q-Analogues of Symmetric Polynomials and nilHecke Algebras

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Symmetric Functions

Definitions

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Define the elementary symmetric functions by:

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_n}$$

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_1 x_3$$

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Define the complete homogenous symmetric functions by:

$$h_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} x_{i_1} \cdots x_{i_n}$$

$$h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_1 x_3$$

Goals and Motivation

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2. The "odd" ($q = -1$) nilHecke algebra can be used in categorification of quantum groups.

   We expect that our q-analogue can also be used in categorification.
3. Our q-bialgebra also has connections to 4D-topology.

Introduction to q-Bialgebras

Definition: Algebra

An *algebra* A is characterized by the following two maps:

$$\eta : \mathbb{C} \to A$$
$$m : A \otimes A \to A$$

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q-Swap and Identity Maps

$$\tau : v \otimes w \to q^{|v||w|} w \otimes v$$
$$1_A : A \to A$$

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We define the multiplication on $A \otimes A$ by

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Multiplication map m_2

The multiplication map $m_2\colon A^{\otimes 4} \to A^{\otimes 4}$ is

$$m_2 = (m \otimes m)(1_A \otimes \tau \otimes 1_A)$$

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Definition: Bialgebra

A *bialgebra* has all four maps η, m, ϵ, and Δ, with the added compatibility that the comultiplication is an algebra homomorphism.

Quantum Noncommutative Symmetric Functions

Description as a q-Bialgebra

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- Let $N\Lambda^q$ be a free, associative, $\mathbb{Z}$-graded $\mathbb{C}$-algebra with generators $h_1, h_2....$ Let $q \in \mathbb{C}$.

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- Define multiplication as:
  $(w \otimes x)(y \otimes z) = q^{\deg(x)\deg(y)}(wy \otimes xz)$.
- Define comultiplication as:
  $\Delta(h_n) = \sum_{m=0}^{n} h_m \otimes h_{n-m}$

Diagrammatics for the Bilinear Form

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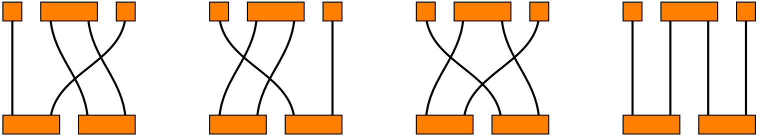

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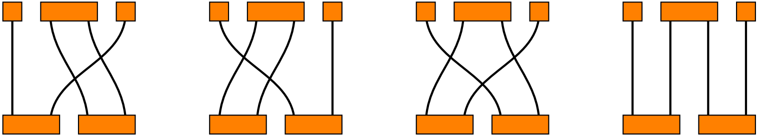

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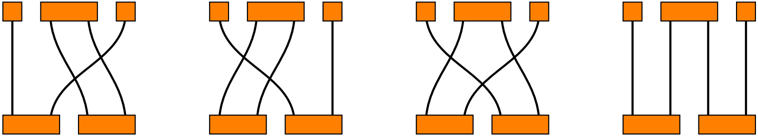

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There are no triple intersections, no critical points with respect to the height function, no instances of two curves intersecting at two or more points, and no crossing between curves originating from the same platform.

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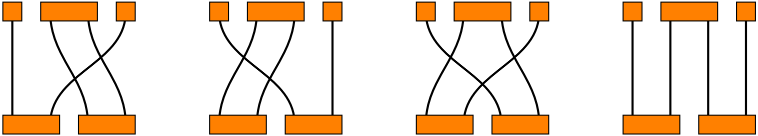

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$(h_1h_2h_1, h_2h_2) = 1 + 2q^2 + q^3$

q-Symmetric Functions

Definition

Define $\text{Sym}^q \cong N\Lambda^q/R$, where R is the radical of the bilinear form.

- The "odd case" refers to $q = -1$, studied in [EK].
- The "even" case refers to $q = 1$, studied in [GKLLRT].

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Diagrammatic Property

1. No strands from different tensor factors intersect:

$$(w \otimes x, y \otimes z) = (w, y)(x, z).$$

The Elementary Symmetric Functions

Definitions

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Theorem

$(h_\lambda, e_k) = 0$ if $|\lambda| = k$, unless $\lambda = 1^k$.

Diagrammatics for the Bilinear Form

Idea of Proof

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- Show that

$$(h_m x, e_n) = \begin{cases} (x, e_{n-1}) & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases}$$

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- Use strong induction on n to find $(h_m x, e_k h_{n-k})$

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- Use strong induction on n to find $(h_m x, e_k h_{n-k})$
- By definition:

$$(-1)^{n+1} q^{\binom{n}{2}} (h_m x, e_n) = \sum_{k=0}^{n-1} (-1)^k q^{\binom{k}{2}} (h_m x, e_k h_{n-k})$$

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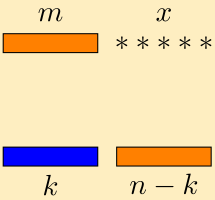

There are two cases to consider by the inductive hypothesis applied to $k < n$. Either there is a strand connecting h_m and e_k, or there is not.

