decreasing ρ satisfying $\operatorname{Hilb} \mathcal{A}_\rho = \operatorname{Hilb} \mathcal{B}_\rho$ not of the forms given by Proposition 7.1, Proposition 7.2, and Corollary 8.2.

These observations, along with results of computer experiments, suggest the following conjecture:

Conjecture 8.3. *If ρ is a strictly decreasing sequence, $\operatorname{Hilb} \mathcal{A}_\rho = \operatorname{Hilb} \mathcal{B}_\rho$ holds if and only if ρ is of the form $(l + (n-1)k + c, l + (n-2)k, \ldots, l)$, where $c \in \{-1, 0, 1\}$, or $(l + 3k + c, l + 2k + c, l + k, l)$, where $c \in \{-1, 1\}$.*

Yet another approach is to examine power ideals of the form $\mathcal{J}'_G = \langle p'_I \rangle$ where I ranges over all subsets of $\{1, \dots, n\}$ and

$$p'_I = \left(\sum_{i \in I} x_i \right)^{D_I + k_I},$$

for some set of constants $k_I \in \{0, 1, 2\}$. A monomization theory on these ideals may generalize to a broader monomization theory. However, we expect that a monomization theory on these power ideals will require changes to aspects of our current monomization theory, as many results in our monomization theory do not have analogues for these power ideals. For instance, if we take $G = K_{n+1}^{k,l}$, $k_I = 2$ for single-element subsets I, and $k_I = 0$ elsewhere, then the power ideal $\mathcal{J}'_G$ is the power ideal $\mathcal{J}_\rho$, where $\rho = (l + (n-1)k + 2, l + (n-2)k, \dots, l)$; this power ideal does not always (and conjecturally, never) admits the monomization $\mathcal{I}_\rho$; if a monomization exists, modifications of the ideal $\mathcal{I}_\rho$ will be needed.

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