Lower Central Series Ideal Quotients Over $\mathbb{Z}$ and $\mathbb{F}_p$

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Abstract

Given a graded associative algebra A, this project studies its lower central series defined by $L_1 = A$ and $L_{i+1} = [L_i, A]$. We may consider successive quotients $N_i(A) = M_i(A)/M_{i+1}(A)$, where $M_i(A) = AL_i(A)A$. These quotients are direct sums of graded components. Our purpose is to describe the $\mathbb{Z}$-module structure of the components; i.e., their free and torsion parts. Following computer exploration using *MAGMA*, two main cases are studied. The first considers $A = A_k/(f_1(x_1^{p^{m_1}}), \ldots, f_k(x_k^{p^{m_k}}))$, with noncommutative polynomial relations f_i, and A_n the free algebra defined on k generators $\{x_1, \ldots, x_k\}$ over a field of characteristic p. For primes $p > 2$, we prove that $p^{\sum n_j} \mid \dim(N_i(A))$. Moreover, we determine polynomials dividing the Hilbert series of each $N_i(A)$. The second concerns $A = \mathbb{Z}\langle x_1, x_2, \rangle/(x_1^m, x_2^n)$. For $i = 2, 3$, the bigraded structure of $N_i(A_2)$ is completely described.

1 Introduction

Algebraic geometry is technically based on commutative algebra as one can reconstruct an affine algebraic variety from its commutative algebra of functions. This suggests to define a noncommutative "space" via a noncommutative algebra which plays the role of the algebra of functions on this nonexistent space.

This can seem a very daring postulate, but it has proven to be a powerful one. It lies at the heart of the theory of noncommutative geometry of Alain Connes and Quantum groups of Vladimir Drinfeld.

Feigin and Shoikhet [FS07] initiated a new approach to the study of a given noncommutative algebra. Their idea was to approximate it by pieces whose degree of noncommutativity is controlled. This parallels the idea of approaching a function by polynomials in its Taylor expansion. One gains through these "more commutative" approximations an access to tools of classical geometry.

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To be more precise, the first approximation of a noncommutative algebra A is its abelianization $A_{ab} := A/A[A,A]A$. A way to generalize this construction to higher orders is to consider the *lower central series* $(L_i)_{i\in\mathbb{N}}$. It is defined inductively: the first term L_1 is A itself, while the following ones are defined as $L_{i+1} = [A, L_i]$. In particular the abelianization of A can be interpreted as $A_{ab} = A/A[A,A]A = M_1/M_2$, where M_i denotes the ideal generated by L_i, i.e. $M_i := AL_iA$. This suggests to define $N_i := M_i/M_{i+1}$ as a generalization of A_{ab}. Note that some other papers on the same subject define and study directly $B_i := L_i/L_{i+1}$, without first forming an ideal.

The innovative work of Feigin and Shoikhet spawned a new line of research. The structure of $B_i(A)$ was first studied by [FS07], then by [DKM08], [DE08], [AJ10], [BJ13], [BB11], [BJ11], and [BEJ$^+$12]. Shortly after came the study of the $N_i(A)$, including papers by [Ker13], [BEJ$^+$12], [JO13], and lastly [CFZ13].

In their paper, [FS07] considered $A = A_n(\mathbb{C})$, the free associative algebra on n letters, over the field of complex numbers, but their results remain valid over any field of characteristic zero, in particular over $\mathbb{Q}$. They have discovered that A/M_3 can be identified with the algebra $\Omega_{even}(\mathbb{C}^n)$ of even differential forms on $\mathbb{C}^n$ with Fedosov product. Thus, one can wonder whether there are other incarnations of classical geometric objects hidden in the $N_i(A)$'s.

This is a difficult question, and a first approach to understand the $N_i(A)$'s is to determine their dimensions. We do not want to restrict ourselves to free algebras, but consider instead algebras with relations. We work with fields or rings different than $\mathbb{Q}$, for example over the integers $\mathbb{Z}$ or a finite field k of characteristic p, as these are more accessible to computer assisted exploration.

In the first section, we consider algebras of the form $A := A_n/(f_1, f_2, \ldots, f_m)$. We show in Theorem 2.8 that $W_N(k)$, the Weyl algebra with divided powers, acts on $N_i(A)$. More generally there is an action of $W_{n_1}(k)\otimes, .. \otimes W_{n_r}(k)$, and one obtains (corollary 2.9) that $\dim(N_i(A))$ is divisible by $p^{\sum n_j}$. We also deduce (corollary 2.10) that the Hilbert series of $N_i(A)$ with respect to the corresponding variables $X_1, \ldots, X_r$ is divisible by $(1 + X_1 + \cdots + X_1^{p^{n_1}-1}) \cdots (1 + X_r + \cdots + X_r^{p^{n_r}-1})$.

In the second section, we work over $\mathbb{Z}$ and consider algebras of the form $A := A_2/(x_1^m, x_2^n)$. We prove that the $\mathbb{Z}$-module structure of $N_2(A)$ and $N_3(A)$ are given by the tables

(m,n)	0	1	2	$\cdots$	$\cdots$	$n-1$	n
0	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
1	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$	$\cdots$	$\mathbb{Z}$	$\mathbb{Z}_n$
s 2	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$	$\cdots$	$\mathbb{Z}$	$\mathbb{Z}_n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\ddots$	$\vdots$	$\vdots$
$m-1$	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$	$\cdots$	$\mathbb{Z}$	$\mathbb{Z}_n$
m	$\vdots$	$\mathbb{Z}_m$	$\mathbb{Z}_m$	$\cdots$	$\cdots$	$\mathbb{Z}_m$	$\mathbb{Z}_{(m,n)}$

Table 1: Bigraded Description of $N_2(A)$

and

(m,n)	0	1	2	$\cdots$	$\cdots$	$n-1$	n	$n+1$
0	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
1	$\vdots$	0	$\mathbb{Z}$	$\cdots$	$\cdots$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_{f(n)}$
2	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\cdots$	$\cdots$	$\mathbb{Z}^3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_n$	$\mathbb{Z}_n \oplus \mathbb{Z}_{f(n)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\ddots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\cdots$	$\cdots$	$\mathbb{Z}^3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_n$	$\mathbb{Z}_n \oplus \mathbb{Z}_{f(n)}$
m	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_m$	$\cdots$	$\cdots$	$\mathbb{Z}^2 \oplus \mathbb{Z}_m$	$\mathbb{Z}_m \oplus \mathbb{Z}_n$	$\mathbb{Z}_{f(n)} \oplus \mathbb{Z}_{(m,n)}$
$m+1$	$\vdots$	$\mathbb{Z}_{f(m)}$	$\mathbb{Z}_m \oplus \mathbb{Z}_{f(m)}$	$\cdots$	$\cdots$	$\mathbb{Z}_m \oplus \mathbb{Z}_{f(m)}$	$\mathbb{Z}_{f(m)} \oplus \mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(m,n)}$

Table 2: Bigraded Description of $N_3(A)$

We give an explicit basis of the non-torsion part and also compute the torsion in terms of m and n.

