Highly Non-convex Graph Crossing Sequences

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Introduction: Graphs

A graph is a tuple (V, E), where E is a collection of pairs of V.

V is called the set of vertices, and E is called the set of edges.

A drawing D of a graph G is a mapping from vertices to points on the plane and edges to curves joining the points corresponding to end-points of the edges.

Example: K_4

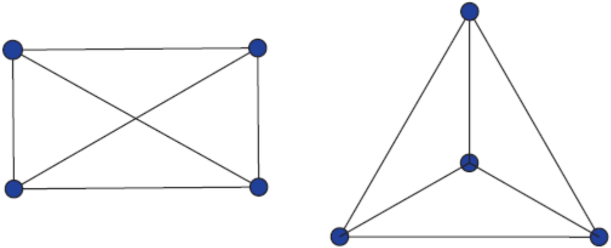

Figure: Two different drawings of K_4

Notice that one drawing has an intersection (or "crossing") between edges, where ther other does not.

Crossing Number

A crossing in a graph drawing is an intersection between curves that does not occur at an end-point of edges (curves).

The crossing number of a graph G, $cr(G)$, is the minimum possible number of crossings in all drawings of G.

Determining the exact crossing number of a graph is a central problem in topological graph theory.

Crossing Number

Graphs that can be drawn on the plane without crossings are called planar graphs.

The crossing number of a graph measures the non-planarity of the graph.

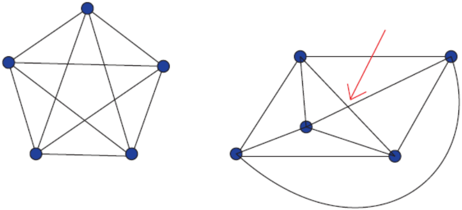

Figure: A non-planar graph (K_5)

Crossing Number on Surfaces

Question. *What happens if we add several handles?*

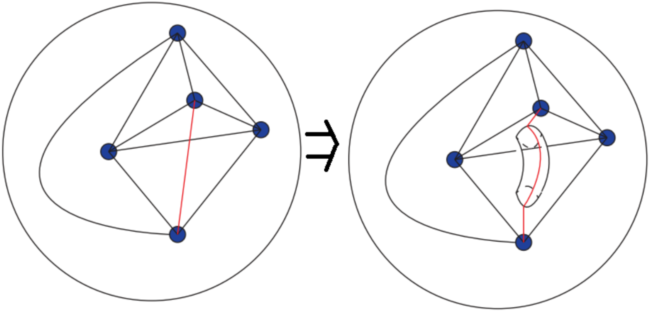

Figure: "Lifting" a crossing edge using a handle.

Crossing Number on Surfaces

The (orientable) surface of genus g is the surface obtained by "adding g handles" to the sphere. The sphere has genus 0.

The k-th crossing number $cr_k(G)$ of graph G is the minimum number of crossings among all drawings of G on the orientable surface of genus k.

The crossing number of a graph G is smaller on a surface of higher genus, and there always exists a g such that $cr_k(G) = 0$ when $k \geqslant g$.

Crossing Number on Surfaces

The genus g of a graph G is the minimum genus of the surface on which G can be drawn without crossings. i.e., $cr_g(G) = 0$. The genus of a graph always exists and is well-defined.

Crossing Sequence

The genus g of a graph G is the minimum genus of the surface on which G can be drawn without crossings. i.e., $cr_g(G) = 0$. The genus of a graph always exists and is well-defined.

The (orientable) crossing sequence of graph G is the sequence $cr_0(G), cr_1(G), ..., cr_g(G)$, where g is the genus of G.

All graph crossing sequences are strictly decreasing.

Attempt at Characterization of Crossing Sequences

Question: What sequences are crossing sequences of some graphs?

A sequence $\mathbf{a} = a_1, a_2, ..., a_n$ is convex if for all $1 \leqslant i \leqslant n-2$, $a_i - a_{i+1} \geqslant a_{i+1} - a_{i+2}$.

Example:

- $5, 3, 2, 1$: convex $(5 - 3 \geqslant 3 - 2 \geqslant 2 - 1)$
- $9, 7, 3, 1$: *not* convex $(9 - 7 < 7 - 3)$

Attempt at Characterization of Crossing Sequences

Theorem (Širáň, 1983)

Any convex, strictly decreasing sequence of nonnegative integers is a crossing sequence of some graph.

A graph obtained by joining multiple $K_{3,3}$'s with a cut vertex was used to prove this theorem.

Attempt at Characterization of Crossing Sequences

Conjecture (Širáň)

All crossing sequences of graphs are convex.

