Maximal Monochromatic Geodesics in an Antipodal Coloring of Hypercube

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Abstract

A geodesic in the hypercube is the shortest possible path between two vertices. Leader and Long (2013) conjectured that, in every antipodal 2-coloring of the edges of the hypercube, there exists a monochromatic geodesic between antipodal vertices. For this and an equivalent conjecture, we prove the cases $n = 2, 3, 4, 5$. We also examine the *maximum* number of monochromatic geodesics of length k in an antipodal 2-coloring and find it to be $2^{n-1}(n-k+1)\binom{n-1}{k-1}(k-1)!$. In this case, we classify all colorings in which this maximum occurs. Furthermore, we explore the maximum number of antipodal geodesics in a subgraph of the hypercube with a fixed proportion of edges, providing a conjectured optimal configuration as a lower bound, which, interestingly, contains a constant proportion of geodesics with respect to n. Finally, we present a series of smaller results that could be of use in finding an upper bound on the maximum number of antipodal geodesics in such a subgraph of the hypercube.

1. Introduction

Graph theory is a burgeoning field of mathematics centered on the study of mathematical structures called graphs, and is intimately related to Ramsey Theory, another field of mathematics concerned with showing that patterns must emerge in sufficiently large systems. Often, such determinations are made by coloring the elements of these systems, in our case the edges or vertices of graphs.

Within graph and Ramsey theory, we consider 2-colorings of the edges of the hypercube Q_n with vertices $\{0,1\}^n$, where edges connect vertices that differ only in a single coordinate. We call a path on this hypercube a *geodesic* if it is the shortest possible path between two points.

We also define the *antipodal* vertex x^a of x to be the unique vertex on Q_n that is farthest from x. In other words, given $x = (e_1, e_2, \ldots, e_n)$, where each of $e_1, e_2, \ldots, e_n$ is either 0 or 1, $x^a = (1-e_1, 1-e_2, \ldots, 1-e_n)$. Furthermore, given an edge xy, its antipodal edge is defined to be $x^a y^a$. We call a coloring of Q_n antipodal if antipodal edges are colored different colors.

A classic result of Dirac [1] states that in a graph of average degree d, there must exist some path of length d. As of late, this kind of result has been expanded to the specific case of Q_n; Long recently showed that every subgraph G of the hypercube with average degree

d contains some path of length $2^{d/2} - 1$ [2], and Leader and Long complemented this result by showing a similar statement for geodesics: that every subgraph G with average degree d must contain some geodesic of length d [3].

A natural Ramsey-theoretical question to ask of all of this research is whether, given these geodesics in the hypercube, we can also guarantee that in some 2-coloring of the edges of the hypercube, there exists some geodesic with all of its edges the same color, or monochromatic. Specifically, we examine the following conjectures, proposed by Leader and Long and based off of those of Norine, Feder, and Subi [4] [3].

Conjecture 1.1. *Given an antipodal* 2*-coloring of* Q_n*, there exists a monochromatic geodesic between some pair of antipodal vertices.*

Conjecture 1.2. *Given a* 2*-coloring of* Q_n*, there exists a geodesic between antipodal vertices that changes color at most once.*

As it turns out, these two conjectures are equivalent. We present a proof below, similar to that of Leader and Long.

Proof. We must prove it in both directions. If Conjecture 1.1 is true, consider some 2-coloring of the cube Q_n, and in particular consider the larger hypercube Q_{n+1}, which Q_n is a subcube of where $x_{n+1} = 0$: we color the remaining edges of Q_{n+1} such that the overall coloring in the $(n + 1)$-dimensional hypercube is antipodal. Now, consider the path GG^a, where G is a monochromatic, say red, geodesic on Q_{n+1} (therefore G^a is also monochromatic, say blue). Now let $GG^a = x_1x_2x_3 \dots x_{2n+3}$, where $x_{2n+3} = x_1$ and $x_{n+2} = x_1^a$. Note that $G = x_1x_2x_3 \dots x_{n+2}$ contains all red edges and $G^a = x_{n+2} \dots x_{2n+3}$ contains all blue edges. Now, we know that between exactly two pairs of vertices connected by an edge in this path, the $(n+1)$st coordinate changes: that is, the edge goes between vertices $(e_1, e_2, e_3, \dots, e_n, 0)$ and $(e_1, e_2, e_3, \dots, e_n, 1)$ or vice versa. Without loss of generality let this occur on the edge x_ix_{i+1} where $1 \le i \le n+1$. Then we also know that the edge $x_{n+1+i}x_{n+2+i}$ also traverses that direction, since G^a is antipodal to G. Now, consider the path in the cube $x_{i+1} \dots x_{n+1+i}$. This path has length n, and, noting that we have excluded both edges on GG^a that traverse in the $(n+1)$st direction, it is on the subcube Q_n. Moreover, $x_{i+1} \dots x_{n+2}$ is a red geodesic, and $x_{n+2} \dots x_{n+1+i}$ is a blue geodesic traversing exactly the remaining directions, so $x_{i+1} \dots x_{n+1+i}$ is a geodesic between antipodal vertices that changes color at most once.

