Extremal Functions of Pattern Avoidance in Matrices

Jonathan Tidor
under the mentorship of Jesse Geneson

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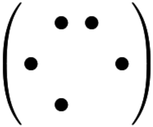

PATTERN AVOIDANCE IN MATRICES

Definition

*A **0-1 matrix** is an array of zero and one entries.*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

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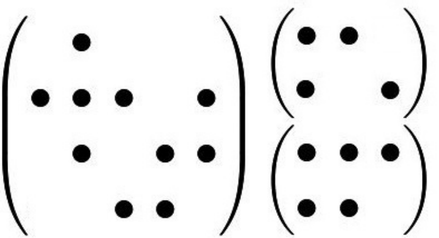

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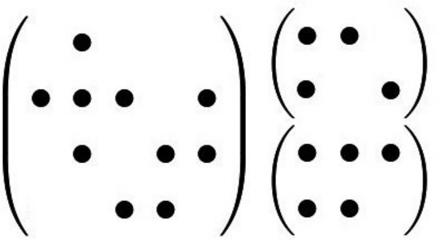

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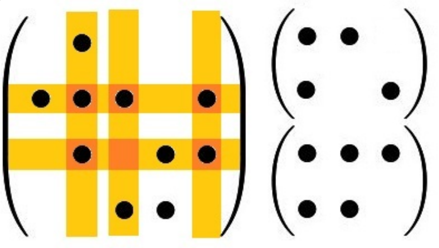

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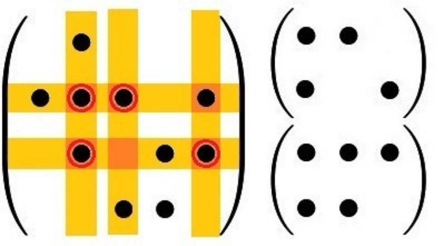

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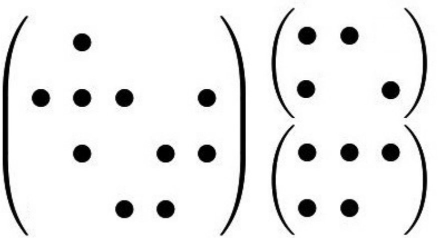

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MATRIX EXTREMAL FUNCTIONS

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Definition

The ***weight*** *extremal function,* $ex(n, M)$*, is defined as the maximum number of one entries in an* $n \times n$ *matrix that avoids* M*. The* ***rectangular weight*** *extremal function,* $ex(m, n, M)$*, is defined the same for an* $m \times n$ *matrix.*

Motivation

1. Unit distances in convex polygons
2. Stanley-Wilf Conjecture

Unit distances in convex polygons

Problem

(Erdös and Moser, 1959) What is the maximum number of unit distances that can be formed between the vertices of a convex n-gon?

- They conjectured that the answer would be linear in n, which matches the current lower bound
- The current upper bound is $n \log_2 n + 3n$, found by Aggarwal using the weight extremal functions of two matrices

$$\begin{pmatrix} \bullet & & \bullet \\ & \bullet & \bullet \end{pmatrix} \quad \begin{pmatrix} \bullet & \bullet \\ \bullet & \\ & \bullet \end{pmatrix}$$

Stanley-Wilf Conjecture

Conjecture

(Stanley and Wilf) For any permutation π, the number of permutations length n that avoid π is at most exponential in n.

- For example, 24315 contains the permutation 123
- In 2004, Marcus and Tardos proved that all permutation matrices have linear extremal functions
- This proves the conjecture by a theorem of Klazar (2000) that demonstrates the equivalence of the two

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- For example, **24**31**5** contains the permutation 123
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An example

Problem

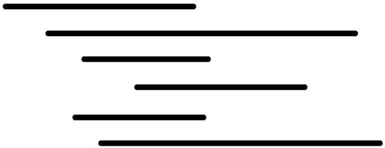
What is the value of the extremal function $ex(m, n, L_1)$*?*

$$\begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & & \end{pmatrix}$$

L_1: Bar-visibility graphs

Definition

*A **bar-visibility graph** has horizontal bars for the vertices. The edges are vertical lines that connect two bars without crossing any other.*



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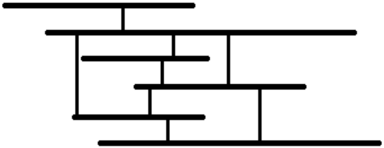

