

On the Algorithmic and Theoretical Exploration of Tiling-Harmonic Functions

Yilun Du

Mentor: Dr. Sergiy Merenkov

Abstract

Potential theory has powerful implications in physics such as in studies of natural phenomena such as gravitation, electro-statics, and fluid dynamics. It is also important in mathematics and has deep connections to differential geometry and holomorphic functions in the complex plane. The study of discrete harmonic functions is one interesting aspect of potential theory that has important applications such as image processing and stochastic processes.

In this paper, we explore a new class of harmonic functions defined on a tiling T , a square tiling of a region D , in \mathbb{C} . Consider the set of functions $\mathcal{F}(T)$ such that the energy of a tiling is minimized, with energy given by:

$$E_T(u) = \sum_{t \in T} (\max_p u(p) - \min_p u(p))^2.$$

We define these functions as tiling harmonic functions. We develop an efficient algorithm for computing interior values of tiling harmonic functions and graph harmonic functions in a tiling. Using our algorithm, we find that in general tiling harmonic functions are not generally equivalent to graph harmonic functions. In addition, we prove some theoretical results on the structure of tiling harmonic functions and classify one type of tiling harmonic function.

Introduction

Potential theory, the study of Laplace's equation, is a significant field in both mathematics and physics. It has direct application in physical potential fields such as gravitation and electro-statics, as well as velocity potential fields in fluid dynamics [2]. Solutions to Laplace's equations, harmonic functions, are the coincidentally the real and imaginary parts of holomorphic functions in \mathbb{C} , and Laplace's equation solutions have impacts in many fields of mathematics such as differential geometry [2]. In addition, methods used in potential theory have applications in mathematical physics and differential equations [2].

Functions that satisfy Laplace's equation are called harmonic functions. For a function on the 2D coordinate plane that satisfies Laplace's equation, the value of the function at a point is found to be the average of the function's values on the points of a circle of radius r around the particular point. Thus, one way to define a discrete harmonic function on a graph is to define each vertex as the average of the connected vertices. These functions, called graph harmonic functions have many applications such as in image processing and many probabilistic counterparts [3]. We explore discrete harmonic functions on square tiling T .

Suppose that T is a *square tiling* of a region D in \mathbb{C} , i.e., T is a finite collection of squares with edges parallel to the x and y axes, that have mutually disjoint interiors and their union is all of \bar{D} . Figure 1 shows some pictures of tilings. A *standard tiling* is a tiling where every square has side-length 1 and the vertices are of the form $(m, n) \in \mathbb{Z}^2$.

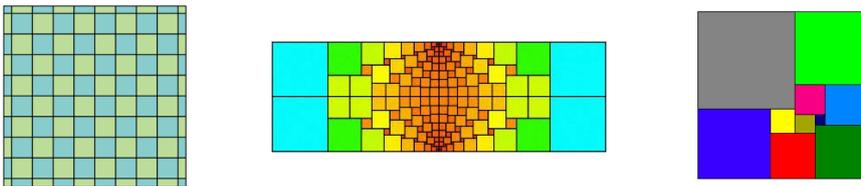


Figure 1: Standard tiling and non-standard tilings

Denote with $\mathcal{F}(T)$ the set of all real valued functions defined on the set of vertices of T . For such a function u , define the *energy of u on T* to be the non-negative number

$$E_T(u) = \sum_{t \in T} (\max_p u(p) - \min_p u(p))^2,$$

where t is a square in T and p runs over all vertices of T that lie on t . We call $(\max_p u(p) - \min_p u(p))^2$ for each tiling the oscillation of $\mathcal{F}(T)$ on the tile. A function $u \in \mathcal{F}(T)$ is called T -harmonic (or **tiling-harmonic**) if for every subtiling T' (i.e., a subset of T) and every function $u' \in \mathcal{F}(T)$ such that $u' \equiv u$ on the boundary vertices of T' , we have $E_u(T') \leq E_{u'}(T')$.

In addition we call a function $u \in \mathcal{F}(T)$ **graph-harmonic** on a tiling if for every interior vertex p of T , the value $u(p)$ is equal to the average of u on the neighbor vertices of p . This graph harmonic function definition on square tiling is analogous to the definition of a graph harmonic function on a graph. An interesting connection between tiling and graph harmonic functions is that if we were to define the oscillation of a grid as the square of all adjacent vertices in a square, then the function that minimizes the energy given the boundaries would be the graph harmonic function. This is because for the energy of each interior vertex to be maximized with respect to neighboring vertices, then the vertex must be equal to average of all values of neighboring vertices. We construct an algorithm to calculate both tiling and graph harmonic functions on tilings given boundary conditions and compare the values of interior points of both functions.

