The PRIMES-USA 2013 problem set

Dear PRIMES-USA applicant,

This is the PRIMES-USA 2013 problem set. Please send us your solutions as part of your PRIMES-USA application by November 15, 2012. For complete rules, see **check link:** http://web.mit.edu/primesusa/apply.shtml

Please solve as many problems as you can.

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF (preferred), DOC, or JPG file. The name of the attached file must start with your last name, for example, "smith-solutions." Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES-USA.

You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day, and think about them, on and off, over some time (several days). We encourage you to apply if you can solve at least 50% of the problems. [1]

Enjoy!

[1]We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

Problem 1. You toss a coin n times. What is the probability that the number of heads you'll get is divisible by 3? (Find an exact formula, not involving sums of unbounded length; it may depend on the remainder of n modulo 6).

Solution. We have $p = 2^{-n}(1 + \binom{n}{3} + \binom{n}{6} + \ldots)$. This can be written as

$$p = \frac{(1+z)^n + (1+z^{-1})^n + 2^n}{3 \cdot 2^n},$$

where z is a cubic root of 1. This implies that

$$p = \frac{w^n + w^{-n} + 2^n}{3 \cdot 2^n},$$

where w is a 6th root of 1. So we get that

$$p = \frac{1}{3} + \frac{b_n}{3 \cdot 2^n},$$

where $b_n = 2, 1, -1, -2, -1, 1$ if $n = 0, 1, 2, 3, 4, 5$ modulo 6, respectively.

Problem 2. (a) Let $c < 2\pi$ be a positive real number. Show that there are infinitely many integers n such that the equation

$$x^2 + y^2 + z^2 = n$$

has at least $c\sqrt{n}$ integer solutions.

(b) Find a constant $C > 0$ such that there are infinitely many n for which the equation

$$x^5 + y^3 + z^2 = n$$

has $\geq Cn^{1/30}$ nonnegative solutions.

Solution. (a) Let B_n be the number of solutions of $x^2 + y^2 + z^2 \leq n$. It is easy to see that $B_n \sim \frac{4}{3}\pi n^{3/2}$ as $n \to \infty$ (volume of the ball). This means that $B_n - \frac{2}{3}cn^{3/2} \to +\infty$. So there are infinitely many n for which $B_n - B_{n-1} \geq \frac{2}{3}c(n^{3/2} - (n-1)^{3/2})$. But $\frac{2}{3}c(n^{3/2} - (n-1)^{3/2}) \sim cn^{1/2}$ as $n \to \infty$. This implies the statement.

(b) The solution is similar. Indeed, the number of nonnegative integer solutions of $x^5 + y^3 + z^2 \leq n$ behaves as the volume of the region defined by this inequality in the first octant. This volume is easy to compute: after making dilations $x \mapsto n^{1/5}x, y \mapsto n^{1/3}y, z \mapsto n^{1/2}z$, this region maps to $x^5 + y^3 + z^2 \leq 1$. Let V be the volume of the latter. Then the volume of the former is $Vn^{\frac{1}{5}+\frac{1}{3}+\frac{1}{2}} = Vn^{\frac{31}{30}}$. So we can take $C < (31/30)V$.

Problem 3. A finite string of 0s and 1s is called **admissible** if it occurs in one of the rows of the Pascal triangle modulo 2. I.e., a string is admissible if it has the form

$$\binom{n}{m}, \binom{n}{m+1}, \ldots, \binom{n}{m+k},$$

where the binomial coefficients are taken modulo 2.

(a) Which strings of length ≤ 4 are not admissible? Why?

(b) Give an explicit description of all admissible strings of length n.

(c) What is the number $a(n)$ of admissible strings of length n? (Write a recursion, guess the answer and prove it by induction; the recursive formula may be different for $n = 2r$ and $n = 2r + 1$).

Solution. (a) It is easy to see that all strings but 1011 and 1101 are admissible. To show that these strings are not admissible, note that such a string cannot occur in an even row (since those have 0 at every other place). But $\binom{2r+1}{k+1} = \binom{2r+1}{k}\frac{2r+1-k}{k+1}$, so if k is odd, these numbers have the same parity. So every other element in a string occuring in an odd row is followed by the same element. But this is not satisfied for the two strings in question.