Alternatively, if Conjecture 1.2 is true, consider some antipodal 2-coloring of Q_n. We know that there exists some geodesic G that changes color at most once, say at x: denote the red section as R and the blue section as B: G, thus, is equivalent to RB. Let the two end vertices be v and v^a, where R starts at v and ends at x and B starts at x and ends at v^a. But then we know that R^a is a geodesic with endpoints at v^a and x^a, traversing the same directions as R, and is blue. Therefore BR^a is a monochromatic geodesic between x and x^a, antipodal vertices, so we are done. □

In this paper, we prove Conjecture 1.1, and thereby Conjecture 1.2, for $n = 2, 3, 4, 5$.

We also consider the opposite question, which has particular significance in extremal graph theory. Specifically, in addition to looking into this minimization of the number of geodesics on the hypercube, we also explore the *maximum* number of monochromatic geodesics of length k in an antipodal 2-coloring, showing that this is $2^{n-1}(n-k+1)\binom{n-1}{k-1}(k-1)!$ in Q_n. In addition to showing this maximum, we classify all of the colorings in which the maximum occurs, calling such colorings *subcube colorings* because of the nature of the construction.

Finally, we generalize this idea and look at the maximum number of geodesics in a subgraph of the hypercube with some constant proportion of edges, deriving a conjectured optimal configuration for this maximum (and thereby obtaining a lower bound) and computing the number of edges in this so-called *middle-layer subgraph.* We also obtain the trivial upper bound of $p \cdot 2^{n-1}n!$.

2. Minimum Number of Geodesics in an Antipodal Coloring of Q_n

In this section, we prove Conjectures 1.1 and 1.2 for $n = 2, 3, 4, 5$. We have also proved the conjectures for $n = 6, 7$ using a computer, but have not been able to develop a general proof strategy.

Our first lemma shows the existence of a monochromatic geodesic of length a given one of length $a - 1$ in Q_a, and thereby facilitates the casework that will occur later.

Lemma 2.1. *If there is a monochromatic geodesic G of length $a - 1$ in Q_a, then there also exists a monochromatic geodesic of length a.*

Proof. Let G be a red geodesic between vertices x_1 and x_2. Consider the edge e incident to x_2 in the coordinate direction not already traversed by G, connecting x_2 to another vertex, say x_3. If e is red, then we are done. If e is blue, then consider the antipodal red edge $x_3^a x_2^a$. Because we are considering Q_a, $x_3^a = x_1$, so the edge $x_1 x_2^a$ is red. Therefore, concatenating this edge to G, the resulting geodesic between x_2 is x_2^a is monochromatic red and of length a. □

This lemma established, we can begin our cases.

Lemma 2.2. *Conjecture 1 is true for $n = 2$*

Proof. We evidently have a monochromatic geodesic of length 1 (considering any edge). Therefore, by Lemma 2.1, we have a monochromatic geodesic of length 2. □

Lemma 2.3. *Conjecture 1 is true for $n = 3$*

Proof. Consider any vertex of Q_n. Two of its three incident edges must have the same color, and thus we have a monochromatic geodesic of length 2. Therefore, by Lemma 2.1, we have a monochromatic geodesic of length 3. □

Lemma 2.4. *Conjecture 1 is true for $n = 4$*

Proof. First, consider any vertex of Q_n. Two of its three incident edges must have the same color, and thus we have a monochromatic geodesic of length 2.

Without loss of generality, assume that this is the geodesic $(1,0,0,0) - (0,0,0,0) - (0,1,0,0)$. Let this geodesic be colored red. Because we are considering an antipodal coloring, the antipodal geodesic $(0,1,1,1) - (1,1,1,1) - (1,0,1,1)$ is also monochromatic, but blue instead of red. Now, let us consider the edges $(1,0,0,0)-(1,0,1,0)$ and $(1,0,0,0)-(1,0,0,1)$. If either is red, we can add that edge to our original geodesic and have a monochromatic geodesic of length 3, and, therefore, by Lemma 2.1, we would be done. Otherwise, both are blue. Now, consider the antipodal version of this: the edges $(1,0,1,1) - (1,0,1,0)$ and $(1,0,1,1) - (1,0,0,1)$. Again, if either is blue, we have a monochromatic blue geodesic of length 3 and, by Lemma 2.1, are done. Therefore, we consider the case when both are red.

Now, consider the square $ABCD$ where $A = (1,0,0,0)$, $B = (1,0,1,0)$, $C = (1,0,1,1)$ and $D = (1,0,0,1)$. We know that AB and AD are red, and BC and CD are blue. Therefore, B is the endpoint of both a monochromatic red geodesic of length 2 and a monochromatic blue geodesic of length 2. Considering any incident edge to B in a direction not already covered by these geodesics, no matter its color, we have a monochromatic geodesic of length 3 by concatenation to one or the other geodesic, and therefore have one of length 4 by Lemma 2.1, so are done.

□

Lemma 2.5. *Conjecture 1 is true for* $n = 5$

Proof. First, consider any vertex. Two of its three incident edges must have the same color, and thus we have a monochromatic geodesic of length 2. We first want to show that there exists a monochromatic geodesic of length 3.