1 Computer Algorithm

We developed algorithms to calculate values of interior points in graph and tiling harmonic functions efficiently given boundary points through an iterative process that looped through all interior points. We ran all our algorithms through C++ while all graphical images were generated through Mathematica.

1.1 Algorithm for a tiling harmonic function on a 2×2 tiling

Suppose that T is a 2×2 standard tiling consisting of 4 squares S_1, \dots, S_4 and u is defined on the boundary vertices of T . We want to find a value of u in the interior vertex so that the energy is as small as possible.

$$E_T(u) = (M_1 - m_1)^2 + (M_2 - m_2)^2 + (M_3 - m_3)^2 + (M_4 - m_4)^2,$$

where M_i, m_i are the maximum, minimum respectively, of u on S_i .

Let X be the value of u on the interior vertex that gives the smallest energy. Con-

sidering only one square S_i , the most ideal value of X is in the interval $[m_i, M_i]$, so that X does not affect the energy of S_i . We sort the values $m_1, M_1, \dots, m_4, M_4$ from smallest to largest and generate 7 intervals from those sorted numbers. If, for example,

$$m_1 < m_2 < M_1 < m_4 < M_2 < m_3 < M_3 < M_4,$$

then the 7 intervals will be $I_1 = [m_1, m_2]$, $I_2 = [m_2, M_1]$, \dots , $I_7 = [M_3, M_4]$.

In each interval I_k , the algorithm generates a value X_k as the candidate for the central value as follows. Suppose that $[a, b]$ is the interval I_k . For each square S_i we compare a, b with m_i, M_i .

- If $b \leq m_i$ then let $E_i(x) = (M_i - x)^2$, $\alpha_i = 1$ and $c_i = M_i$.
- If $m_i \leq a \leq b \leq M_i$ then let $E_i(x) = (M_i - m_i)^2$, $\alpha_i = 0$ and $c_i = 0$.
- If $M_i \leq a$ then let $E_i(x) = (x - m_i)^2$, $\alpha_i = 1$ and $c_i = m_i$.

If $\alpha_1 + \dots + \alpha_4 = 0$ then let $X_i = a$. If $\alpha_1 + \dots + \alpha_4 \neq 0$, then define

$$c = \frac{c_1 + c_2 + c_3 + c_4}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4},$$

and $E(x) = E_1(x) + E_2(x) + E_3(x) + E_4(x)$. If c is not between a and b then X_k is the one of a, b that minimizes E , or a if both have the same energy. If c is between a and b then X_k equals c .

Let X be the X_k , $k = 1, \dots, 7$ which gives the smallest energy, or, if more than one, the smallest X_k that gives the smallest energy.

1.2 Algorithm of Tiling Harmonic Functions for a $n \times m$ tiling

Consider a $n \times m$ tiling, with $(n + 1)(m + 1)$ vertices. We denote these as (i, j) , with $0 \leq i \leq n$ and $0 \leq j \leq m$. We are given the value of u on $(0, i), (j, 0), (n, i), (m, j)$ with $0 \leq i \leq n$ and $0 \leq j \leq m$, and the goal is to find the values of u for (i, j) with $1 \leq i \leq n - 1$ and $1 \leq j \leq m - 1$ so that the energy is minimal.

We first set up initial values for the interior vertices. We usually set the initial values as the values of the graph harmonic functions (algorithm below) defined by boundary points. We then apply the 2×2 algorithm to replace the initial value of $(1, 1)$ with one

that gives the smallest energy. Then we apply the 2×2 algorithm to replace the initial value of $(2, 1)$ with one that gives the smallest energy. We repeat this procedure until we replace all nm initial values of the interior vertices with new values. In step 3, we return to point $(1, 1)$ and apply again the 2×2 algorithm and we keep the same procedure. The algorithm terminates when the difference of the energy after $N - 1$ steps minus the energy after N steps is less than a prescribed error.

Figure 2-6 show our algorithm running through randomly generated interior values with given specified boundary points $u(i, 0) = 0$, $u(i, 10) = 10$, $u(0, j) = j$, $u(10, j) = j$ to determine the minimal energy grid.