Diagrammatics for the Bilinear Form

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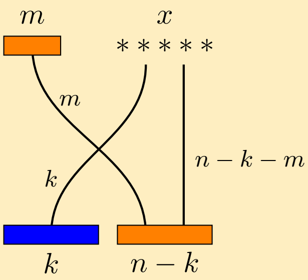

If no strand connects h_m and e_k.
This contributes $q^{km}(x, e_k h_{n-k-m})$.

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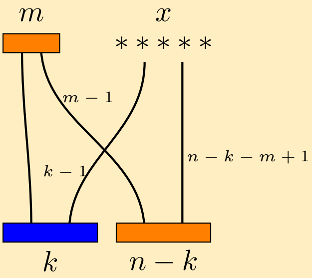

If a strand connects h_m and e_k.
This contributes $q^{(k-1)(m-1)}(x, e_{k-1}h_{n-k-m+1})$.

Summary of Diagrammatic Rules for any q

Theorem

$(e_n, e_n) = q^{-\binom{n}{2}}$

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Diagammatics

- There is at most one strand connecting an orange (h) platform and a blue (e) platform.
- There is a sign as given above when n strands connect two blue platforms.

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Theorem

h_1^n is in the center of $N\Lambda^q$, if $q^n = 1$.

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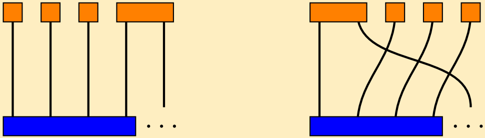

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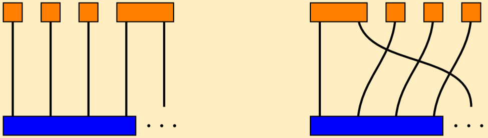

Other Relations (for $q^3 = 1$)

$v_1 = h_{11211} + h_{12111} + h_{21111}$
$v_2 = h_{1122} - 2h_{1221} + 3h_{2112} + h_{2211}$
$v_3 = 2h_{1131} - 2h_{114} + 2h_{1311} - 2h_{141} + 3h_{222} + 2h_{1113} - 2h_{411}$
$v_1 + q^2 v_2 + q v_3 = 0$

q-divided Difference Operators

Definition

The ring of q-symmetric polynomials ($q\mathrm{Pol}_a$):
$\mathbb{Z}\langle x_1, x_2, ..., x_a\rangle / \langle x_j x_i - q x_i x_j = 0 \text{ if } j > i\rangle$

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$\partial_i(x_i) = q$
$\partial_i(x_{i+1}) = -1$
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Leibniz Rule: $\partial_i(fg) = \partial_i(f)g + r_i(f)\partial_i(g)$

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Leibniz Rule: $\partial_i(fg) = \partial_i(f)g + r_i(f)\partial_i(g)$
Note that these definitions account for the odd case as well.

Properties of the q-divided Difference Operators

Lemma

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$$\partial_i(x_{i+1}^k) = -\sum_{j=0}^{k-1} q^{-j} x_i^j x_{i+1}^{k-1-j}$$

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$$e_k(x_1,\dots,x_n) = \sum_{1\le i_1<\ldots<i_k\le n} x_{i_1}\cdots x_{i_n}$$

and the k-th *twisted elementary q-symmetric polynomial*:

$$\widetilde{e}_k(x_1,\dots,x_n) = \sum_{1\le i_1<\ldots<i_k\le n} \widetilde{x}_{i_1}\cdots \widetilde{x}_{i_n},$$

where $\widetilde{x}_j = q^{j-1}x_j$.

Properties of the q-divided Difference Operators

Theorem

$\partial_i(\widetilde{e}_k) = 0$.
Hence $\widetilde{\Lambda}_n^q \subseteq \bigcap_{i=1}^{n-1} \ker(\partial_i)$.

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Conjecture

$\bigcap_{i=1}^{n-1} \ker(\partial_i) \subseteq \widetilde{\Lambda}_n^q$.

More properties

nilHecke Relations

$\partial_i x_i - q x_{i+1} \partial_i = q$
$\partial_i x_{i+1} - \frac{1}{q} x_i \partial_i = -1$

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Braiding Relation

$$\partial_i \partial_{i+1} \partial_i \partial_{i+1} \partial_i \partial_{i+1} = -\partial_{i+1} \partial_i \partial_{i+1} \partial_i \partial_{i+1} \partial_i$$

References

- A.P. Ellis and M. Khovanov. The Hopf algebra of odd symmetric functions. 2011. http://arxiv.org/abs/1107.5610
- Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon. Noncommutative symmetric functions. 1994. http://arxiv.org/abs/hep-th/9407124

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