Without loss of generality, assume that our monochromatic geodesic of length 2 is $(1,0,0,0,0)-(0,0,0,0,0)-(0,1,0,0,0)$. Let this geodesic be colored red. As before, because we are considering an antipodal coloring, the antipodal geodesic $(0,1,1,1,1)-(1,1,1,1,1)-(1,0,1,1,1)$ is also monochromatic, but blue instead of red. Now, let us consider the edges $(1,0,0,0,0)-(1,0,1,0,0)$, $(1,0,0,0,0)-(1,0,0,1,0)$, and $(1,0,0,0,0)-(1,0,0,0,1)$. If any of these are red, we can add that edge to our original geodesic and have a monochromatic geodesic of length 3. Therefore, we consider the case where all three of these are blue. By a similar argument, the edges $(1,0,1,1,1) - (1,0,1,1,0)$, $(1,0,1,1,1) - (1,0,1,0,1)$, and $(1,0,1,1,1)-(1,0,0,1,1)$ are all red, or else we have a monochromatic geodesic of length 3. Now, let us consider the edge $(1,0,1,0,0)-(1,0,1,1,0)$. If it is blue, we have the monochromatic geodesic of length 3: $(1,0,0,0,1)-(1,0,0,0,0)-(1,0,1,0,0)-(1,0,1,1,0)$. Otherwise, if it is red, we have the monochromatic geodesic of length 3: $(1,0,0,1,1) - (1,0,1,1,1) - (1,0,1,1,0) - (1,0,1,0,0)$.

Therefore, we can assume we have a monochromatic geodesic of length 3 in any (antipodal) coloring. Let this geodesic be, without loss of generality, red and the geodesic $(1,0,0,0,0) - (0,0,0,0,0) - (0,1,0,0,0) - (0,1,1,0,0)$. Because this is an antipodal coloring, we also know that this is a monochromatic blue geodesic: $(0,1,1,1,1) - (1,1,1,1,1) - (1,0,1,1,1) - (1,0,0,1,1)$ Now, consider the two edges: $(1,0,0,0,0) - (1,0,0,0,1)$ and

$(1, 0, 0, 1, 0$. If either is red, then we have a monochromatic geodesic of length 4 so are done by Lemma 2.1. Therefore, we consider the case where both are blue. By similar logic, we are done unless the two edges $(1, 0, 0, 1, 1) - (1, 0, 0, 0, 1)$ and $(1, 0, 0, 1, 1) - (1, 0, 0, 1, 0)$ are blue.

Now, consider the geodesic $G_s = (1, 0, 0, 1, 0) - (1, 0, 0, 0, 0) - (1, 0, 0, 0, 1)$. There are three edges incident to $(1, 0, 0, 0, 1)$ and three edges incident to $(1, 0, 0, 1, 0)$ that would each form a geodesic of length three with this geodesic. I claim that all of the edges incident to $(1, 0, 0, 0, 1)$ must be the same color (and the same for $(1, 0, 0, 1, 0)$) or else we have a monochromatic geodesic of length 4. For the sake of contradiction, consider the case where, without loss of generality, two of the edges incident to $(1, 0, 0, 0, 1)$ are blue and the other is red. Consider the red edge. There are two edges incident to $(1, 0, 0, 1, 0)$ that, along with the red edge, would form a geodesic of length 4 if added onto G_s. If either is red, we have a monochromatic geodesic of length 4, contradiction. Otherwise, they must be blue. However, in turn, one of these edges could form a geodesic of length 4 when added to G_s along with one of the other edges incident to $(1, 0, 0, 0, 1)$, which we know to blue: this is a contradiction, however, since this is monochromatic, and therefore the claim is true, or else we are done by Lemma 2.1.

Without loss of generality let those edges incident to $(1, 0, 0, 0, 1)$ be red and those incident to $(1, 0, 0, 1, 0)$ be blue. Because this is an antipodal coloring, we know the three edges incident in the directions at hand to $(0, 1, 1, 0, 1)$ are themselves red. Note that these edges occupy the same cube as those incident to $(1, 0, 0, 0, 1)$. We know that the remaining edges of this cube must themselves be blue, as otherwise we can form a monochromatic geodesic of length 4 with them, one of the edges incident to $(1, 0, 0, 0, 1)$ and G_s. But now consider some vertex on this cube that is both incident to one of the incident edges to $(1, 0, 0, 0, 1)$ and incident to a blue geodesic of length 3 on the cube, in the direction that both is not present on the cube and is not in the direction between $(1, 0, 0, 0, 1)$ and $(1, 0, 0, 0, 0)$. If this is blue, we have a blue geodesic of length 4 so are done by Lemma 2.1. Otherwise, if it is red, concatenating it with one of the incident edges to $(1, 0, 0, 0, 1)$, the edge connecting this to $(1, 0, 0, 0, 0)$, and edge between this and $(0, 0, 0, 0, 0)$, we have a red geodesic of length 4 so are done by Lemma 2.1. Thus, we are done.

□

3. Maximum Number of Geodesics in an Antipodal Coloring of Q_n

In this section, we wish to find the maximum number of monochromatic geodesics between antipodal vertices in an antipodal 2-coloring on the hypercube $Q_n = \{0, 1\}^n$, which has 2^n vertices. We will denote this maximum as M_n. Note that all of our lemmas and theorems assume an antipodal edge coloring with two colors of Q_n, unless stated otherwise.

Lemma 3.1. *The number of geodesics on Q_n between antipodal vertices is $2^{n-1}n!$.*

Proof. There are 2^{n-1} pairs of antipodal vertices. Between these vertices, we have n directions to traverse (or, if we think about this in the sense of coordinates, n coordinates to change). Thus, there are evidently $n!$ different orders in which we can traverse these directions, and thus $n!$ geodesics between each pair of antipodal vertices, so $2^{n-1}n!$ in total. □

Now, we define a *cycle* on Q_n to be the path GG^a, where G is some geodesic.