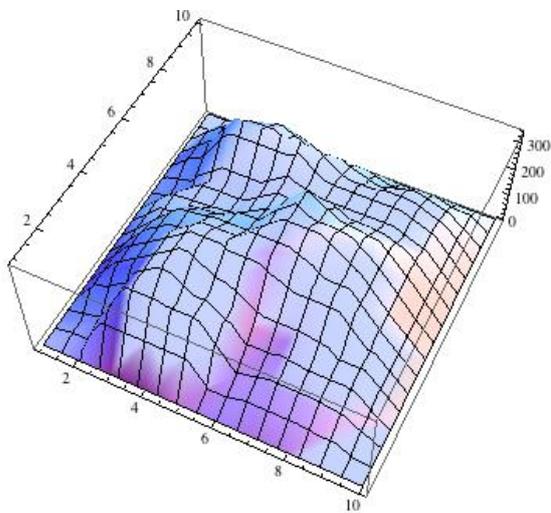


Figure 2: Grid After 5 Iterations

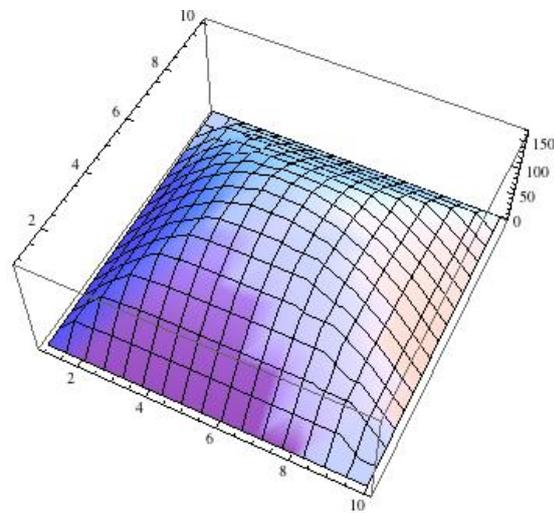


Figure 3: Grid After 10 Iterations

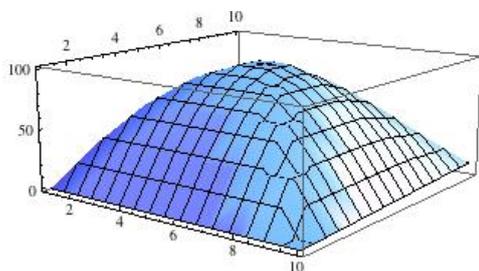


Figure 4: Grid After 15 Iterations

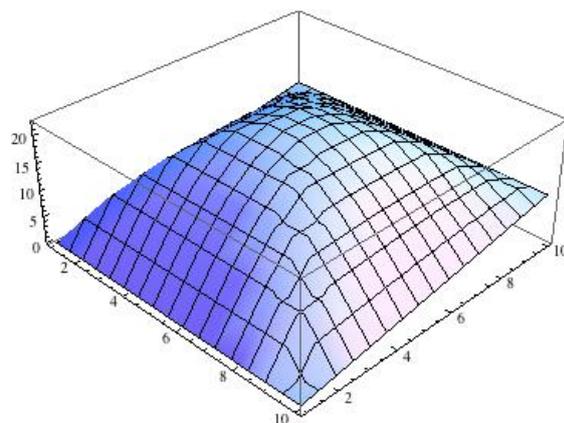


Figure 5: Grid After 25 Iterations

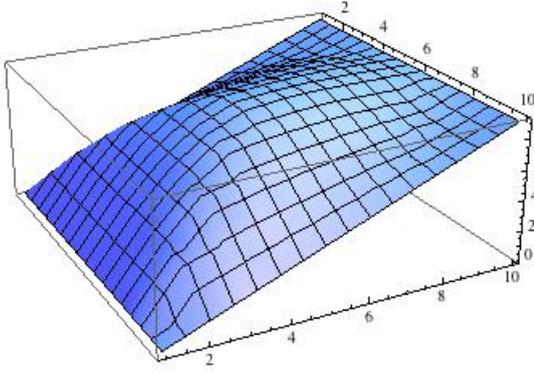


Figure 6: Grid After 35 Iterations

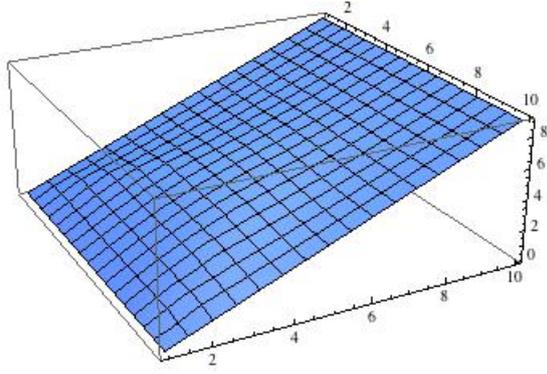


Figure 7: Grid After 45 Iterations

1.3 Algorithm for generalized square tilings

We first initialize each point to be the value of the average of all boundary points. We apply an analog of our algorithm for 2×2 tiles on each interior vertices so that it chooses the value of the vertices that minimizes the energy of the tiles that contain the interior point. We then iterate through all interior points from left to right and then from top to bottom continuously until our energy converges.