(b),(c) Assume for simplicity that n is even: $n = 2r$. Then there are the following kinds of admissible strings of length n.

1) (even rows) $0b_1....0b_r$, where $b_1...b_r$ is an admissible string of length r. The number of such is $a(r)$.

2) (even rows) similarly, $b_10...b_r0$. The number of such is also $a(r)$.

3) (odd rows) $b_1b_1...b_rb_r$. The string above of length $2r + 1$ would then be $0c_10c_2...0c_r0$, where $c_1...c_r$ is admissible of length r. There are $a(r)$ of such strings.

4) (odd rows) $b_0b_1b_1....b_{r-1}b_{r-1}b_r$. The string above is $c_10c_20...0c_{r+1}$, where $c_1...c_{r+1}$ is admissible. The number of such strings is $a(r + 1)$.

So at first sight we get the equation

$a(2r) = 3a(r) + a(r + 1)$.

But it is not quite right because the classes (1)-(4) have intersections. Namely, the string of zeros belongs to all 4, and the string of ones to the last two, and 0...01 and 10...0 belong to 1) and 4) and 2) and 4) respectively. So we overcount by 6, and the correct equation is

$a(2r) = 3a(r) + a(r + 1) - 6$.

In a similar way we get $a(2r + 1) = 2a(r) + 2a(r + 1) - 6$.

Then using that $a(0) = 1$, $a(1) = 2$, $a(2) = 4$, it is easy to show by induction that $a(n) = n^2 - n + 2$.

Problem 4. Positive solutions of the equation $x \sin(x) = 1$ form an increasing sequence $x_n, n \geq 1$.

(a) Find the limit

$$c_1 = \lim_{n\to\infty} n(x_{2n+1} - 2\pi n).$$

(b) Find the limit

$$c_2 = \lim_{n\to\infty} n^3(x_{2n+1} - 2\pi n - \frac{c_1}{n}).$$

Solution. (a) Let $x_{2n+1} = 2\pi n + b_n$. Clearly, $b_n \to 0$, and $\sin b_n = \frac{1}{2\pi n + b_n}$. This shows that $b_n \sim 1/2\pi n$, so the required limit is $c_1 = 1/2\pi$.

(b) We have $\sin x = x - x^3/6 + ...$ Thus, we get that up to quartic terms in $1/n$,

$$b_n = \frac{1}{b_n + 2\pi n} + \frac{b_n^3}{6} = \frac{1}{(2\pi n)^{-1} + 2\pi n} + \frac{1}{6(2\pi n)^3} = \frac{1}{2\pi n} - \frac{5}{6(2\pi n)^3}.$$

So $c_2 = -5/48\pi^3$.

Problem 5. Let us say that a polynomial f with complex coefficients is **degenerate** if there exists a square matrix B such that $B \neq f(A)$ for any square matrix A. What are all the degenerate polynomials of degree 2? degree 3? degree 4? any degree?

Solution. We claim that f is degenerate if and only if there exists $b \in \mathbb{C}$ such that $f(z) = b$ implies $f'(z) = 0$. Such polynomials have the form $f(z) = b + (z - a_1)^{n_1}...(z - a_k)^{n_k}$, where $n_i > 1$ for all i. So for degree 2, all polynomials are degenerate; for degree 3, the degenerate polynomials are those with a degenerate critical point, and for degree 4, the degenerate polynomials are those of the form $b + g(z)^2$, where g is a quadratic polynomial.

Indeed, if f is of this form then the matrix $\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$ is not of the form $f(A)$. Indeed, A could not be diagonalizable, so it is conjugate to a Jordan block with some eigenvalue z. But then $f(A) = b \cdot \mathrm{Id}$, as $f'(z) = 0$. On the other hand, suppose that f is not of the above form. Let $J_{b,n}$ be the Jordan block of size n with eigenvalue b. Pick z such that $f(z) = b$ and $f'(z) \neq 0$. Then f is invertible near z. Let $A = f^{-1}(J_{b,n})$. This makes sense, as f^{-1} has a Taylor expansion $f^{-1}(w) = z + c_1(w - b) + c_2(w - b)^2....$ And we have $f(A) = J_{b,n}$. Then by the Jordan normal form theorem, for any B there is A such that $f(A) = B$.