Lemma 3.2. *There are at maximum* 2 *monochromatic geodesics in each cycle.*

Proof. Consider some cycle $C = x_1x_2 \dots x_nx_{n+1} \dots x_{2n+1}$, with $x_1 = x_{2n+1}$ and $x_{n+1} = x_1^a$, which contains $2n + 1 - 1 = 2n$ distinct vertices. Now consider all of the shifts of these cycle $x_k \dots x_{2n+1}x_1 \dots x_{k-1}$. There are two possibilities. First, none of $x_k \dots x_{n+k} = x_k^a$ are monochromatic for $1 \leq k < n+1$, which implies that none of their antipodal versions $x_{n+k} \dots x_{2n+1}x_1 \dots x_k$ are either.

Otherwise, consider the case where there exists some k such that that geodesic is monochromatic. Then we know that each of the edges $x_kx_{k+1}, x_{k+1}x_{k+2}, \dots x_{n+k-1}x_{n+k}$ are the same color, say red, and because the coloring is antipodal, each of the edges $x_{n+k}x_{n+k+1} \dots x_{k-1}x_k$ are the other color, say blue. Thus, we know we have two monochromatic geodesics within our cycle C, call them G and G^a. But now consider any other geodesic $x_j \dots x_{n+j} = x_j^a$ for $1 \leq j < n+1, j \neq k$. If $j < k$, then x_jx_{j+1} is contained within G^a so is blue. However, $x_{n+j-1}x_{n+j}$ is contained within G, as $k - 1 < n + j - 1 < n + k - 1$ and thus is red. Therefore, this geodesic cannot be monochromatic. Oppositely, if $j > k$, then x_jx_{j+1} is contained within G, as $k < j < n + k$, so is blue. But $x_{n+j-1}x_{n+j}$ is contained within G^a, as $n + k - 1 < n + j - 1 < 2n + 1$, so is red. Therefore, this geodesic also cannot be monochromatic.

Thus, in each cycle C, there are at maximum 2 monochromatic ones, so we are done. □

Lemma 3.3. *Given some two coloring of the edges of* Q_n *where* $n > 2$*, a monochromatic red geodesic between* x *and* x^a *on* Q_n *and that each geodesic between* x *and* x^a *is monochromatic, all of the edges of* Q_n *must be red.*

Proof. Let G be our original red geodesic, and consider the final edge of this geodesic: between some vertex y and x^a. Without loss of generality, assume that this edge traverses the nth direction: that is, it is located between two points $(0, e_1, e_2, \dots, e_n)$ and $(1, e_1, e_2, \dots, e_n)$. Consider all geodesics between x and x^a ending with this edge: we know that there exists some geodesic whose first edge is in every direction except for the nth (since no geodesic can traverse the same direction twice) that contains this edge. Since this edge is red, and these geodesics must be monochromatic, we know every edge emitting from x except for the one in the nth direction must be red. And, of course, considering the second-to-last edge of G, there exists some geodesic between x and x^a both containing this edge and our nth direction edge emitting from x, and therefore all edges emitting from x are the same color. This implies that every geodesic between x and x^a is monochromatic red, as each contains some edge emitting from x (since they start at x), and, since every edge of the hypercube is in one of these geodesics, all of the edges of Q_n are red. □

Theorem 3.4. $M_n = 2^{n-1}(n-1)!$

Proof. First, we show that $M_n \leq 2^{n-1}(n-1)!$. By Lemma 3.2, we know that each cycle contains at maximum two geodesics. Now, notice that, clearly, each cycle is self-repeating/containing: that is, there are no two cycles that contain the same geodesic. This means that we can partition the $2^{n-1}n!$ overall geodesics into subsets containing $2n$ geodesics where the number of monochromatic geodesics is at maximum 2. Thus,

$$M_n \leq \frac{2}{2n} \cdot 2^{n-1}(n)! = 2^{n-1}(n-1)!$$

Thus, it suffices to present a case where $M_n = 2^{n-1}(n-1)!$: where every cycle contains 2 monochromatic geodesics. To do this, consider two subcubes of Q_n, $G_0 = (0, \{0,1\}^{n-1})$ and $G_1 = (1, \{0,1\}^{n-1})$. Let us color each edge of G_0 red; because the edges of G_1 are antipodal to those of G_0, it follows that each edge of G_1 is colored blue.

All that remains are the 2^{n-1} edges connecting G_0 to G_1. Consider some arbitrary coloring of $E_1, E_2, \ldots, E_{2^{n-1}}$ that satisfies the antipodal coloring. Now, consider the $(n-1)$-dimensional G_0 as its own separate hypercube. Between each pair of antipodal vertices on G_0 (the antipodal definition as it applies to the cube G_0), there are $(n-1)!$ geodesics, which are all monochromatic since all of the edges of G_0 are colored the same color. In particular, there are $(n-1)!$ geodesics ending at a particular endpoint, let's say x. The edge E_x that connects to this endpoint is either red or blue. If is red, then we have $(n-1)!$ monochromatic geodesic by attaching it to the geodesics on G_0 ending at x. Otherwise, if E_x is blue, if $x = (0, e_1, e_2, \ldots, e_n)$, we consider $x' = (1, e_1, e_2, \ldots, e_n)$ on G_1. There are, by the same argument, $(n-1)!$ blue geodesics ending at x' on G_1, and therefore $(n-1)!$ monochromatic geodesics on Q_n if we attach E_i to those geodesics.