1.4 Algorithm for calculating graph harmonic functions

Suppose that T is a standard tiling with given boundary conditions. We initialize the value of each interior vertices as the value of the average of all the boundary points. We then loop through all the points systematically, from left to right and top to bottom, assigning each interior point as the average of the four adjacent neighboring points until an equilibrium is reached.

2 Algorithmic Proofs

We first prove that both of our algorithms for tiling harmonic and graph harmonic functions converge.

Theorem 2.1. *Our algorithm for computing tiling harmonic functions is globally convergent.*

Proof. Each time we minimize a point in our algorithm, we decrease the total energy of the whole grid. Since there is some minimum in the total energy of the grid, our algorithm must converge. Furthermore, at the point of convergence, we must have that

each interior point is the average of a certain number of neighboring points as defined in our algorithm. \square

Theorem 2.2. *Our algorithm for computing graph harmonic functions is globally convergent and converges to the unique graph harmonic solution for a set of boundary points.*

Proof. An alternative definition of a graph harmonic function is a function that minimizes the sum of the squares of the difference between all neighboring vertices (see introduction). Each time our algorithm takes a point as the average of its neighboring point, the sum of the squares of the difference between all neighboring vertices decreases. Therefore, our algorithm must eventually converge. Furthermore, this algorithm must converge to the unique solution since the only time our algorithm will stop modifying a point is if the average of the points next to it is equal to its value defining a graph harmonic function which is unique given a set of boundary conditions. \square

An interesting question is whether our algorithm for computing tiling harmonic functions also appears to converge to the tiling harmonic solution with minimal energy. Unfortunately, it appears that for randomly generated boundary conditions, the function does not necessarily converge to a function where the overall energy is minimized (see qualitative description of tiling harmonic functions). Instead, our algorithm converges to a point such that each point is locally minimized in energy with respect to the neighboring 2 by 2 block. However, there is only a finite number of different energies that our algorithm will eventually converge to.

Theorem 2.3. *There are only a finite number of different energies our algorithm can converge to, given a certain set of boundary values.*

Proof. We have that for our algorithm to converge, each point has to be the average of a certain set of neighbors. Therefore, each tiling that gives an energy that our algorithm converges to can be mapped to some type of graph harmonic function on a graph G . For example, if we have that a point is the average of two of its neighbors, its corresponding vertex on the graph would be connected to the two vertices of its neighbors. Each corresponding graph must have some unique energy E , since graph harmonic functions are unique given specified boundary conditions. Since we can only make a finite number of these graphs, we can only have a finite number of different energy values for our algorithm. \square

3 Qualitative Properties of Tiling Harmonic Functions

Using our algorithm, we explored many of the properties of tiling harmonic functions. We found that given a function u defined on the boundary vertices of T , there may be *more than one* tiling-harmonic extension $U \in \mathcal{F}(T)$ of u . Figure 8 and 9 two different tiling harmonic functions on the same boundary conditions. In this example, any number in the middle vertex between 1 and 2 would give you the same minimal energy.