2 Divisibility of Total Dimensions in characteristic p

The main tool of this section, Proposition 2.6, states that any finite dimensional module over $W_N(k)$, a sub algebra of the Weyl algebra with divided power structure, has dimension divisible by p^N. We show in Theorem 2.8 that $N_i(A)$ can be equipped with an action of $W_N(k)$, and as a corollary, one obtains (corollary 2.9) that $\dim(N_i(A))$ is divisible by $p^{\sum n_j}$.

2.1 Weyl algebra with divided powers

We first recall the definition 2.1 of the algebra $W(k)$ and then give in lemma 2.2 a system of generators in order to formulate the definition 2.3 of $W_N(k)$.

Definition 2.1 *The* **Weyl algebra with divided powers** *over* $\mathbb{Z}$*,* $W(\mathbb{Z})$*, is the algebra of linear operators of the form*

$$\sum_{i,j} a_{ij} x^i \frac{D^j}{j!},$$

where $D := \dfrac{\partial}{\partial x}$ *and the coefficients* a_{ij} *are in* $\mathbb{Z}$*. For a commutative ring* R*, one defines* $W(R) := W(\mathbb{Z}) \otimes R$*.*

Note that the elements of $W(\mathbb{Z})$ define endomorphisms of $\mathbb{Z}[x]$ despite the denominators. If we denote $D_j := D^j/j!$ it is clear that x, together with D_j for all non-negative j generate $W(\mathbb{Z})$. Also one has:

$$D_j D_r = \frac{D^j D^r}{j!r!} = \frac{(j+r)!}{j!r!} \frac{D^{j+r}}{(j+r)!} = \binom{j+r}{j} D_{j+r}. \tag{1}$$

From now on, R will be a field k of characteristic p. We have a well-known Lemma:

Lemma 2.2 *If* k *has characteristic* p*, the algebra generated by* D_j *for all non-negative* j *is also generated by* D_{p^i} *for all non-negative* i*. More precisely, if* a *is a non-negative integer with representation* $a = a_n p^n + \cdots + a_0$ *in base* p*, we have*

$$D_a = \frac{1}{C} \prod_s (D_{p^s})^{a_s}, \ \text{with } C = \prod_s (a_s!). \tag{2}$$

Proof One can write a as the sum of 2 elements b and c, in a compatible way with its decomposition in basis p:

$$a = \underbrace{a_n p^n + \cdots + a_k p^k}_{b} + \underbrace{a_{k-1} p^{k-1} + \cdots + a_0}_{c}.$$

We claim that

$$D_a = D_b D_c.$$

According to eq. (1), we already know that

$$\binom{a}{b} D_a = D_b D_c.$$

Therefore it suffices to prove, that $\binom{a}{b} = 1 \pmod p$.

Let us recall Lucas's Theorem: for all non-negative integers m, n and prime p, we have

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod p, \tag{3}$$

where $m = \sum_{i=0}^{k} m_i p^i$ and $n = \sum_{i=0}^{k} n_i p^i$. In our setting:

$$\binom{a}{b} = \Pi_s \binom{a_s}{b_s} \pmod{p}.$$

But we can decompose this product into two products (for $s \leq k-1$ and for $s > k-1$) and use the remark that by definition of b, $b_s = \begin{cases} a_s & \text{if } s > k-1, \\ 0 & \text{otherwise.} \end{cases}$

In other words (mod p):

$$\begin{aligned} \binom{a}{b} &= \Pi_s \binom{a_s}{b_s} \\ &= \Pi_{s>k-1} \binom{a_s}{b_s} \Pi_{s\leq k-1} \binom{a_s}{b_s} \\ &= \Pi_{s>k-1} \binom{a_s}{a_s} \Pi_{s\leq k-1} \binom{a_s}{0} \\ &= 1. \end{aligned}$$

Iterating this result, one gets

$$D_a = \Pi_s D_{a_s p^s}. \tag{4}$$

We now want to prove by induction that

$$\alpha! D_{\alpha p^i} = (D_{p^i})^\alpha. \tag{5}$$

By eq. (1),

$$\binom{\alpha p^i}{p^i} D_{\alpha p^i} = D_{(\alpha-1)p^i} D_{p^i},$$

so we are looking for the expression of $\binom{\alpha p^i}{p^i}$. But Lukas' theorem gives

$$\binom{\alpha p^i}{p^i} = \binom{\alpha}{1} = \alpha,$$

which completes the induction step. □

Thus, $W(k)$ is generated by x and D_{p^i} for all $i \geq 0$.

Definition 2.3 *Denote by $W_N(k)$ the subalgebra generated by x and $D_p, \cdots, D_{p^{N-1}}$. By Lemma 2.2, it is generated by x and all D_j with $j < p^N$.*

For example, $W_1(k)$ is generated by x and D with relations $[D, x] = 1$ and $D^p = 0$.

We will need the following lemma in the proof of proposition 2.6.

Lemma 2.4 *For $j < p^N$, all $D_j \in W_N(k)$ are nilpotent. Moreover x^{p^N} is central in this algebra.*

Proof To show that all D_j are nilpotent, we first show that all D_{p^i} are nilpotent.