Thus, for each pair of vertices on G_0 and G_1 $y = (0, f_1, f_2, \ldots, f_n)$ and $y' = (1, f_1, f_2, \ldots, f_n)$, we have $(n-1)!$ monochromatic geodesics on Q_n. This accounts for all geodesics (no monochromatic geodesic can have edges on both G_0 and G_1 for obvious reasons, so this covers all the bases), and, as there are 2^{n-1} such pairs of vertices, we have $2^{n-1}(n-1)!$ monochromatic geodesics on such a coloring of Q_n.

To make the proof tight, it remains to show that this monochromatic coloring of subcubes is the unique set of colorings that maximizes the number of geodesics. Call this type of 2-coloring a *subcube coloring*.

We've already shown that these colorings works; it remains to show that there are no other colorings that work. Now, for $n > 4$, assume that there is some non-subcube coloring for contradiction. Note that, by Lemma 3.2, there are at maximum 2 monochromatic geodesics in each cycle: our maximum, thus, occurs when each cycle contains two monochromatic geodesics. Now, consider some cycle GG^a, where G is a monochromatic (say red) geodesic between antipodal vertices. Without loss of generality assume that the endpoints of G are $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ and that the first edge in this geodesic is from $(0, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$ and the last is from $(1, 1, \ldots, 0)$ to $(1, 1, \ldots, 1)$ (or, to be more concise, the first edge is in the 1st direction and the last edge is in the nth direction). Note that both of these edges are red: consider all geodesics (and thus cycles) containing these two edges. In other words, consider all of the possible geodesics between $(1, 0, \ldots, 0)$ and $(1, 1, \ldots, 0)$ on the $(n-2)$-dimensional hypercube J $(1, \{0,1\}^{n-2}, 0)$. Note that, because our two outer edges surround this hypercube, for each cycle containing these edges to have a monochromatic geodesic, each geodesic on this $(n-2)$-dimensional hypercube between the two points at hand. must be monochromatic. This, in turn, by Lemma 3.3, implies that all edges on our $(n-2)$-dimensional hypercube are red.

Note further that this in turn implies that the hypercube J^a $(0, \{0,1\}^{n-2}, 1)$ is blue.

Now, consider some other monochromatic cycle not already covered with $C = HH^a$, where H and H^a are monochromatic and their endpoints are $v = (e_1, e_2, \ldots, e_n)$ and $v_a = (1-e_1, 1-e_2, \ldots, 1-e_n)$. Without loss of generality assume H is red. Now, we claim that either the first edge or the last edge of H must traverse either the 1st direction or the nth direction. Assume that this is not true for contradiction. Then we have the two inner vertices x and y, where $v - x$ is an edge on H with $x = (e_1, e_2, \ldots, (e_i + 1) \bmod 2, \ldots, e_n)$ and $v^a - y$ is an edge on H with $y = (1-e_1, 1-e_2, 1-e_3, \ldots (2-e_j) \bmod 2, \ldots, 1-e_n)$ for $i \neq j$. But this means, by the same logic as before, that the hypercube K in Q_n with the ith and jth coordinates set is monochromatic red. But note that, clearly, this hypercube shares edges with J^a (by setting the four coordinates that are fixed in one or the other hypercube and then varying the remainder), so we have a contradiction (as all of the edges of J^a are blue).

This means that for each such cycle, one of the end edges must traverse one of the original cycle's end directions (in our case the 1st and nth directions). Note that it cannot always be both directions, as that would not cover all of our cycles. For the first cycle C_a such that it cannot consist of both directions, without loss of generality let the direction of them that is traversed be the nth direction. From then on, for every cycle, one of the directions must be the nth direction (as otherwise we can intersect the $(n-2)$-dimensional hypercube created with the hypercube produced by C_a or our original cycle-produced hypercube).

Therefore, for every cycle, without loss of generality, if our coloring preserves maximality, one of the end edges must be the nth direction. Now, let us retreat to our original cycle GG^a. We now know that each geodesic ending in the edge from $(1, 1, \ldots, 0)$ to $(1, 1, \ldots, 1)$ must be monochromatic. That is, each geodesic in the hypercube $(\{0,1\}^{n-1}, 0)$ must be monochromatic. However, evidently, because we have once such geodesic red, it follows easily by Lemma 3.3 that all of the edges of the hypercube $(\{0,1\}^{n-1}, 0)$ must be red, and, because this is an antipodal coloring, we have that the hypercube $(\{0,1\}^{n-1}, 1)$ is blue. But this is a subcube coloring, a contradiction.

Thus, we have that no coloring that is not a subcube coloring can achieve our maximum, so we are done. Note that for our small cases $n = 2, 3, 4$, our proof does not necessarily function because of the limitations of Lemma 3.3. We leave these cases as exercises for the reader, noting that each of them can either be proved through casework, which is fairly easy in such small cases, or through slightly tweaking the above logic.