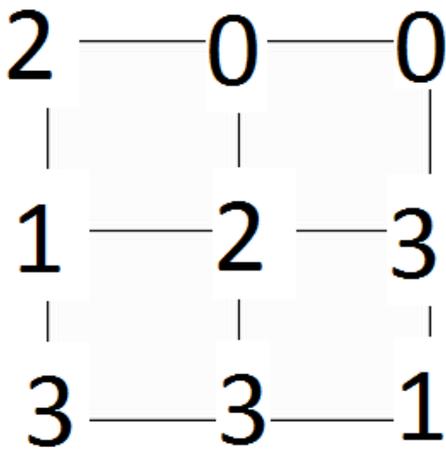


Figure 8: Tiling Harmonic Function 1

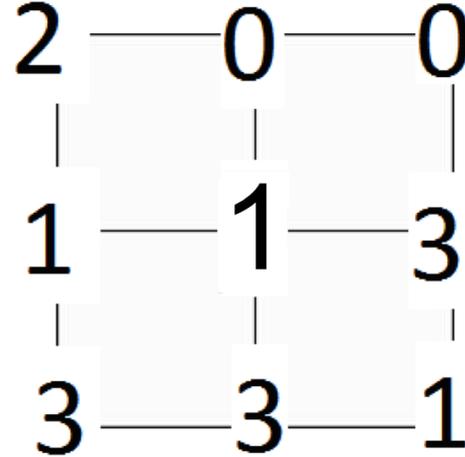
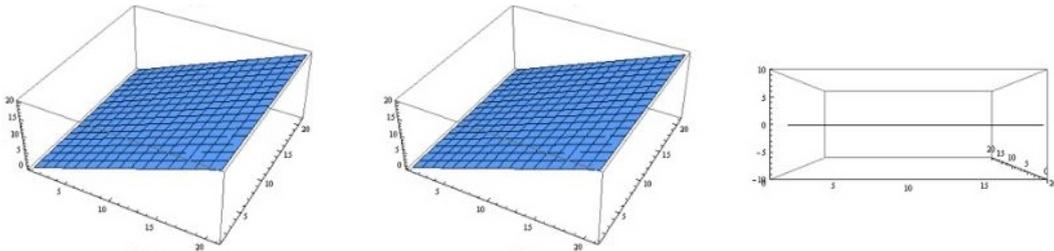


Figure 9: Tiling Harmonic Function 2

We compare the behavior of tiling harmonic functions and graph harmonic functions on several different boundary values in the following diagrams. In each of the following three examples, given the specified boundary values, we calculate both the graph-harmonic function and tiling-harmonic function. We then calculate the difference of the two functions. In the following figures, we present the tiling-harmonic function, graph-harmonic function and the difference, respectively.

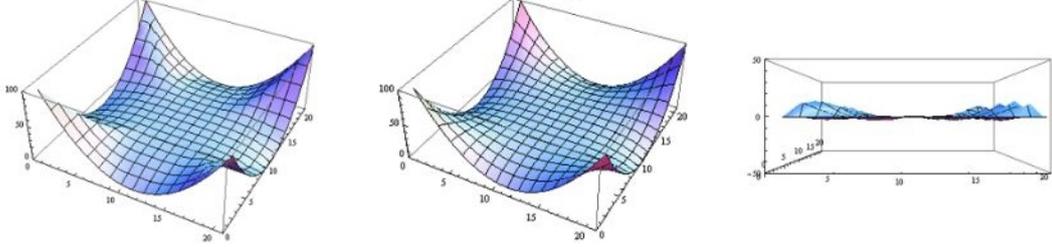
- Example 1: $u(0, j) = j$, $u(20, j) = j$, $u(0, i) = 0$, $u(20, i) = 20$.



Notice that both the tiling-harmonic and the graph-harmonic lie on a plane and

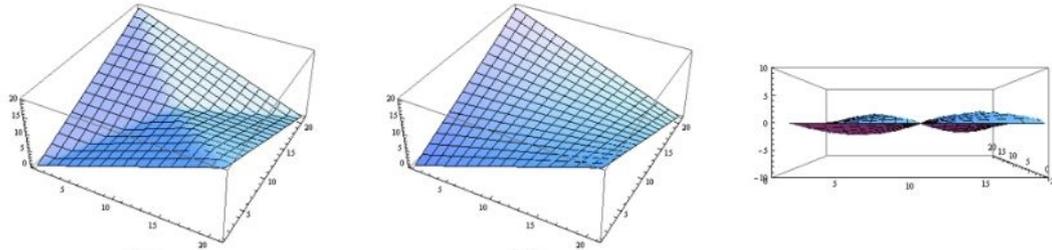
are equal.

- Example 2: $u(0, j) = (10 - j)^2$, $u(20, j) = (10 - j)^2$, $u(0, i) = (10 - i)^2$, $u(20, i) = (10 - i)^2$.



The tiling-harmonic and graph-harmonic functions have similar shapes. The differences between functions are relatively large on the corners of the function but soon approach the zero at the center. Notice, however, that the graph-harmonic function is much smoother than tiling harmonic function, especially at the edges.

- Example 3: $u(0, j) = j$, $u(20, j) = 20 - j$, $u(0, i) = i$, $u(20, i) = 20 - i$.



The graphs of the two functions are also very close. However, the surface of tiling-harmonic around the diagonal lines is not smooth while that of the graph-harmonic function is. The tiling-harmonic function in this case is the union of four triangles and is piecewise linear.