Since $D_{(m-1)p^i} D_{p^i} \overset{(1)}{=} \binom{mp^i}{p^i} D_{mp^i}$, an induction with Lucas's theorem $\binom{mp^i}{p^i} \overset{(3)}{=} \binom{m}{1} = m$ shows that $m! D_{mp^i} = D_{p^i}^m$. In particular, for $m = p$, we have that

$$D_{p^i}^p = p! D_{p^{i+1}} = 0. \tag{6}$$

For arbitrary $0 \le j < p^N$, we have by the proof of Lemma 2.2 that

$$D_j^p \overset{(4)}{=} \left(\prod_s D_{j_s p^s} \right)^p = \prod_s (D_{j_s p^s})^p.$$

It remains to show that one of the terms in this product vanishes. Choosing any term in the product and noting that since all $j_s < p$, $j_s! \neq 0$, one has

$$D_{j_s p^s}^p \overset{(5)}{=} \left(\frac{D_{p^s}^{j_s}}{j_s!} \right)^p = \frac{(D_{p^s}^{j_s})^p}{j_s!^p} = \frac{(D_{p^s}^p)^{j_s}}{j_s!^p} \overset{(6)}{=} \frac{0^{j_s}}{j_s!^p} = 0. \tag{7}$$

Thus, we have shown that all D_j are nilpotent.

It is clear that x^{p^N} commutes with x. We now show that it commutes with D_j as well. According to lemma 2.2 it suffices to show it for D_{p^i}, with $p^i < p^N$. To this end, note that

$$[D_{p^i}, x^{p^N}] x^\ell = D_{p^i}(x^{p^N} x^\ell) - x^{p^N}(D_{p^i} x^\ell) = \binom{p^N + \ell}{p^i} x^{p^N + \ell - p^i} - \binom{\ell}{p^i} x^{p^N + \ell - p^i}.$$

Now, we show that $0 = (\binom{p^N+\ell}{p^i} - \binom{\ell}{p^i})$. But, by Lucas's Theorem we have that

$$\binom{p^N + \ell}{p^i} - \binom{\ell}{p^i} = \binom{1}{0} \prod_s^{N-1} \binom{\ell_s}{p_s^i} - \prod_s^{N-1} \binom{\ell_s}{p_s^i} = 0. \ \square$$

2.2 Divisibility of dimensions of $W_N(k)$-modules

For the rest of this section, we assume that k is algebraically closed.

Lemma 2.5 *Let V be a $W_N(k)$ module. Then all of the D_j share a common null vector $v \in V$. Moreover, if V is irreducible, x^{p^N} acts by a scalar $s \in k$.*

Proof The D_j's commute with each other and, by Lemma 2.4, are nilpotent. We will prove by induction that they all share a common null vector. Our base case is true: as D_1 is nilpotent, for any v, there exists a certain power n for which $v_1 := D_1^n v_k$ is nonzero, but $D_1^{n+1} v_k$ vanishes, so one has $D_1(v_1) = 0$. Now, suppose that $D_1, \ldots, D_k$ all share a common null vector $v_k \in V$. Since

D_{k+1} is nilpotent, there exists some integer ℓ such that $D_{k+1}^{\ell}(v_k) = 0$ and $v_{k+1} := D_{k+1}^{\ell-1}(v_k) \neq 0$. In particular $D_{k+1}(v_{k+1}) = 0$. Additionally, for any $j \leq k$, we have $D_j(v_{k+1}) = D_j(D_{k+1}^{\ell-1}v_k) = D_{k+1}^{\ell-1}D_j(v_k) = 0$, so our induction is done.

In addition, since x^{p^N} is central in $W_N(k)$, and since V is an irreducible $W_N(k)$-module, Schur's lemma asserts that x^{p^N} acts by multiplication by a scalar. □

This lemma enables to derive the main result of this section:

Proposition 2.6 *Any finite dimensional module over $W_N(k)$ has dimension divisible by p^N.*

We recall the following basic result whose proof we omit:

Lemma 2.7 *Let E, F be subspaces of vector spaces V, W respectively. Given a linear mapping $\phi : V \to W$ such that $\phi(E) \subset F$, the map $\bar{\phi} : V/E \to W/F$ given by $\bar{\phi}([v]) = [\phi(v)]$ for $v \in V$ is well defined and linear.*

Proof Let M be a finite dimensional module over $W_N(k)$. If M is not already irreducible, then we may find an irreducible submodule V_1 of M; then, we have that $M \cong V_1 \oplus M/V_1$. If M/V_1 is not yet irreducible, then we may find an irreducible submodule $V_2 \subset M/V_1$; this implies the existence of a module $F_2 \subset M$ such that $F_1 := V_1 \subset F_2$ and $F_2/F_1 \cong V_2$. By continuing this process we build in a finite number of steps an exhausting filtration $F_1 \subset \cdots \subset F_n = M$ of M. The associated successive quotients $V_i := F_i/F_{i-1}$ are by construction irreducible modules and together form the Jordan-Hölder decomposition of M :

$$M = V_1 \oplus V_2 \oplus \cdots \oplus V_d.$$[1]

To prove the proposition, it suffices to show that each V_i has dimension divisible by p^N. Let V be one of these V_i.

Our strategy is to show that $V \cong k[x]/(x^{p^N} - s)$, with s given by lemma 2.5. A filtration on $k[x]$ is given by spaces F_i of polynomials of degree less than i. It induces a filtration $\bar{F}_i$ on $Q = k[x]/(x^{p^N} - s)$, such that $\bar{F}_i = \bar{F}_{p^N}$ for $i \geq p^N$. One has $k[x]/(x^{p^N}) = gr(Q)$ where $gr(Q)_i := \bar{F}_i/\bar{F}_{i-1}$. It is clearly of dimension p^N. One will conclude with the property that a filtered k-module and its associated graded share the same dimension.

Therefore, we want to build an injective map $\bar{\bar{f}} : k[x]/(x^{p^N} - s) \xrightarrow{\sim} V'$ for V' a non-zero submodule of V: since V is irreducible, one will get $V \cong V'$ and hence the result.

By Lemma 2.5, there exists a common null-vector v to all the D_j. Set $V' = W_N(k) \cdot v$ to be the $W_N(k)$ submodule of V generated by v.

Consider $W_N(k) \cdot b$, the one-dimensional free $W_N(k)$ module generated by a symbol b. Then, we have a map

$$f : W_N(k) \cdot b \longrightarrow W_N(k) \cdot v.$$

[1] V_2 may not be a submodule of M, as this is a decomposition as vector spaces.