□

Theorem 3.5. *The maximum number of monochromatic geodesics of length k where $k \in \mathbb{Z}^+, k > 1$ in an antipodal 2-coloring is $2^{n-1}(n-k+1)\binom{n-1}{k-1}(k-1)!$. This maximum occurs only in a subcube coloring.*

Proof. It is clear that in a cycle, the maximum number of monochromatic geodesics of length k occurs when the cycle consists of two monochromatic antipodal geodesics. Thus our maximum occurs when every cycle satisfies this property, which we already know, by Theorem 3.4, only happens in subcube colorings.

Now, note that in such a cycle, there are $2(n-k+1)$ monochromatic geodesics, and that each of these geodesics of length k is part of $(n-k+1)!$ larger geodesics. These larger geodesics can be separated into groups of $n-k+1$ which are all part of the same cycle; therefore we are overcounting each geodesic $(n-k)!$ times. From there, since there are $2n$ geodesics in each cycle, the total number of geodesics of length k is:

$$2^{n-1}(n)! \cdot \frac{1}{2n} \cdot 2(n-k+1) \cdot \frac{1}{(n-k)!} = \frac{2^{n-1}(n-1)! \cdot (n-k+1)}{(n-k)!}$$

Note that $\frac{(n-1)!}{(n-k)!} = \binom{n-1}{k-1}(k-1)!$, so our expression is equivalent to:

$$2^{n-1}(n-k+1)\binom{n-1}{k-1}(k-1)!.$$

□

4. Maximum Number of Geodesics in a Subgraph of Q_n

In this section, we seek the maximum number of geodesics in a subgraph of the hypercube with some constant proportion of edges. In particular, we conjecture an optimal asymptotic configuration, which we call a *middle-layer subgraph*, and then compute the number of geodesics in such a subgraph. After providing this lower bound, we then provide a trivial upper bound.

To begin, we must define the concept of hamming weight.

Definition 4.1. *The* hamming weight $w(v)$ *of a vertex* $v \in Q_n$ *is the number of* 1*'s in the binary representation of* v.

From this, we define the middle layer of the hypercube. From here on, when defining our subgraph, we will only be considering hypercubes of even dimension, or of the form Q_{2n}. The construction is similar for odd dimension but introduces some unnecessary technicalities for our discussion, since we are mainly discussing the asymptotic case.

Definition 4.2. *The* middle layer *of the hypercube* Q_{2n} *is the set of vertices with hamming weight* n.

Using this, we formally define our middle-layer subgraph.

Definition 4.3. *A* k-middle layer subgraph *of* Q_{2n} *is the induced subgraph consisting of all vertices* v *for which* $n - k\sqrt{n/2} \le w(v) \le n + k\sqrt{n/2}$.

Lemma 4.4. *Given the k-middle layer subgraph H of Q_{2n} with $p2^{2n}$ vertices, let e is the number of edges incident to the vertices in our subgraph. Then $\lim_{n\to\infty} \frac{e}{n2^{2n}} = p$.*

Proof. Given that this is a middle-layer subgraph, we know that for each vertex, our subset contains all n edges incident to it except for vertices at the edge of the subset. There are $2\binom{2n}{n-k\sqrt{n/2}}$ such vertices. This gives us that the number of edges is greater than or equal to $np2^{2n} - 2n\binom{2n}{n-k\sqrt{n/2}}$. So our limit is at least

$$\lim_{n\to\infty} \frac{np2^{2n} - 2n\binom{2n}{n-k\sqrt{n/2}}}{n2^{2n}} = p - \lim_{n\to\infty} \frac{\binom{2n}{n-k\sqrt{n/2}}}{2^{2n-1}}$$

From the normal approximation to the binomial distribution, we have that

$$\frac{\binom{2n}{n-k\sqrt{n/2}}}{2^{2n}} = \frac{e^{-k^2/2}}{\sqrt{\pi n}}$$

Thus, given that k is a positive integer, we have that $e^{-k^2/2} < 1$, so our expression is less than or equal to $\frac{1}{\sqrt{n}}$, and thus our overall limit is equivalent to

$$p - \lim_{n\to\infty} \frac{1}{\sqrt{n}} = p$$

Notice that, even though we bounded the number of edges from above, the limit would remain the same with any constant less than $\binom{2n}{n-k\sqrt{n/2}}$ because the right side of the limit ended up going to 0. □

Note that the converse is also clearly true. From this lemma, we can establish the proportion of edges in a k-middle layer subgraph.

Lemma 4.5. *A k-middle layer subgraph of Q_{2n} contains $(\int_{-k}^{k} \frac{e^{-x^2/2}}{\sqrt{n\pi}} + o(1))2n2^{2n-1}$ edges.*

Proof. Notice that the proportion of vertices for which $n - k\sqrt{n/2} < w(v) < n + k\sqrt{n/2}$ is asymptotically equal to

$$\frac{\sum_{i=n-k\sqrt{n/2}}^{i=n+k\sqrt{n/2}} \binom{2n}{n-i}}{2^{2n}} = \int_{-k}^{k} \frac{e^{-x^2/2}}{\sqrt{\pi n}}$$