In the above diagram and in additional computations, we found that when boundary points lie on a plane P , the tiling harmonic is equal to the graph harmonic function. Moreover, both lie on the plane P . Given some boundary values, the graph-harmonic function and the tiling-harmonic function are **not equal**. For example, when our function is piecewise smooth, such as shown in Example 3, our graph harmonic function appears to be smooth, while the tiling harmonic function is composed of 4 triangles that converge at a point. However, in many smooth functions such as those seen in Example

3, we find that the functions are pretty close to each other near the center. In general, we find that both graph and tiling harmonic functions are fairly similar to each other in terms of graphical shape. However, we find that tiling harmonic functions, unlike graph harmonic functions, often tend to be piecewise smooth, especially near the edges.

3.1 Multiple Energies Given by Algorithm

Usually when our boundaries are relatively smooth functions, our program converges to one final grid, regardless of the initialization of interior points. Unfortunately, when boundary points are randomly generated, there appears to be multiple different functions that our algorithm converges to, depending on the initialization of interior points. Furthermore, it appears that these local minima have different energy. Usually, we initialize the interior points as the corresponding graph harmonic function, and the final output grid that results appears to always have the minimal energy compared to other grids generated from different internal initialized points. Following are two tables of two outputs from our program, one initialized from the graph harmonic function, and the other one where the interior points were all initialized to the minima of the boundary points. Figure 10 shows a graphical display of values of Table 1 and Figure 11 shows a graphical display of the values of Table 2. On both tables, we see that each interior point is minimized relative to all the neighboring points. Table 1 was initialized by setting all points as the minimal element of the boundary value while Table 2 was initialized from the graph harmonic function defined by the boundaries. The one initialized by the minimal value on the boundary appears to have much lower values on the interior. Table 2 had the lowest energy out of all different initializations while Table 1 had significantly higher energy.

8	24	9	18	21	23	19	28	15	29
27	9	9	14	14	14	14	14	14	22
20	9	8.625	8.625	8.625	8.625	8.625	8.625	7.5	2
15	9	8.5	8.25	8.25	8.25	8.25	8.0625	7.5	1
11	9	8.25	8	7.875	7.875	7.875	7.875	7.5	21
0	7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5	27
26	13	10	25	9.33333	9.33333	9.33333	9.33333	9.33333	4
27	13	11.1667	11.1667	11.1667	11.1667	11.1667	11.1667	13	29
25	13	13	13	13	13	13	13	13	29
13	20	29	15	20	26	28	18	20	4

Table 1

8	24	9	18	21	23	19	28	15	29
27	19	14	14	14	14	14	14	14	22
20	19	14	14.25	14.25	14.25	14.25	14.25	14.25	2
15	15	14	14.5	14.5	14.5	14.5	14.5	14.25	1
11	11	14	14.825	15.2	15.3333	15.75	17	20.625	21
0	11	14	15	15.65	15.9	16.1667	17	20.625	27
26	19	17	15	16.15	16.475	16.6	17	17	4
27	20	17.6	15	17.3	17.3	17.3	17.3	18	29
25	20	20	15	20	20	20	18	18	18
13	20	29	15	20	26	28	18	20	4