It is defined on b by $f(b) := v$, and extended to $w \cdot b \in W_N(k) \cdot b$ by $f(w \cdot b) = w \cdot f(b) = w \cdot v$. This map is clearly surjective. We want to show that $k[x]/(x^{p^N} - s) \cdot b$ is a quotient of $W_N(k) \cdot b$ and that f will induce the map $\bar{\bar{f}}$ that we are looking for. More precisely we will show that f produces a surjective module morphism

$$\bar{f} : k[x] \cdot b \longrightarrow W_N(k) \cdot v,$$

which in turn will induce

$$\bar{\bar{f}} : k[x]/(x^{p^N} - s) \cdot b \longrightarrow W_N(k).v.$$

Let (D_{p^i}) be the left ideal generated by all D_{p^i} for $0 \leq i < N$. Then, $(D_{p^i}) \cdot b$ is a submodule of $W_N(k) \cdot b$. If we show that $(D_{p^i}) \cdot b \subset Ker(f)$, then by Lemma 2.7, there is an induced map

$$\bar{f} : W_N(k) \cdot b/(D_{p^i}) \cdot b \twoheadrightarrow W_N(k) \cdot v.$$

Since $W_N(k) \cdot b/(D_{p^i}) \cdot b \cong \big(W_N(k)/(D_{p^i})\big) \cdot b \cong k[x] \cdot b$, we will have the desired map $\bar{f} : k[x] \cdot b \longrightarrow W_N(k) \cdot v$.

We therefore show that $(D_{p^i}) \cdot b \subset Ker(f)$. Consider an arbitrary element in $(D_{p^i}) \cdot b$. It is of the form $uD_{p^i} \cdot b$ for some $u \in W_N(k)$. Since $f(uD_{p^i} \cdot b) = uD_{p^i} \cdot f(b) = uD_{p^i} \cdot v$, and since we have chosen v so that $D_{p^i}v = 0$, we are done.

It remains to show that this map

$$\bar{f} : k[x] \cdot b \longrightarrow W_N(k) \cdot v$$

that we have just built indeed descends to a map

$$\bar{\bar{f}} : k[x]/(x^{p^N} - s) \cdot b \longrightarrow W_N(k).v.$$

By Lemma 2.5, $x^{p^N}(v) = s(v)$, so we have that $(x^{p^N} - s) \cdot b \in Ker(\bar{f})$, and we have a map $\bar{\bar{f}} : k[x]/(x^{p^N} - s) \cdot b \twoheadrightarrow W_N(k) \cdot v$ by Lemma 2.7.

To show that $\bar{\bar{f}}$ is injective, it suffices to show that $k[x]/(x^{p^N})$ is irreducible. So, let B be a non-trivial submodule of $k[x]/(x^{p^N})$, we will show that it coincides with $k[x]/(x^{p^N})$. We want to show that B contains an element of the form 1+higher terms, since such an element generates the module $k[x]/x^{p^N}$ over $k[x]$. Let b be an arbitrary nonzero vector in B. Let x^n be the lowest monomial it contains. (We normalize b so that $b = x^n +$ higher terms.) As

$$D_n b = 1 + \text{higher terms},$$

we have that $B = k[x]/(x^{p^N})$. $\square$

2.3 Applications

Suppose that we work over an algebraically closed field k of characteristic p. Denote the algebra $A_n := k\langle x_1, x_2, \dots, x_n\rangle$. Our noncommutative algebra is $A := A_n/(f_1, f_2, \dots, f_m)$, where each $f_i = g_i \circ h$ for $g_i \in A_n$ and $h(x_1, \dots, x_n) := (x_1^{p^N}, \dots, x_n)$.

Theorem 2.8 *In the above setting, the algebra $W_N(k)$ acts on $N_i(A)$.*

Proof We first describe the action on A. Let x act by $x(a) = x_1 a$. To define the action of D_m, consider the automorphism T of the algebra $A \otimes k[t]/t^{p^N}$ given by

$$T(x_i) := \begin{cases} x_1 + t & i = 1 \\ x_i & i > 1. \end{cases}$$

To show that this is well-defined, we check that $T(f) = f$ for $f = g \circ h$. Since f is in $k\langle x_1^{p^N}, x_2, \dots, x_n\rangle$, it is of the form $f = \sum (\Pi_j \alpha_j x_1^{p^N})\alpha_l$, with $\alpha_k \in k\langle x_2, \dots, x_n\rangle$

Therefore

$$T(f) = \sum (\Pi_j \alpha_j T(x_1^{p^N}))\alpha_l = f,$$

by definition of T and since $T(x_1^{p^N}) = (T(x_1))^{p^N} = (x_1 + t)^{p^N} = x_1^{p^N} + t^{p^N} = x_1^{p^N}$.

Now, define the action of D_j on $N_i(A)$, for $j < p^N$, to be the coefficient of t^j of T acting on $N_i(A)$. So, $T(v) = \sum t^j D_j(v)$. This defines the required action. □

Corollary 2.9 *With the conditions of Theorem 2.8, if $N_i(A)$ is finite dimensional (i.e. if the abelianization A_{ab} is finite dimensional), then dim$(N_i(A))$ is divisible by p^N. More generally, if we instead define h as*

$$h(x_1, \dots, x_r, \dots, x_n) := (x_1^{p^{n_1}}, \dots, x_r^{p^{n_r}}, x_{r+1}, \dots, x_n)$$

(i.e. the relations are noncommutative polynomials of the first r variables $x_1^{p^{n_1}}, \dots, x_r^{p^{n_r}}$), then dim$(N_i(A))$ is divisible by $p^{\sum n_i}$.

Proof According to Proposition 2.6, each finite dimensional representation of $W_N(k)$ has dimension divisible by p^N. In the case of the relations being polynomials of $x_i^{p^{n_i}}$ with $1 \le i \le r$, the tensor product algebra $\bigotimes_i W_{n_i}(k)$ acts on $N_i(A)$. Because this is a tensor product of irreducible representations of $W_{n_i}(k)$, each of its irreducible representations has dimension divisible by $p^{\sum n_i}$. □

Corollary 2.10 *Except for finite dimensionality of $N_i(A)$, suppose that in the situation of Corollary 2.9, the relations are homogeneous in $x_1, \dots, x_r$. Then, the Hilbert series of $N_i(A)$ with respect* *to the corresponding variables $X_1, \ldots, X_r$ is divisible by*

$$(1 + X_1 + \cdots + X_1^{p^{n_1}-1}) \ldots (1 + X_r + \cdots + X_r^{p^{n_r}-1}),$$

in the sense that the ratio is a power series with non-negative integer coefficients.