Problem 6. The sequence $b(n)$ is defined by the recursion

$$b(2n + 1) = 2b(n) + 2,\ b(2n) = b(n) + b(n - 1) + 2,$$

for $n \geq 1$, with $b(0) = 0$, $b(1) = 1$.

(a) Find a generating function for $b(n)$ and deduce a formula for $b(n)$, as explicit as you can.

(b) Let x be a positive real number, and $[a]$ denote the floor (integer part) of a. Find the limit as $m \to \infty$ of $b([2^m x])/[2^m x]$ as a function of x.

Solution. Let $\sum b(j)z^j = f(z)$. Then the recursion implies that

$$f(z) = (1+z)^2 f(z^2) + z\frac{1+z}{1-z}.$$

Let $g(z) = f(z)(1-z)^2$. Then

$$g(z) = g(z^2) + z - z^3,$$

so

$$g(z) = \sum_{i\geq 0}(z^{2^i} - z^{3\cdot 2^i}).$$

So

$$f(z) = \frac{1}{(1-z)^2}\sum_{i\geq 0}(z^{2^i} - z^{3\cdot 2^i}).$$

Thus,

$$b(n) = \sum_{0\leq i\leq \log_2(n+1)}(n - 2^i + 1) - \sum_{0\leq i\leq \log_2(n+1)-\log_2(3)}(n - 3\cdot 2^i + 1).$$

(b) Denote by $L(x)$ the limiting function (we will see that it exists). Clearly, $L(2x) = L(x)$, so we may assume that $1 < x \leq 2$. Let $z = 1/x$. Then $1/2 \leq z < 1$.

We have

$$\frac{b(n-1)}{n-1} \sim \sum_{0\leq i\leq \log_2(n)}(1 - \frac{2^i}{n}) - \sum_{0\leq i\leq \log_2(n)-\log_2(3)}(1 - 3\cdot\frac{2^i}{n}).$$

We may replace n with $2^m/z$. Then we have

$$\frac{b(n-1)}{n-1} \sim \sum_{0\leq i\leq m-\log_2(z)}(1 - 2^{i-m}z) - \sum_{0\leq i\leq m-\log_2(3z)}(1 - 3\cdot 2^{i-m}z).$$

So for $z \leq 2/3$, $i = m-1$ participates in the second sum, while if $z > 2/3$, it does not. So for $z \leq 2/3$, we get

$$\frac{b(n-1)}{n-1} \sim 4z + (1-3z) = 1+z,$$

while for $z > 2/3$ we get

$$\frac{b(n-1)}{n-1} \sim 1 + z + (1 - 3z/2) = 2 - z/2.$$

So $L(x) = 1 + 1/x$ if $3/2 \leq x \leq 2$, $L(x) = 2 - 1/2x$ if $1 \leq x \leq 3/2$, and it it extended to all positive numbers by $L(2x) = L(x)$ (so we get a continuous function).

Problem 7. Let f be a continuous real function on $[0,\infty]$. Show that if $\lim_{n\to\infty}(f(na)) = 0$ for all $a > 0$ then $\lim_{x\to+\infty} f(x) = 0$.

Solution. If f does not go to zero then there exists $\delta > 0$ and a sequence of closed intervals $J_k = [a_k, b_k]$, $b_k < a_{k+1}$, $b_k \to \infty$, such that $|f| \geq \delta$ on J_k.

Also note that if I is any closed interval of positive numbers then the union of nI for $n \geq 1$ covers the semiaxis $[c, +\infty)$ for large enough c.

Now let us inductively construct a nested sequence of closed intervals $I_N, N \geq 1$. Let $I_1 = J_1$. Given I_N, take c_N such that union of $nI_N, n \geq 1$ covers $[c_N, +\infty)$, and take J_k with some $k = k_N > N$ contained in this semiaxis. So J_k has positive length intersection with $n_N I_N$ for some n_N. Take $I_{N+1} = I_N \cap (J_k/n_N)$. Then if x is contained in the intersection of all I_N, we have $|f(n_N x)| \geq \delta$ for all N (and clearly n_N goes to infinity). This gives a contradiction.

Note. The final three problems are more difficult than the previous seven. They have more of the flavor of research problems, and are more open-ended. Even when the original problem can not be solved, partial progress is encouraged, and can take many forms: finding (useful!) ways to model the problem, coming up with reasonable conjectures, identifying the crucial missing steps, and more.