Table 2

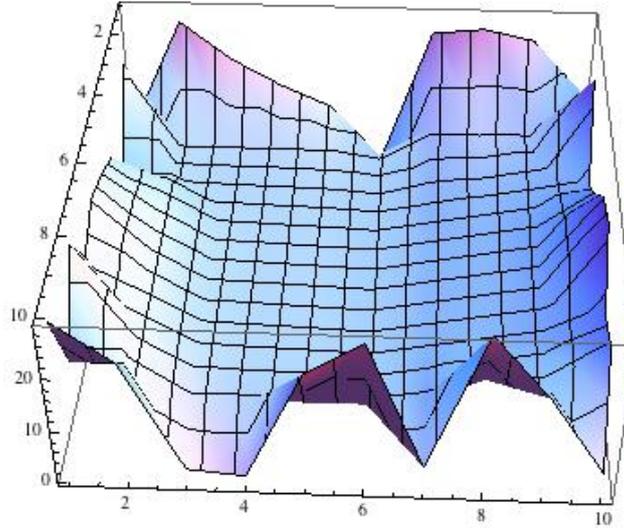


Figure 10: Graph Corresponding to Table 1

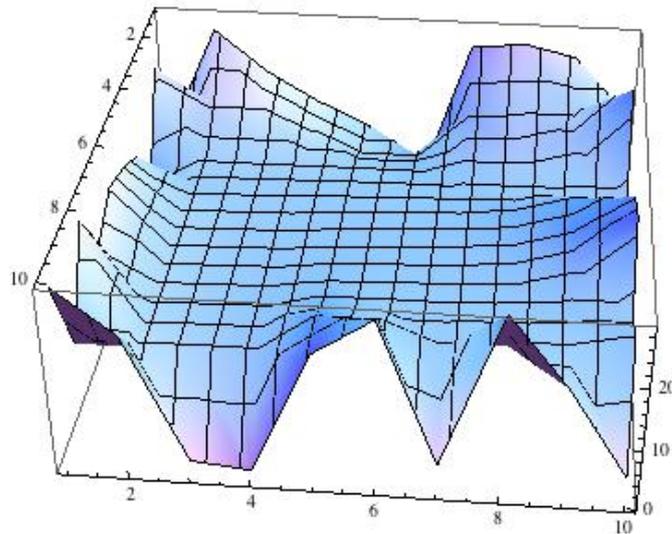


Figure 11: Graph Corresponding to Table 2

We have several proposed solutions to this problem. One possible solution that could solve this problem is to continuously perturb the points until we reach the global

minimum. Another possible method would be to randomly generate many different interior values, and see which one gives us the lowest energy. In addition, it appears that as long as interior points are generated from the values of the graph harmonic function, we appear to always get the minimal possible energy of the tiling harmonic function with our algorithm.

4 Theoretical Properties of Tiling Harmonic Functions

We also explored some of the theoretical structures of tiling harmonic functions.

Theorem 4.1. *The function $f(x,y)=cy$ is tiling harmonic for a real number c .*

Proof. It suffices to show that $f(x, y) = y$ is tiling harmonic. Consider any subtiling of any arbitrary square tiling. We define energy of a tile in terms of an area. Let us say we have a square with total energy of E . Consider splitting a square into two halves. We have then that each halve has energy $\frac{E}{2}$.

Now consider a narrow vertical strip of our subtiling, such that no two portions of two horizontally adjacent tiles are contained in the vertical strip (so we would have a vertical stack of portions of tiles in our strip). Consider the top and bottom tiles that are contained in our vertical strip or if the vertical strip goes through the subtiling more than once, consider the top and bottom portions of each separate portion. Let the boundary values of the subtiling be given by $f(x, y) = y$.

Let the top tile have a boundary vertex with value x and bottom tile have a boundary vertex with value of y , where the boundary vertices are not necessarily in the vertical strip. We have that the height of the part of subtiling we cut with the pair of vertical lines as $x - y$. In addition, for each portion of a tile except for the portion of the top tile, we have that the top pair of one tile's set of vertices must be on the same horizontal line as bottom vertices of the above tile (none of these vertices are necessarily inside our vertical strip). So there exists a set of vertices going down through the tiles, such that each tile contains two adjacent vertices in the set (where the vertex with largest value would be x and the vertex with least value would be y). Call the differences between two adjacent vertices d_i . Then we have that the sum of the d_i is $x - y$. In addition, note that the total energy of each tile included in our vertical strip is at least d_i^2 (since the vertices that make d_i are located in the i^{th} square). We define the width of the

vertical strip be W_1 , and the width of the i^{th} tile we are considering as w_i , so we have the fraction of the tile inside our vertical strip as $r_i = \frac{W_1}{w_i}$. Therefore the energy of the tiles in our vertical strip is $r_i * d_i^2$. Finally, we have that the total energy of all the tiles in that vertical strip is going to be at most $(r_0 d_0^2 + \dots + r_n d_n^2)$. By Cauchy-Schwartz, we have that $(r_0 d_0^2 + \dots + r_n d_n^2)(\frac{1}{r_0} + \dots + \frac{1}{r_n}) \geq (d_0 + \dots + d_n)^2 = (x - y)^2$. Note also that $(\frac{1}{r_0} + \dots + \frac{1}{r_n}) = (x - y)/(W_1)$.

Thus we have that the minimal energy of that vertical strip is greater than or equal $(x - y) * W_1$ which is just the area of the part of the square subtiling inside our vertical strip. We can take arbitrarily many vertical strips until we have the whole subtiling, so the energy of whole subtiling must be at least the area of the subtiling, but the function $f(x, y) = y$ gets you a tiling with energy equal to the area of the subtiling and satisfies the boundary condition. Therefore, we have that the function $f(x, y) = y$ is tiling harmonic. By scaling $f(x, y) = y$ by a real number c , we have $f(x, y) = cy$ is tiling harmonic for all real numbers c .