Proof Consider the case $r = 1$, as the general proof follows similarly. Let $M = N_i(A)$. It is a $\mathbb{Z}$-graded module over $W_N(k)$, with a grading given by $deg(x) = 1$, $deg(D) = -1$, and nonnegative degrees of the vectors. Because of this, we may take any homogeneous vector and apply D_j until getting 0; thus, there exists a common null vector of D_j, namely $v_1 \neq 0$. Let $M_1 = F_1$ be the submodule generated by v_1, then it has a basis of $\langle v_1, xv_1, x^2v_1 \ldots \rangle$. Thus, we have two cases for M_1. First, if none of these $x^s v_1 = 0$, then $M_1 \cong k[x]$. Second, if $x^s v_1 = 0$, where s is minimal but positive, then we have that s is a multiple of p^{n_1} as by Theorem 2.8. Thus, $M_1 \cong k[x]/(x^{jp^{n_1}})$ for some positive integer j.

Next, let $v_2 \neq 0$ be a common null vector of $D_j \in M/F_1$. We define M_2 as the submodule in M/F_1 generated by v_2, and F_2 as the preimage of M_2 in M. Continuing this construction, we make an exhaustive filtration $F_1 \subset F_2 \subset F_3 \subset \ldots$ of M such that $F_i/F_{i-1} = M_i$, and all $M_i \cong k[x]$ or $k[x]/(x^{jp^{n_1}})$.

If E is a graded vector space, denote h_E as the Hilbert Series of E, i.e., if $E = \bigoplus_i E_i$, then $h_E = \sum_i \dim(E_i) X^i$.

Since $h_{N_i(A)} = h_{M_1} + h_{M_2} + \cdots$, we are done if each h_{M_i} is divisible by the desired polynomial. To this end, note that if $M_i \cong k[x] \cdot v = \langle v, xv, \ldots \rangle$ and $deg(v) = \ell$, then $h_{M_i} = X^\ell + X^{\ell+1} + \cdots = (1 + X + \cdots + X^{p^{n_1}-1})(X^\ell + X^{\ell+p^{n_1}} + \cdots)$. And, if $M_i \cong k[x]/(x^{p^{n_1}j}) \cdot v'$, where $deg(v') = \ell'$, then $h_{M_i} = X^{\ell'} + X^{\ell'+1} + \cdots + X^{\ell'+(j-1)p^{n_1}} = (1 + X + \cdots + X^{p^{n_1}-1})(X^{\ell'} + X^{\ell'+p^{n_1}} + \cdots + X^{\ell'+(j-1)p^{n_1}})$. $\square$

3 Bigraded Structure of N_2 and N_3 over $\mathbb{Z}$

In this section, we give complete descriptions of the abelian group of $N_i(A)$ for $i = 2, 3$ and $A = A_2/(x_1^m, x_2^n)$, where $A_2 = \mathbb{Z}\langle x_1, x_2 \rangle$. A bigrading of A_k, the free algebra with k generators, is given by the total degree in $x_1, x_2, \ldots, x_k$. This gives us more information about the inherent structure of the algebra.

However, with the added relations from the ideal (x_1^m, x_2^n), which is generated by homogeneous terms in x_1, x_2, A inherits a bigrading from A_2 which is bounded by $_m, n)$. More precisely, the bigrading of a monomial P is given by $(|P|_{x_1}, |P|_{x_2})$, where $|P|_{x_1}$ denotes the total degree in x_1 of P and $|P|_{x_2}$ denotes the total degree in x_2 of P. For example, the bigrading of the term $x_1 x_2^3 x_1$ is given by $(2, 3)$.

In fact, the bigrading over A_2 and A induce a grading over $N_2(A)$ and $N_3(A)$.

When we view $N_i(A_2)$ as finite-dimensional abelian groups, we may induce a bigrading based upon the degrees of each generator.

Since these are abelian groups, they may be decomposed into a free part (copies of $\mathbb{Z}$) and a torsion part (direct sum of $\mathbb{Z}_m$ for integral m) by the Fundamental Theorem of Finitely Generated Abelian Groups. Thus, using the data generated by our *MAGMA* program, we conjecture and prove the structures of N_2 and N_3.

We will use the simple but well-known Leibniz Rule throughout:

Lemma 3.1

$$[a_1 \dots a_n, b] = \sum_{i=1}^{n} a_1 \dots a_{i-1}[a_i, b]a_{i+1} \dots a_n$$

and

$$[a, b_1 \dots b_n] = \sum_{i=1}^{n} b_1 \dots b_{i-1}[a, b_i]b_{i+1} \dots b_n.$$

3.1 Structure of N_2

The aim of this section is to show that the abelian group structure of N_2 is given by the following table:

(m,n)	0	1	2	$\cdots \quad \cdots$	$n-1$	n
0	0	$\cdots$	$\cdots$	$\cdots \quad \cdots$	$\cdots$	$\cdots$
1	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots \quad \cdots$	$\mathbb{Z}$	$\mathbb{Z}_n$
2	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots \quad \cdots$	$\mathbb{Z}$	$\mathbb{Z}_n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$m-1$	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots \quad \cdots$	$\mathbb{Z}$	$\mathbb{Z}_n$
m	$\vdots$	$\mathbb{Z}_m$	$\mathbb{Z}_m$	$\cdots \quad \cdots$	$\mathbb{Z}_m$	$\mathbb{Z}_{(m,n)}$

Table 3: Bigraded Description of $N_2(A)$

where $(m,n) = gcd(m,n)$. In other terms we want to show that

Theorem 3.2 *The free part of $N_2(A)$ as a $\mathbb{Z}$-module has a basis $\{x_1^i x_2^j y \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$. (Free part description)*

and that

Theorem 3.3 *As a $\mathbb{Z}$-module, the elements $x_1^i x_2^{n-1} y$ for $0 \leq i \leq m-2$ (resp $x_1^{m-1} x_2^j y$) are of torsion of order n (resp m), except when $i = m-1$ for which $x_1^{m-1} x_2^{n-1} y$ is of order (m,n). (Torsion part description)*

Our chain of reasoning in proving Theorem 3.2 starts with forming a basis of $M_2(A_2)$ (Lemma 3.5). This induces a generating family of $N_2(A_2) = M_2(A_2)/M_3(A_2)$ with eventually some redundancy. In order to eliminate this redundancy, we will rewrite these elements using R to arrive to a normal form and obtain a basis of $N_2(A_2)$. Finally, if we take into account the extra relations of A to find as basis of $N_2(A)$ (Theorem 3.2), then some torsion appears. This torsion part induced by the relations will be separated from the free part of $N_2(A)$.

Let us recall a presentation of A/M_3 from [BEJ+12], inspired by the seminal paper [FS07] by Feigin and Shoikhet.