Problem 8. Two countries A and B share a national border which is a straight line. Along this border are $2n$ wells. An architect from each country is chosen to dig n canals within that country. The canals within each country cannot intersect, and each well is the endpoint of exactly one canal in each country.

(a) The architects go home and draw up all the possible plans for canals. How many plans does each architect have?

Once the canals are dug, if one ignores the national borders, the result is a collection of "lakes" (i.e., circular canals). Both countries want lots of lakes, but for political reasons, A wants an even number of lakes, and B wants an odd number. They pay their architects accordingly. If there are k lakes and k is even then architect A earns p^k dollars and B earns nothing. If k is odd then B earns p^k dollars and A earns nothing. For now, assume $p = 2$.

You are architect A. Unfortunately for you, architect B has spies everywhere, and if you settle on a plan in advance, B will counter it. Instead, you decide upon a mixed strategy, i.e. you assign a probability to each plan, and will choose randomly when the time comes. We call a mixed strategy *Nash* if the expected value of the difference in salary (between the two architects) does not depend on B's choice of plan. We call a Nash strategy a *tied strategy* if the expected value of the salary difference is zero.

(b) A spy from country A infiltrates the house of architect B with a mission: find exactly one plan, and destroy it. B will no longer be able to use that plan. It turns out that you now have a unique tied strategy! You may assume this fact, for any plan. You should also consider the assumption granted in part (c).

The spy returns to you and tells you he destroyed a plan where exactly one pair of adjacent wells was attached by a canal. Can you find your unique tied strategy?

(Hint: An exact answer will be difficult, but a recursive formula depending on n is enough. A recursive formula will only be valid if it implies that, for each n, the result is a valid strategy, i.e. that all probabilities are nonnegative and add up to 1)

(c) Unfortunately, architect B was too clever, and had an extra copy of the plan. Just to spite you, B builds precisely the strategy the spy had destroyed. Which architect is expected to benefit most? You may assume that the answer to this question is the same for any plan.

(d) Continue the setup of part (b). Suppose that $p = 1$. Find an n for which multiple tied strategies exist, and demonstrate them. Suppose that $p = \sqrt{2}$. Find an n such that there are multiple tied strategies.

(e) (continued). For each n find a finite list of values of p for which, outside of that list, tied strategies in the setup for (b) are guaranteed to be unique.

(Hint: Try to solve part (b) again for small n, with $p = 2.5$, $p = 3.3333$, $p = 4.25$. None of these are in your list, however.)

(f) Let $p = 2$. Spies inform you that architect B has recovered his missing plan, and has every plan available. Is there a Nash strategy? Which architect does it benefit?

(Hint: First, answer this question - is there a weighting of A's plans, by "probabilities" which need not lie between 0 and 1, such that the weighted average for each of B's plans is equal? Now, what would you need to do to show that each weight can be chosen between 0 and 1? How can you use your answer to part (b)?)

Solution.

(a) This is just the n-th Catalan number.

Let $\{a_i\}$ denote the set of A's possible canal plans. Let $e = a_0$ be the plan corresponding to the longest element of the symmetric group: the first and the last wells are attached, the second and the penultimate are attached, and so forth until the central pair. This is the only plan which the spy could destroy in part (b).

Consider the vector space $V = V_n$ spanned by $\{a_i\}$. The problem posits a bilinear form on V, where $\langle a_i, a_j \rangle$ is computed by flipping a_j

across the border, concatenating it with a_i, and replacing circles by a multiplicative scalar $-p$.

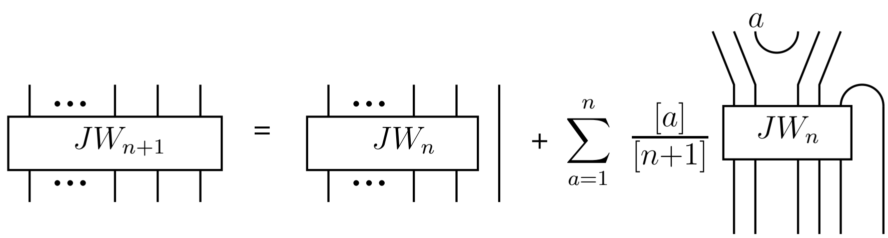
It is trivial to demonstrate that V_n is isomorphic as a vector space to the Temperley-Lieb algebra TL_n with parameter $-p$, and the bilinear pairing corresponding to the trace pairing on TL_n. The plan e corresponds to the identity element of TL_n. Many of the questions are trivial when the theory of the Temperley-Lieb algebra is applied, and the questions are really just questions about that theory. We try to provide answers which do not rely upon Temperley-Lieb theory or terminology.