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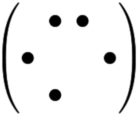
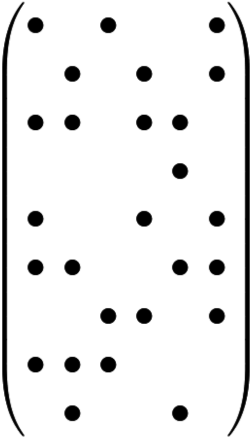
L_1: Characterization



- Draw a bar from the leftmost to the rightmost one entry in each row except the bottom one
- Mark every one entry that's not at the end of a bar nor is one of the bottom two in its column
- Draw an edge from that one entry to the next bar below it
- $ex(m, n, L_1) \leq 2(n-2) + 2m + (3(n-1) - 5) = 5n + 2m - 12$



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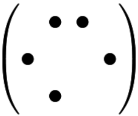

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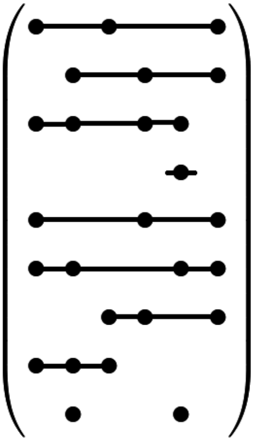

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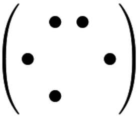

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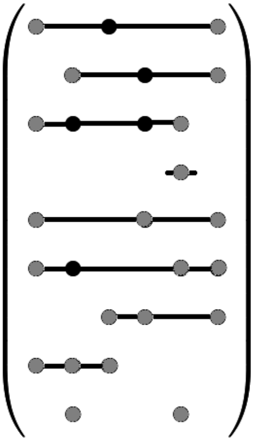

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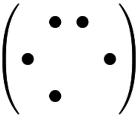

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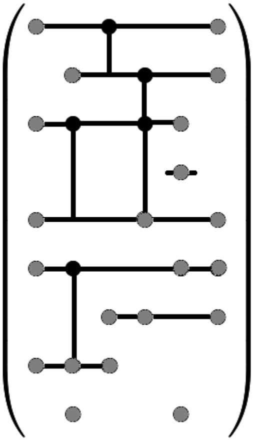

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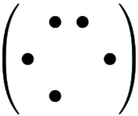

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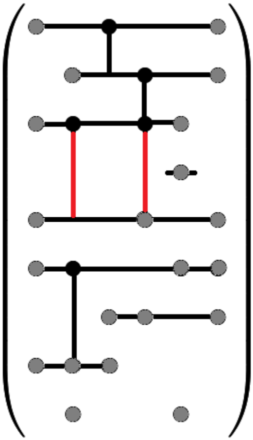

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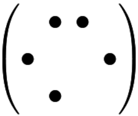

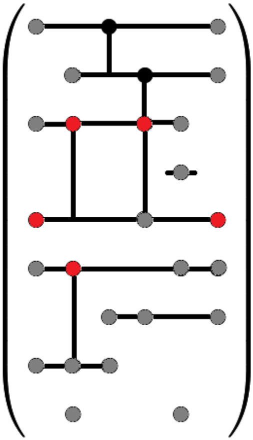

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  - $ex(n,M) = ex(n,n,M)$
  - $ex(\min(m,n),M) \leq ex(m,n,M) \leq ex(\max(m,n),M)$

Separability

Definition

*A matrix M is called **separable** if there exist functions f and g and some constant c such that $ex(m, n, M) = f(m) + g(n) + O(1)$ for all $m, n \geq c$.*

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Theorem

If a matrix M is separable, then it is linear.

Equivalent Definitions

Definition

*The **finite difference** $\Delta_1 ex(m, n, M)$ is defined to be $ex(m, n, M) - ex(m-1, n, M)$. The difference $\Delta_2 ex(m, n, M)$ is defined equivalently on the second coordinate.*

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Theorem

The following are equivalent:

- *M is separable*
- *$\Delta_1 ex(m,n,M)$ is a function of m only*
- *$\Delta_2 ex(m,n,M)$ is a function of n only*
- *$ex(m,n,M) = ex(m,c,M) + ex(c,n,M) + O(1)$ for $m,n \geq c$*

Lower Bounds

Theorem

For any matrix M, $ex(m,n,M) \geq ex(m,c,M) + ex(c,n,M) - 2c^2$.