Theorem 3.4 *$A_2/M_3 = \langle x_1, x_2, y\rangle/(R)$ where R is the set of relations*

$$[x_1, x_2] = y, \tag{8}$$

$$[x_1, y] = [x_2, y] = y^2 = 0. \tag{9}$$

Below are three tools we use intermediately to prove 3.2:

Remark A basis of A_n, denoted $\mathcal{B}(A_n)$, is given by monomials in the generators $x_1, \cdots, x_n$.

Lemma 3.5 *A basis of $M_2(A_2)$ is given by $\{vyw \mid v, w \in \mathcal{B}(A_n)\}$.*

Proposition 3.6 *A basis of $N_2(A_2)$ as a $\mathbb{Z}$-module is given by $\{x_1^i x_2^j y\}$.*

Proof Recall the definition of $M_2(A_2) = A_2L_2(A_2)A_2 = A_2[A_2, A_2]A_2$. Any element in this M_2 is a linear combination of $u[v, w]z$ where $u, v, w, z \in \mathcal{B}(A_2)$.

The Leibniz rule gives $u[v, w]z = u(\sum v_i y w_i)z$ for some $v_i, w_i \in \mathcal{B}(A_2)$; note this means that there is at least one y term in each monomial, and M_2 is spanned by $\{v'yw' \mid v', w' \in \mathcal{B}(A_2)\}$. It is simply a routine checking to verify the linear independence of this basis. □

We now prove Proposition 3.6.

Proof Starting with a basis $\mathcal{B}_2$ of $M_2(A_2)$ given by Theorem 3.4, we use the relations from Theorem 3.2 to rewrite the elements of its image $\bar{\mathcal{B}}_2$ in $N_2(A_2) = M_2(A_2)/M_3(A_2)$ in a normal form. Using relation (2), we may commute y anywhere within, so we push them to the right of every term by convention.

We now show that x_1 and x_2 commute in monomials which contain a y. Let $u, w \in \mathcal{B}(A_2)$:

$$ux_2x_1wy \overset{(8)}{=} u(x_1x_2 - y)wy = ux_1x_2wy - uy^2 \overset{(8)}{=} ux_1x_2wy.$$

Thus, any element of $\bar{\mathcal{B}}_2$ may be rewritten in the form of $x_1^i x_2^j y$; the set of all such elements is still a generating family, but now is linearly independent in the quotient. □

These set us up for the proof of Theorem 3.2.

Proof Now, we finally work with $N_2(A)$. To show that $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, recall that A has the additional relations x_1^m, x_2^n, so if $i \geq m$ or $j \geq n$, then $x_1^i x_2^j y$ vanishes. But, for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, no torsion can occur in total degree $i+j < m$ or n. This is because m and n are the degrees of A's relations. □

And Theorem 3.3:

Proof For each bidegree with torsion, we specifically calculate the terms causing torsion. For example, to find those with bidegree $(m,1)$, we first note that the term must be of the form $x_1^m y$ by Proposition 3.6. The generators of $N_2(A)$ are the images of the generators of $N_2(A_2)$ modulo relations $x_1^m = 0, x_2^n = 0$. To show its torsion, note that:

$$0 = [x_1^m, x_2] = \sum_{s=0}^{m-1} x_1^s [x_1, x_2] x_1^{m-s-1} = \sum_{s=0}^{m-1} x_1^s y x_1^{m-s-1} = \sum_{s=0}^{m-1} x_1^{m-1} y = m x_1^{m-1} y.$$

Similarly, we find that $n x_2^{n-1} y = 0$.

Thus, for all $j < n$, we have that $m x_1^{m-1} x_2^j y = 0$, so there is $\mathbb{Z}_m$ torsion there. Likewise, we find $n x_1^i x_2^{n-1} y = 0$ for $i < m$, so there is $\mathbb{Z}_n$ torsion there.

However, let us consider what happens with $x_1^{m-1} x_2^{n-1} y$. We know that $m x_1^{m-1} x_2^{n-1} y = n x_1^{m-1} x_2^{n-1} y = 0$. Let k be the order of $x_1^{m-1} x_2^{n-1} y$; then, since for all a, b, $am x_1^{m-1} x_2^{n-1} y = bn x_1^{m-1} x_2^{n-1} y = 0$, by Bezout's Lemma we have $k \mid (m,n)$. Thus, the term generates the group $\mathbb{Z}_{(m,n)}$. □

3.2 Structure of N_3

In this section, we prove that the non-zero terms in the bigraded structure of N_3 are given by the following table:

(m,n)	0	1	2	$\cdots \quad \cdots$	$n-1$	n	$n+1$
0	0	$\cdots$	$\cdots$	$\cdots \quad \cdots$	$\cdots$	$\cdots$	$\cdots$
1	$\vdots$	0	$\mathbb{Z}$	$\cdots \quad \cdots$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_{f(n)}$
2	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\cdots \quad \cdots$	$\mathbb{Z}^3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_n$	$\mathbb{Z}_n \oplus \mathbb{Z}_{f(n)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\cdots \quad \cdots$	$\mathbb{Z}^3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_n$	$\mathbb{Z}_n \oplus \mathbb{Z}_{f(n)}$
m	$\vdots$	$\mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_m$	$\cdots \quad \cdots$	$\mathbb{Z}^2 \oplus \mathbb{Z}_m$	$\mathbb{Z}_m \oplus \mathbb{Z}_n$	$\mathbb{Z}_{f(n)} \oplus \mathbb{Z}_{(m,n)}$
$m+1$	$\vdots$	$\mathbb{Z}_{f(m)}$	$\mathbb{Z}_m \oplus \mathbb{Z}_{f(m)}$	$\cdots \quad \cdots$	$\mathbb{Z}_m \oplus \mathbb{Z}_{f(m)}$	$\mathbb{Z}_{f(m)} \oplus \mathbb{Z}_{(m,n)}$	$\mathbb{Z}_{(m,n)}$

Table 4: Bigraded Description of $N_3(A)$

Where $(m,n) = gcd(m,n)$ and

Definition 3.7 *The function* $f : \mathbb{N} \longrightarrow \mathbb{N}$ *is defined by*

$$f(k) := \begin{cases} k & k \text{ odd} \\ \frac{k}{2} & k \text{ even.} \end{cases}$$

In addition to the notation $y := [x_1, x_2]$ from the previous section, we introduce the following two terms: $z_1 := [x_1, y]$, $z_2 := [x_2, y]$.