Part (b) asks for a vector $v_0 = \sum_i c_i a_i$ with $c_i \geq 0$ and $\sum c_i = 1$, such that $\langle v_0, a_j \rangle = 0$ for all $j \neq 0$. It does not claim that $\langle v_0, a_0 \rangle \neq 0$. However, we will construct such a vector v_0 where $\langle v_0, a_0 \rangle \neq 0$, having sign $(-1)^n$. This benefits architect A when n is even, and architect B when n is odd. This answers question (c).

The assumptions of parts (b) and (c) imply that, for each i, there is a vector v_i (with probabilistic coefficients) with $\langle v_i, a_j \rangle = z_i \delta_{ij}$ for some $z_i \neq 0$. In particular, the bilinear form is non-degenerate. Moreover, the sign of z_i is equal to $(-1)^n$ for each i. Rescaling each v_i by z_i^{-1} and letting it be the column of a matrix A, we obtain the inverse matrix of the bilinear form. Note that A has either all positive or all negative signs, depending on the parity of n. For any vector E of expected values that have the same sign as A, one can find a vector $w \in V$ with positive coefficients, for which the expected payoffs of w are equal to E. A Nash strategy is a particular example of a vector E. This answers question (f).

Attaching two adjacent wells on B's side will give a map $V_n \to V_{n-1}$. Clearly v_0, if it exists, will be in the kernel of this map for any pair of wells except the central pair. In fact, v_0 is already specified up to scalar by this condition. This takes a small argument to demonstrate, which we omit here. In Temperley-Lieb language, v_0 is the Jones-Wenzl projector. It is easy to construct by hand for small n (say, $n = 1, 2, 3$).

There is a simple recursive formula for a scalar multiple JW of v_0, for which the coefficient of e is 1. We write JW_n for this vector when there are $2n$ wells. The national border in this picture goes from left to right on top, then from right to left on bottom, so that the central pair of wells are the top right and bottom right endpoints. This formula is originally due to Frenkel and Khovanov(??).



In this formula, $[2] = p = 2$ and $[k] = k$. One important feature of this formula is that all coefficients are positive.

Moreover, calculating $\langle JW_{n+1}, e\rangle := d_{n+1}$ is not difficult. The first term yields $(-p)d_n$, every term in the sum except $a = n$ contributes nothing because JW lies in the kernel of adjacent connections, and the $a = n$ term contributes $\frac{n}{n+1}d_n$. Combined with the base case $d_1 = -p = -2$ we see that $d_n = (-1)^n(n+1)$.

To check (d) and (e), plug in $[2] = p = q + q^{-1}$ and show that $[n](q - q^{-1}) = q^n - q^{-n}$. In particular, one calculates that $d_n = (-1)^n[n+1]$. Now $[n] = 0$ exactly when $q \neq 1$ and $q^{2n} = 1$. So long as $[1], [2], \ldots, [n+1]$ are all invertible, JW_n can be defined and $d_n \neq 0$, and the form is non-degenerate. This implies that tied strategies are unique. Only when q is a nontrivial $2k$-th root of unity for $k \leq n+1$ is there a possibility for multiple tied strategies. (In fact, they do exist in these cases, as the form is extremely degenerate. Temperley-Lieb theory will imply this fairly quickly.)

Problem 9. Elsie and Fred are playing SHUFFLE DUEL. In this game, two standard 52-card decks are shuffled: the "playing" deck and the "target" deck. Both decks are visible. After being shuffled, the target deck is then laminated and mounted on the wall, and is not touched again.