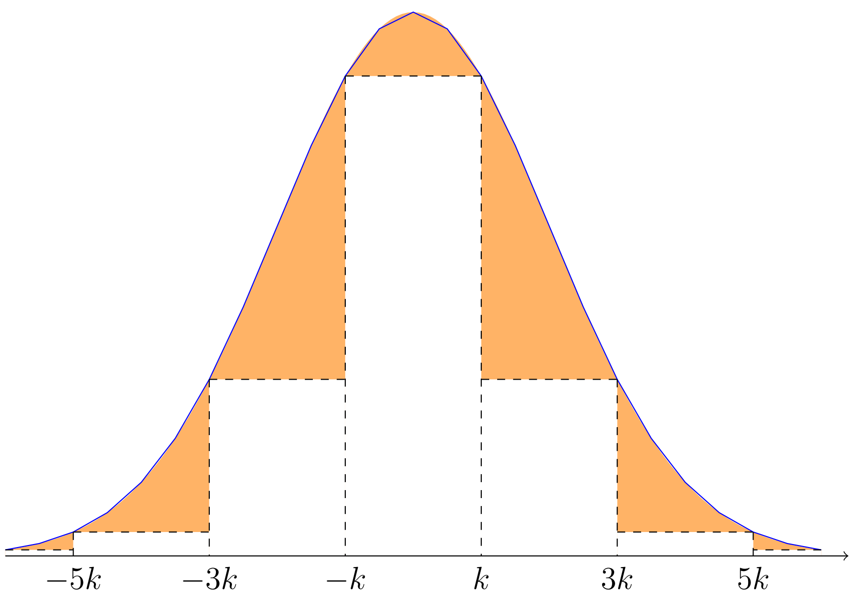
from the normal approximation to the binomial distribution. From Lemma 4.4, this is asymptotically the same as the proportion of edges involved, so the lemma follows. □

With this established, we can compute the number of antipodal geodesics in a middle-layer subgraph with a fixed proportion p of edges. The entire computation is not detailed here; the important aspect is that the antipodal geodesic must follow a path of edges that stays within the subgraph, and thus can be modeled as a random walk. Each value of j in the summation corresponds to a starting vertex of weight $n+j$ or $n-j$: the summation then calculates the probability that a random geodesic starting at that vertex will stay within the middle-layer. The major technique used in the calculation is the reflection principle, which allows us to equate the distribution of a given walk to one that is not constrained by the boundaries of the middle layer, a much simpler calculation.

Theorem 4.6. *The proportion of geodesics in a middle-layer subgraph of Q_{2n} with $2pn2^{2n-1}$ edges for $p = \int_{-k}^{k} \frac{e^{-x^2/2}}{\sqrt{\pi n}}$ edges is equal to:*

$$1 - \frac{(2k\sqrt{\frac{n}{2}}+1)\sum_{j\in\mathbb{Z}} \binom{2n}{n+(2j-1)k\sqrt{\frac{n}{2}}}}{2^{2n}}$$

There are two important aspects of this result that are worth noting. First, it is constant with respect to n, which is what leads us to believe that this coloring is optimal or at least close to an optimal lower bound. Second, interestingly, the proportion itself is equivalent to the area under the normal curve shown below, given, as before, $p = \int_{-k}^{k} \frac{e^{-x^2/2}}{\sqrt{2\pi}}$.



Conjecture 4.7. *For large n and a given proportion of edges p, a k-middle layer subgraph for appropriate k yields the maximum number of antipodal geodesics.*

Note that our result only establishes a lower bound on the proportion of geodesics and is conjectured to be optimal. The only upper bound, however, is the following trivial result:

Theorem 4.8. *A subgraph of the hypercube Q_n with $pn2^{n-1}$ edges contains at most $p2^{n-1}n!$ geodesics.*

Proof. Any given edge can be in $n!$ geodesics in a subgraph of Q_n: in particular, there are n places for the edge in the sequence of directions of the geodesic and $(n-1)!$ possible orderings of the remaining directions, so $n!$ geodesics in total. Thus, given that each geodesic contains n edges, a subgraph with $pn2^{n-1}$ edges has at most $\frac{pn2^{n-1} \cdot n!}{n} = p2^{n-1}n!$ geodesics. □

5. Expected Number of Geodesics in Subgraphs of Q_n

In this section, we present some results on the expected number of geodesics of length k in the hypercube. We include them here because they may be important in finding an upper bound on the proportion of geodesics in a subgraph discussed in the previous section.

Lemma 5.1. *The number of geodesics of length k in Q_n is $2^{n-1}\binom{n}{k}(k)!$.*

Proof. There are 2^n ways to choose a starting vertex, and $\binom{n}{k}$ ways to choose an ending vertex, since there are n possible coordinates to change. From there, there are $k!$ ways to arrange the directions traversed in the geodesic. Multiplying these and dividing by 2, since we double count each geodesic, gives us $2^{n-1}\binom{n}{k}(k)!$ geodesics of length k in total. □

Proposition 5.2. *The expected number of monochromatic geodesics of length k in an antipodal 2-coloring is $2^{n-k}\binom{n}{k}(k)!$.*

Proof. Note here that there are $2^{n2^{n-2}}$ ways to color the cube antipodally. Given one monochromatic geodesic (and its antipodal pair), there are $2^{n2^{n-2}-k}$ ways to color the rest of the cube antipodally. Thus, we have that the expected number of monochromatic geodesics of length k is, using Lemma 5.1, is, as we are counting two monochromatic geodesics in this case (he original and its antipodal pair),

$$2\left(\frac{2^{n2^{n-2}-k}}{2^{n2^{n-2}}} \cdot 2^{n-1}\binom{n}{k}(k)!\right) = 2^{n-k}\binom{n}{k}(k)!.$$