In this section, we prove the following lemmas about the structure of $N_3(A)$ using tools similar to those from the previous section:

Lemma 3.8 *The free part of* N_3 *is generated as a* $\mathbb{Z}$*-module by the following terms:* $x_1^i x_2^j z_1$, $x_1^i x_2^j z_2$, $x_1^i x_2^j y^2$, *for* $0 \leq i \leq m-1$, $0 \leq j \leq n-1$. *(Free part description)*

Lemma 3.9 *As a* $\mathbb{Z}$*-module, the* $x_1^{m-1} x_2^j y^2$ *and* $x_1^{m-1} x_2^{j+1} z_1$ *(resp* $x_1^i x_2^{n-1} y^2$ *and* $x_1^{i+1} x_2^{m-1} z_2$*) terms are of torsion of order* m *and* $f(m)$ *(resp* n *and* $f(n)$*), except when* $j = n-1$*, for which* $x_1^{m-1} x_2^{n-1} y$ *is of order* (m, n)*. (Torsion part description)*

First, we show that

Proposition 3.10 M_3 *is generated by* $u[x_1, y]v$, $u[x_2, y]v$, *and* $uyvyw$, *for* $u, v, w \in A$.

Proof We first show that M_3 is generated by $u[g, y]v$ and $uyvyw$, for $u, v, w \in A$ and $g \in \{x_1, x_2\}$. By definition, $M_3 = A[A, [A, A]]A$, so any of its elements may be written as $u[a, [b, c]]v$ for some u, a, b, c, v in A_2.

We will concentrate on showing that $[a, [b, c]]$ can be written as a sum of $u[g, [b, c]]u'$. Consider $a = a_1 \cdots a_k$, where each of $a_i \in \{x_1, x_2\}$. We are done if we use the Leibniz Rule:

$$[a, [b, c]] = [a_1 \cdots a_k, [b, c]] = \sum_{i=1}^{k} a_1 \cdots a_{i-1}[a_i, [b, c]]a_{i+1} \cdots a_k.$$

Next, we will show that $[g, [b, c]]$ can be written as a sum of $u[g, [g', d]]v$, with $g, g' \in \{x_1, x_2\}$ and $u, d, v \in A$. We apply the Jacobi identity to get $[g, [b, c]] = [b, [g, c]] - [c, [g, b]]$. Looking at the first term $[b, [g, c]]$, we can apply the Leibniz rule as before to show that it can be written as a sum of $u[g, [g', d]]v$. Since the second term is the same up to order as the first term, we are done.

Finally, we consider terms of the form $[g, [g', d]]$, showing that they can be written as a sum of the desired basis terms of $u[g, y]v$ and $uyvyw$. Let $d = d_1 \cdots d_j$, with each $d_i \in \{x_1, x_2\}$. We apply the Leibniz rule once again, this time to d. Thus,

$$[g, [g', d]] = \sum_{i=1}^{j} [g, u_i[g', d_i]v_i] = \sum_{i=1}^{j} (u_i[g, [g', d_i]]v_i + u_i[g', d_i][g, v_i] + [g, u_i][g', d_i]v_i)$$

for some $u_i, v_i \in A$. The term $u_i[g,[g',d_i]]v_i$ is of the form of $u[g,y]v$ already, as $[g',d_i]=y$ or 0. To show that $[g,v_i]$ (and simultaneously $[g,u_i]$) is in the form of uyw (or is equal to 0) with $u,w \in A$, we apply the Leibniz rule to $v_i = v_{i,1}\cdots v_{i,\ell}$ with $v_{i,j} \in \{x_1,x_2\}$.

$$u_i y[g,v_i] = u_i y \sum_{j=1}^{\ell} v_{i,1}\cdots v_{i,j-1}[g,v_{i,j}]v_{i,j+1}\cdots v_{i,\ell} = \sum_{\substack{j=1 \\ v_{i,j}\neq g}}^{\ell} u_i y w_j y v_j$$

with $w_j, v_j \in A$, which completes the proof. □

Then, we recall a theorem by [EKM09] that is the M_4 analogue of Theorem 3.4.

Theorem 3.11 *A presentation of A_2/M_4 is given by the generators x_1, x_2 the following relations:*

$$[x_1,z_2]=[x_1,z_1]=[x_2,z_1]=[x_2,z_2]=0, \quad yz_1=yz_2=y^3=0, \quad z_1^2=z_1z_2=z_2^2=0.$$

Armed with Lemma 3.10 and Theorem 3.11, we can find a basis of $M_3/M_4=N_3$.

Proof Our aim is to rewrite the terms E and F in a normal form using rewriting rules from M_4's basis, where $E := u[x,y]v$ and $F := uyvyw$ for $u,v,w \in A$ and $x \in \{x_1,x_2\}$.

Using methods similar to those in Theorem 3.2, we find that x_1 and x_2 in monomials like E commute, and so $E = x_1^i x_2^j z_1$ or $x_1^i x_2^j z_2$.

Next, we will rewrite F. We first note that if there is more than one y present in any monomial, then all the $y's$ commute with everything within that term, so F may be rewritten as $uvwy^2$. Like previously, if $F = uvwy^2 \neq x_1^i x_2^j y^2$, then we can also commute each x_1 and x_2 in these terms. □

We will use a fact in the proof of Lemma 3.9:

Proposition 3.12 *Let $i \geq 1$. Then, $yx_1^i = x_1^i y - ix_1^{i-1}z_1$ and $yx_2^i = x_2^i y - ix_2^{i-1}z_2$.*

Proof To find the torsion, we identify all relations for bidegree $(m+1,2)$, and work our way up from there.

We start off with some algebraic manipulation to get that

$$0 = mx_1^{m-1}y + \frac{m(m-1)}{2}x_1^{m-2}z_1. \tag{10}$$

Let E be the right hand of equation (1).

First, we would like to prove $m(m-1)x_1^{m-1}y^2=0$. Starting with $0=[E,x_2]$, we get that

$$0 = m(m-1)x_1^{m-2}y^2.$$

Multiplying on the right by x_1 yields our first relation.

Second, we would like to show that $mx_1^{m-1}y^2 = 0$. Right multiplication on equation (1) by y yields the relation.

Third, we will show that $mx_1^{m-1}x_2z_1 = 0$. With right multiplication by x_2 on the equation $mx_1^{m-1}z_1 = 0$, commutativity of z_1 with everything yields our desired relation.