The players alternate turns (Elsie goes first because she is older, Fred goes last). On your turn, you must take two adjacent cards in the playing deck, and switch them. Once a certain configuration of the playing deck has been reached, it can never be repeated! You win if the playing deck and the target deck are the same after you have made your move. If your opponent makes a move and you can prove that it is impossible to reach the target deck without passing through a repeated configuration, you may challenge your opponent, and your opponent loses. If the playing deck and target deck are equal to begin with, the last player wins.

a) Elsie and Fred are both very experienced, perfect players. How can you tell who will win?

b) What is the probability that Elsie will win, and why?

Elsie and Fred are "interest gamblers." That is, before the game they put \$ 1 in a bank account. Every time they take a turn, the bank pays interest, multiplying the value of the account by q. When someone wins, they get to cash out from the bank. (Examples: If the game takes 0 turns, Fred wins \$ 1. If the game takes 1 turn, Elsie wins \$ q. If the game takes 2 turns, Fred wins \$ q^2)

Of course, the bank knows that games of SHUFFLE DUEL can take a very long time, so it refuses to pay out unless the game reaches a successful conclusion (the playing deck equals the target deck), and unless the game was AS SHORT AS POSSIBLE for a successful game, given the initial shuffling. Let us call this new game QUICK SHUFFLE DUEL: players may only make moves that are on a shortest path to the target deck.

c) (Easier) What is the expected length of the game? (Harder) What is the expected value of the bank account, at the time of payoff? (Write an explicit product formula).

(Hint: You needn't solve for the 52-card deck. Solve for smaller decks then come up with a formula. Make sure that the expected value is \$ 1 when $q = 1$.)

d) Suppose that Ginger and Harold and Ivana and John and Kelly want to play QUICK SHUFFLE DUEL as well. The play rotates between the 7 players. Which player is most likely to win? What about 10 players? At what number of players will the behavior change?

(Hint: Look more closely at your formula from part (c).)

Solution.

This game is about a walk through the Bruhat graph of $S_{52} = S_n$, starting at the identity, and ending at a randomly-selected permutation w. A successful game is an expression for w in terms of simple reflections, that has no connected subexpressions for the identity.

(a) The parity of the length of an expression for w is determined by the length of the permutation. Therefore any successful game will yield a win for Elsie if $l(w)$ is odd, and Fred if $l(w)$ is even. Only the loser ever has an incentive to play a challengeable move, and they lose anyway if they do.

(b) The probability of winning is $\frac{1}{2}$, so long as the size of the deck is at least 2. Choose any simple reflection s. Cosets of s give a bijection between even and odd permutations.

In quick shuffle duel, a successful game is a reduced expression for w, and has length $l(w)$.

(c) The expected length of the game is half the length of the longest element, so $\frac{n(n-1)}{4}$. This is because each permutation x can be paired with $w_0 x$. The expected value of the game (times $n!$) is the unnormalized Poincare polynomial of S_n, $\sum_{x \in W} q^{l(x)}$. This is known to be equal to $(n)_q!$, where $(n)_q = 1 + q + \ldots + q^n$ and $(n)_q! = (1)_q (2)_q \ldots (n)_q$. Therefore, the expected value is $\frac{(n)_q!}{n!}$.

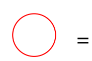
(d) Because $(7)_q$ divides $(n)!_q$ for $n \geq 7$, the lengths will be equidistributed modulo 7. When 53 players show up at the table, the behavior changes.

(Without using this argument, one can still do something. $S_k \subset S_n$ for $k < n$, and thus one can show that S_n is equidistributed over k players if S_k is. One can then attempt to analyze the lengths of elements in cosets of a k-cycle in S_k. Will this work??)

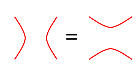
Problem 10. Fix a positive number n. Consider a planar graph where each edge is labelled with a number in $\{1, 2, \ldots, n\}$. This graph need not be connected, and may have connected components which are circles with no vertices. There are only two kinds of vertices in this graph: stars and crosses. A star has 6 incoming edges whose labels alternate $k, k+1, k, k+1, k, k+1$ for some $k \in \{1, 2, \ldots, n-1\}$. A cross has 4 incoming edges whose labels alternate i, j, i, j for some $i, j \in \{1, 2, \ldots, n\}$ with $|i - j| > 1$.

We place an equivalence relation on the set of such graphs, where two graphs are equivalent if they are related by a sequence of the following moves (which can be performed in reverse as well).