□

Proposition 5.3. *The expected number of monochromatic geodesics of length k in an antipodal subgraph of the hypercube with p edges is $\frac{\binom{n2^{n-2}-k}{p-k} \cdot 2^{n-k-1}\binom{n}{k}(k)!}{\binom{n2^{n-2}}{p}}$.*

Proof. First, let us compute the total number of antipodal subgraphs with p edges. Note that we can split all edges into antipodal pairs, of which we can only have 1: there are $n2^{n-2}$ such antipodal pairs. This gives us $\binom{n2^{n-2}}{p}2^p$ possible subgraphs; $\binom{n2^{n-2}}{p}$ possible antipodal pairs chosen and 2^p ways to choose the edges from there. Furthermore, by similar logic, once one geodesic of length k is chosen, we have just $n2^{n-2} - k$ antipodal pairs left, so the number of possible subgraphs is $\binom{n2^{n-2}}{p-k}2^{p-k}$. Therefore, the expected value, using Lemma 5.1 again for the number of total geodesics, is

$$\frac{\binom{n2^{n-2}}{p-k}2^{p-k}}{\binom{n2^{n-2}}{p}2^p} \cdot 2^{n-1}\binom{n}{k}(k)! = \frac{\binom{n2^{n-2}-k}{p-k} \cdot 2^{n-k-1}\binom{n}{k}(k)!}{\binom{n2^{n-2}}{p}}.$$

□

Proposition 5.4. *The expected number of geodesics of length k in a subgraph of the hypercube with p edges is* $\frac{\binom{n2^{n-1}-k}{p-k}\cdot 2^{n-1}\binom{n}{k}(k)!}{\binom{n2^{n-1}}{p}}$.

Proof. Note that the total number of subgraphs is simply $\binom{n2^{n-1}}{p}$, as we are choosing p edges from the entire hypercube. By contrast, the total number of subgraphs with a particular geodesic is $\binom{n2^{n-1}-k}{p-k}$. Thus, by Lemma 5.1, we get that the expected number of geodesics of length k is

$$\frac{\binom{n2^{n-1}-k}{p-k}}{\binom{n2^{n-1}}{p}}\cdot 2^{n-1}\binom{n}{k}(k)!.$$

□

Proposition 5.5. *The expected number of monochromatic geodesics of length k in a coloring of the cube with p red edges and $n2^{n-1}-p$ blue edges is* $\frac{2^{n-1}\binom{n}{k}(k)!\cdot\left(\binom{n2^{n-1}-k}{p-k}+\binom{n2^{n-1}-k}{p}\right)}{\binom{n2^{n-1}}{p}}$.

Proof. Here, we can simply use Proposition 5.4 twice to achieve this expected value, as each of the colors can be thought of as a subgraph of the cube. This gives us that the expected number of monochromatic geodesics of length k is

$$\frac{\binom{n2^{n-1}-k}{p-k}\cdot 2^{n-1}\binom{n}{k}(k)!}{\binom{n2^{n-1}}{p}}+\frac{\binom{n2^{n-1}-k}{n2^{n-1}-p-k}\cdot 2^{n-1}\binom{n}{k}(k)!}{\binom{n2^{n-1}}{n2^{n-1}-p}}$$

Since $\binom{n2^{n-1}-k}{n2^{n-1}-p-k}=\binom{n2^{n-1}-k}{p}$ and $\binom{n2^{n-1}}{n2^{n-1}-p}=\binom{n2^{n-1}}{p}$, this is the same as, after some combining and simplification,

$$\frac{2^{n-1}\binom{n}{k}(k)!\cdot\left(\binom{n2^{n-1}-k}{p-k}+\binom{n2^{n-1}-k}{p}\right)}{\binom{n2^{n-1}}{p}}.$$

□

Proposition 5.6. *There exists a partition of the edges of the hypercube into disjoint antipodal geodesics.*

Proof. We prove a stronger statement: there exists a partition into disjoint antipodal geodesics whose order of directions traversed is exactly the same. Note that this is evidently true for $n=2$, we prove the general statement by induction.

Now, assume we know that it is true for n. Consider Q_{n+1}. Within it, we can consider the n-dimensional hypercube $(\{0,1\}^n,0)$, which we know can be partitioned into such antipodal geodesics. Note that each of these geodesics must end at a different point, since the order of directions traversed is exactly the same and thus the same ending vertex would imply that the geodesics were the same, contradicting their being disjoint. Thus, from here, we can attach an edge in the $n+1$st direction to the end of each of these geodesics; these edges are distinct since the end vertices are distinct, and this addition creates antipodal geodesics, so therefore we have a partition of the edges of Q_{n+1} into disjoint antipodal geodesics.

Thus, our induction is complete, so we are done. □

6. Future Work

There are two major unresolved questions. First, the original conjectures on the existence of a monochromatic geodesic in any antipodal coloring of Q_n still remain open; we have shown their validity for n = 2-7, but the general case is still unresolved. Second, with regard to the question of the maximum number of antipodal geodesics in a subgraph of Q_n with a fixed proportion of edges, we have presented a conjectured optimal configuration as a lower bound, but the upper bound has yet to be lowered from the trivial $p \cdot 2^{n-1}n!$. Resolving this question is of interest, as well as extending any such results to the maximum number of geodesics of a particular length k (other than n) in a subgraph with a fixed proportion of edges.

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