Rationale: "If adding the second handle saves more edges than adding the first handle, why not add the second handle first? (Archdeacon et al.)"

Attempt at Characterization of Crossing Sequences

Conjecture (Širáň)

All crossing sequences of graphs are convex.

Rationale: "If adding the second handle saves more edges than adding the first handle, why not add the second handle first?"

Surprisingly, this is *wrong*!

Non-convex Crossing Sequences

Theorem (Archdeacon et al., 2000)

For every $m > 0$, there exists a graph which has the crossing sequence $\{4\binom{3m}{2}, 3\binom{3m}{2} + 3\binom{m}{2}, 0\}$.

Theorem (DeVos et al. 2010)

If a and b are integers with $a > b > 0$, then there exists a graph G with (orientable) crossing sequence $\{a, b, 0\}$.

Question. Can we find a non-convex crossing sequence of arbitrary length?

Main Result

Theorem

For any $g \geqslant 2$, there exists a graph $G_{m,g}$ with genus g such that for $k < \frac{3}{5}g$,

$$cr_k(G_{m,g}) = (2g - k) \cdot \binom{3m}{2} + 3k \cdot \binom{m}{2}$$

and for $k \geqslant \frac{3}{5}g$,

$$cr_k(G_{m,g}) = 18m^2 \cdot \{g \mod k\}.$$

This presents an example of a non-convex graph crossing sequence of arbitrary length. Archdeacon et al.'s theorem is a special case of this theorem ($g = 2$).

Main Result

Corollary

There exists a family of graphs $G_{m,g}$, $g \geqslant 2$ each with genus g such that for $k < \frac{3}{5}g$,

$$cr_k(G_{m,g}) \sim cr_0(G_{m,g}) \cdot (1 - \frac{k}{3g}) \text{ as } m, g \to +\infty$$

and for $k \geqslant \frac{3}{5}g$,

$$cr_k(G_{m,g}) \sim 2cr_0(G_{m,g}) \cdot (1 - \frac{k}{g}) \text{ as } m, g \to +\infty.$$

This provides the asymptotical lower bound to the "non-convexity" of all graphs in the family of graphs $G_{m,g}$. Therfore, all graphs in this family are highly non-convex.

Main Result

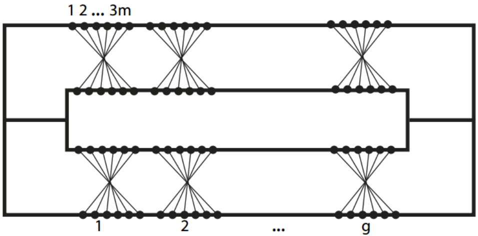

Figure: The graph $G_{m,g}$

The "patch" in the middle can be *flipped*!

Main Result

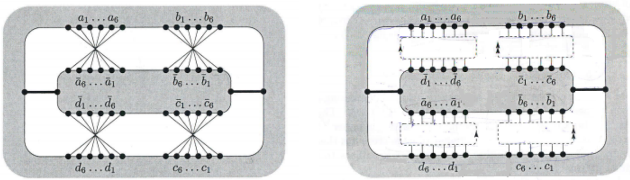

Figure: Embedding of $G_{m,2}$ (Archdeacon et al.'s example) on the plane and on the surface of genus 2

By simple enumeration, $cr_0(G) = 4\binom{3m}{2}$, and $cr_2(G) = 0$.

Main Result

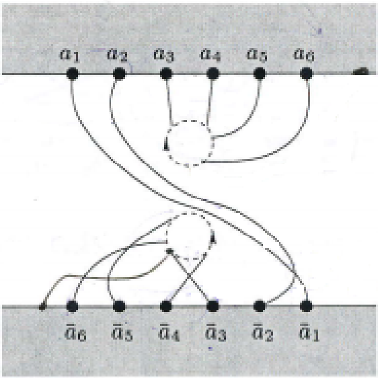

Figure: Method for toroidal embedding of $G_{m,2}$

Further Research

Conjecture (Archdeacon et al. 2000)

Any strictly decreasing (finite) sequence of non-negative integers is the orientable crossing sequence of some graph.

What other examples of non-convex crossing sequences can we find?

Further Research

How non-convex can a graph be?

Question. Does there exist, for any $\epsilon > 0$, a graph G with crossing sequence such that $cr_0(G) - cr_s(G) < \epsilon \cdot (cr_s(G) - cr_{s+1}(G))$?

If there exist such graph for all ϵ, then our 'rational' was completely wrong!

A different direction: determining the exact crossing number of specific graphs (on the plane).

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