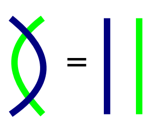
(1) (Circle Removal) Circles with empty interior may be removed.



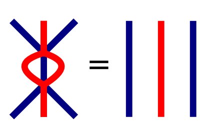
(2) (Bridging) Adjacent edges of the same label can be altered.



(3) (Double Vertex Removal) If two crosses are connected by two edges, both crosses can be removed.

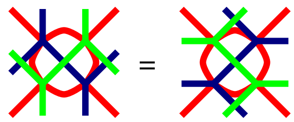


Similarly, if two stars are connected by three edges, both stars can be removed.



(4) (Cross Crossing) A sequence of crosses can be slid to the other side of a cross or star (for any consistent labels).

(5) (Dream Catchers) The following transformation is allowed. In this picture, blue represents the label k, red the label $k+1$, and green the label $k+2$.



Prove that every graph is equivalent to the empty graph.

(Hints: Use induction on n. Given $k \in \{1, 2, \ldots, n\}$ you might consider the k-*subgraph*, consisting only of the edges labelled k. If k is never involved in a star, what does the k-subgraph look like? If k is involved in stars, what does the k-subgraph look like? How might you make the k-subgraph disappear?)

Solution.

The k-subgraph can always be reduced (by ignoring bivalent vertices) to a trivalent graph. This graph is a planar 1-manifold when k is not involved in any stars.

We will show that the graph is equivalent to a graph without the label n. This argument will be general enough that it applies to *graphs with boundary*, which may have edges running to ∞ in the plane, so long as the color n does not run to ∞. Thus we can apply it to subgraphs in simply-connected planar regions which do not have the label n on their boundary.

Suppose that the n-subgraph is a manifold. Consider the subgraph with boundary inside a simply-connected region delineated by this manifold. This is a graph whose labels are contained in $\{1, 2, \ldots, n-1\}$ such that $n-1$ does not appear on the boundary (it can not cross n). Therefore, the label $n-1$ can be removed from the interior. The remainder of the interior can then be slid outside of the n-labelled circle, using Cross Crossing and Double Cross Removal. Now the interior of the n-labelled circle is empty, and it can be removed using Circle Removal. In this way, the label n can be removed from the diagram.

Suppose that the n-subgraph is trivalent. We induct on the number of trivalent vertices. Every trivalent vertex in the n-subgraph comes from a star in the original graph, labelled with n and $n-1$. If the trivalent vertex has a bigon, we claim this bigon can be removed. It is clear that the bigon comes from a part of the original graph whose $\{n, n-1\}$-subgraph looks like the Double Star. If we can remove all other labels from this neighborhood, we can apply Double Star

Removal. Using Cross Crossing and Double Cross Removal for n, we can assume that the only other labels which interfere are labels $\leq n-3$ which cross straight through from right to left, and these can also be removed using Cross Crossing.

Now we need only prove that the n-subgraph can be reduced to one with fewer trivalent vertices. Consider two trivalent vertices which are connected in the n-subgraph, and consider the $n-1$-labelled edges which come from each star in the original graph. If these can be connected, one star to the other, using Bridging, then we can use Double Star Removal to remove both trivalent vertices. Bridging will only fail if there are crosses with labels $\leq n-2$ going through the n-edge between the trivalent vertices. We call these *intervening edges*.

These intervening edges need to be Cross Crossed out of the way. Any edge of label $\leq n-3$ on the side can be Cross Crossed through the original $(n, n-1)$-labelled stars. A combinatorial argument can now be made that one can reduce to a graph where there is at most one intervening edge, and it is colored $n-2$. This combinatorial argument is akin to the one that a permutation in S_{n-2} has an expression which uses the simple reflection s_{n-2} at most once, and therefore there are only two double cosets of S_{n-3}, the identity coset and the coset of s_{n-2}. If there are no intervening edges, we are done by the above paragraph, so suppose there is one.

Using Double Star Addition for colors $(n-1, n-2)$, we may assume that our graph looks locally like a Dream Catcher. Applying the Dream Catcher equivalence, we get a new diagram where the trivalent vertices in the n-graph have been *switched.*The switching operation does not increase the number of trivalent vertices in the n-graph. Using switching, we can turn any closed cycle in the n-graph into a bigon, which can then be removed as above. Now induction finishes the proof.