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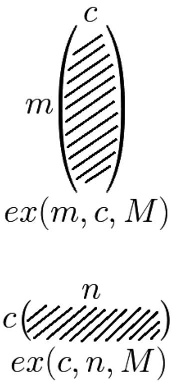

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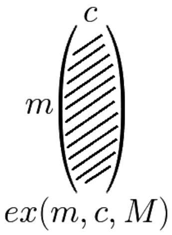

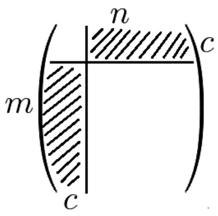

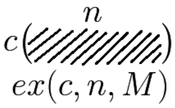

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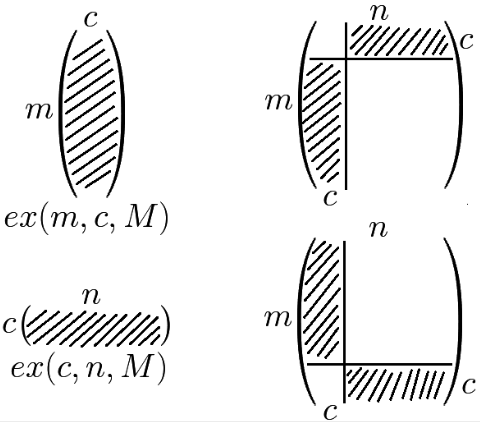

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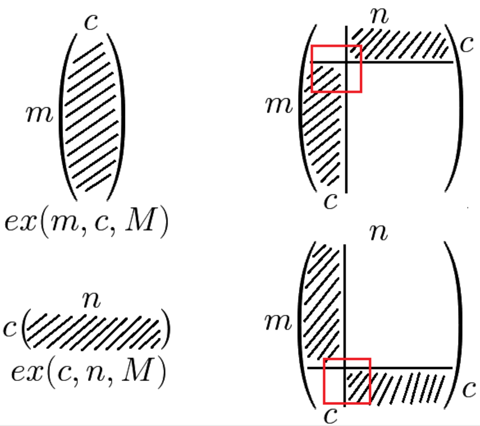

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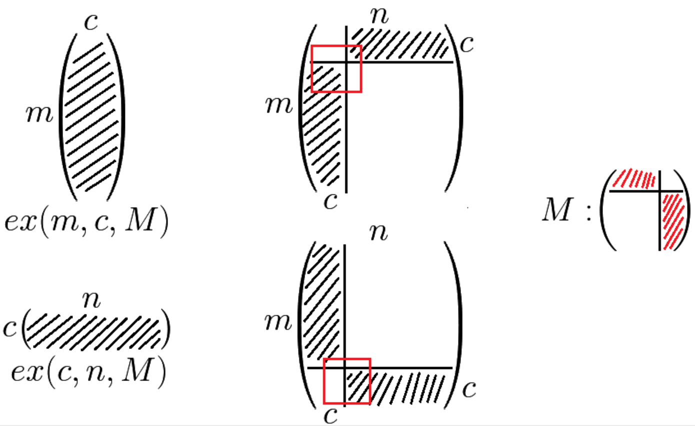

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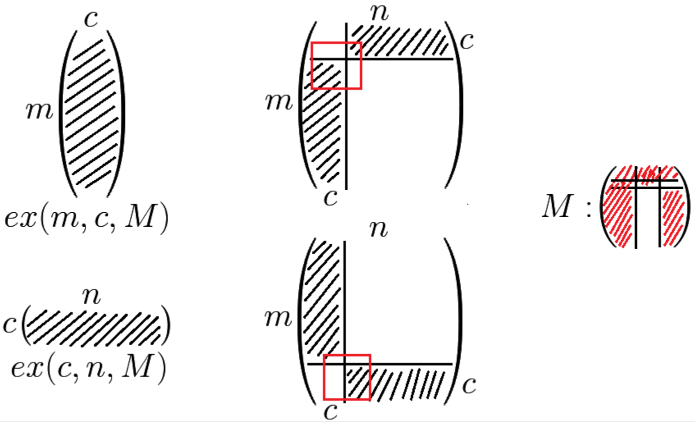

Further Directions

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How small can c be in the definition of separability? Are there any matrices that are not separable for small values of m and n but become separable much later on?

ACKNOWLEDGMENTS

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- MIT PRIMES program
- Jesse Geneson
- My parents

References

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