□

Theorem 4.2. *Tiling harmonic functions satisfy the maximal principle; i.e. that the global maximum or minimum values can not occur in an isolated point in the interior.*

Proof. Assume by contradiction that the maximum or minimum occurred in an isolated interior point. Consider the leftmost and lowest maximal or minimal point in the grid. We then consider the neighboring squares contain it. Using the algorithm tiling harmonic functions above, we can find a point that minimizes the energy of the squares containing the points better than the original point, so then this new interior numbering of our subtiling would have lower energy, negating the fact that the function is tiling harmonic.

□

Another analog we would like to prove about tiling harmonic functions is that they satisfy an analog of Liouville's Theorem, that any bounded tiling harmonic function must necessarily be constant. To this goal, we have gotten the following lemma.

Lemma 4.3. *A tiling harmonic function which is bounded by a certain maximum and minimum value on a infinite standard square tiling will have an average energy of zero on each tile.*

For a function to be harmonic on a tiling, it must be harmonic on all subtilings.

Consider an n times n standard subtiling. Let the average value of the function on the boundary be A with global maximum of the harmonic function as M and the global minimum as m . We then have the total energy of the subtiling will be at most $(4n - 2)(M - m)^2$, since if we set all interior points neighboring the exterior points in a subtiling as A then the standard tiling will have energy of at most $(4n - 2)(M - m)^2$, as the energy of an exterior tiles must necessarily be less than $(M - m)^2$ and there are a total of $4n - 2$ tiles, with the interior tiles having an energy of zero. We then have the average energy of each tile as at most $\frac{(4n-2)*(M-m)^2}{n^2}$ which approaches zero as we take the limit as n approaches infinity. \square

We eventually hope to prove, using the above lemma, that a bounded tiling harmonic function on a infinite set of tiles must be a constant.

Conclusions and Further Work

In our project, we have generated an algorithm that can calculate interior points for both graph and tiling harmonic functions on tilings. We have found that tiling harmonic functions have startling differences with graph harmonic functions, such as a lack of smoothness even in larger tilings and a lack of uniqueness. At the same time, we have found that tiling harmonic functions have many characteristics of typical harmonic functions both in shape and in properties such as the maximal principle. We also believe that tiling harmonic functions posses many additional characteristics found in other general harmonic functions such as an analog of Liouville's Theorem; i.e. that a infinite tiling harmonic functions with bounded values must necessarily be constant. One conjecture that would immediately lead to this result is the conjecture that the maximum oscillation tile on each grid must be a border tile. Initial exploration through our algorithm very strongly supports this hypothesis.

In addition, we also hope to prove that the tiling harmonic functions posses some analog of Harnack's inequality, which could lead to the proof of another conjecture that if T is a general square grid tiling of the upper half plane, then the only non-negative T -harmonic functions u that vanish on the real vertices have the form $u(x, y) = cy$, where $c \geq 0$ is a constant. Initial testing with our algorithm provides a positive result for this question. A positive proof would lead to a parallel result on carpet harmonic

functions which would lead to a simple proof of the quasisymmetric rigidity of Sierpinski Carpets [1].

In the future, we also hope to find all different possible tiling harmonic functions. In this paper, we have shown that $u(x, y) = cy$ is a tiling harmonic function, and algorithmical analysis indicates that all planes are tiling harmonic functions. We would eventually like to either prove that planes are the only tiling harmonic functions or to classify all other possible tiling harmonic functions.

Acknowledgements

I would like to thank the following people in our project: I would like to thank Dr. Merenkov who both proposed my research topic and also advised me throughout the course of the project. In addition, I would like to thank Vellis Vyrion who also mentored me throughout the project. In addition I would like to thank undergraduates Sufei and Lawrence who helped create the diagrams and whom I worked with on the algorithm. Finally, I would like to thank PRIMES-USA for giving me the amazing opportunity to study and research mathematics.

References

- [1] M. Bonk, S. Merenkov. *Quasisymmetric Rigidity of square Sierpinski carpets*. Ann. of Math. 177 (2013), 591–643.
- [2] O. Kellog. *Foundations of Potential Theory*. Springer, New York, 1st Edition, 20 Sep 2010.
- [3] J. Doob. *Classical Potential and Its Probabilistic Counterpart*. Springer, New York, 1st Edition, 12 Jan 2001.