Finally, we will show that $\frac{m(m-1)}{2}x_1^{m-1}x_2z_1 = 0$. If we right multiply equation (1) by x_1, we get the following:

$$0 = \frac{m(m-1)}{2}x_1^{m-1}z_1.$$

To finish, we right multiply by x_2.

Notice that these monomials end with either y^2 or z_1, which both commute with x_2; thus, if we right multiply by x_2^j for $0 \leq j \leq n-3$ we get our desired results.

So, we have found two terms that generate groups: $x_1^{m-1}x_2^jy^2$, and $x_1^{m-1}x_2^{j+1}z_1$, both with bidegrees $(m+1, j+2)$. The first term generates a torsion part of order $gcd(m, m(m-1)) = m$, while the second generates a torsion part of order $(m, \frac{m(m-1)}{2})$. Thus, the torsion in the bidegree is $\mathbb{Z}_m \oplus \mathbb{Z}_{(m, \frac{m(m-1)}{2})}$. For odd m, this is equal to $\mathbb{Z}_m \oplus \mathbb{Z}_m$, and $\mathbb{Z}_m \oplus \mathbb{Z}_{\frac{m}{2}}$ for even m, so our prior Definition 3.7 of $f(k)$ holds.

Since x_1 is symmetric with respect to x_2, we obtain the same results for the bidegrees $(i+2, n+1)$ for $0 \leq i \leq m-3$; i.e., the torsion is $\mathbb{Z}_n \oplus \mathbb{Z}_{(n, \frac{n(n-1)}{2})}$. □

4 Conclusion

In this project, we programmed *MAGMA* [BCP97] to compute data about the dimensions and ranks of these lower central series ideal quotients for various algebras. Using this data, we formulated and proved conjectures concerning these quotients $N_i(A)$. Just like how knowing sufficiently the divisors of an integer, we have proven a partial result about the substructure of an infinite and complex family of algebras in **Section 2**. And, in **Section 3** we characterized the bigraded structure of $N_2(A)$ and $N_3(A)$ for algebras with two generators over $\mathbb{Z}$. In addition, we have gathered over 250 bigraded tables and nearly 100 totally graded tables, which can aid further exploration of these algebraic structures and applications.

5 Future Work

There is still much that may be explored in this topic. Over $\mathbb{Z}$, we could describe the bigraded structure of $N_4(A_2)$ by utilizing a recently published paper by [dCK13] that outlined a basis of A/M_5. In addition, we could try to produce code and explore individual grading of more than just 2 variables. In general, we would like to be able to describe $N_i(A)$, where $A \cong \mathbb{Z}\langle x_1, \cdots, x_k\rangle/(x_1^{m_1}, \cdots, x_k^{m_k})$. Potential further work is to perform individual grading on the $B_i(A)$ defined in the introduction.

There are several conjectures we were not able to prove by the time of submission:

1. By comparing Tables 5 and 6 in Section 5, where the only difference is that they were cal-

culated over $\mathbb{Z}$ versus $\mathbb{F}_p$, we seem to be able to recover Table 6's data from the others'. We mod out the free parts by $p\mathbb{Z}$, leaving a copy of $\mathbb{Z}_p$. If there was torsion $\mathbb{Z}_m$ in the Table over $\mathbb{Z}$, then there would be a copy of $\mathbb{Z}_{(p,m)}$ over $\mathbb{F}_p$. All our tables support this.

2. We have a conjecture about generators for the free part of $N_4(A)$, that they are $x_1^i, x_2^j v$, where $v \in A$ has one of the following bidegrees: $(1,3),(2,2),(3,1),(2,3),(3,2),(3,3)$.

3. Though we have a complete description of $N_2(A)$, with $A = A_2/(x_1^m, x_2^n)$, we have found proofs of the same fashion that allow us to conjecture the number of generating terms there are in the basis $N_2(A_k)$:

$$\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i}.$$

By using a complex filtration, a closed form of this expression can be found:

$$Re((1+i)^k).$$

6 Methods and Tables

In order to calculate free and torsion subgroups of $N_i(A)$, we use preexisting code that calculated $N_i(A)$ over $\mathbb{Q}$ for one relation. This required us to modify the code to allow for multiple relations, calculations over $\mathbb{Z}$ and $\mathbb{F}_p$, and most importantly: to calculate bigraded data (that is, degrees of individual generators in A_2). The code computes each N_i after computing the corresponding L_i and M_i, then moves on to the subsequent N_{i+1}.

However, computers can only handle linear systems of size a few thousands. The dimension of A_2 in degree n is 2^n, so to compute data with degree n about $N_i(A_2)$, we need to solve linear systems of size 2^n. Realistically, our last calculable value $n = 12$, as $2^{12} = 4096$ bigraded entries. So, we work with many data tables of $N_i(A)$ for small $i < 12$, automating the collection process by writing *Java* and *BASH* scripts to convert data to *LaTeX* tables. Below, we present a small selection of our data collection, which contains over 350 tables.

The rows represent m and the columns represent n, where our relations are $x_1^m = x_2^n = 0$. A cell with a small $_0$ represents no free component there, while a blank cell indicates that the computer was not able to calculate data there. Each non-trivial cell is of the form $R,(T)$, where R represents the rank of the free component ($\mathbb{Z}^R$), while (T), in parentheses, represents the torsion structure. For example, in $(2,5)$ of Table 6, $T = (2 \cdot 4)$ represents $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. Absence of parentheses indicates an absence of torsion.

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	0 (7)	0	0	0	
2	0	1	1	1	1	1	1	0 (7)	0	0		
3	0	0 (3)	0 (3)	0 (3)	0 (3)	0 (3)	0 (3)	0	0			
4	0	0	0	0	0	0	0	0				
5	0	0	0	0	0	0	0					
6	0	0	0	0	0	0						
7	0	0	0	0	0							
8	0	0	0	0								
9	0	0	0									
10	0	0										
11	0											

Table 5: N_2 : $\mathbb{Z}\langle x_1, x_2\rangle/(x_1^3, x_2^7)$, Time: 906.16 sec, Memory: 780.78MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	0 $(2\cdot 3)$	0	0	0	0	
2	0	1	1	1	1	1	0 $(2\cdot 3)$	0	0	0		
3	0	1	1	1	1	1	0 $(2\cdot 3)$	0	0			
4	0	0 (4)	0 (4)	0 (4)	0 (4)	0 (4)	0 (2)	0				
5	0	0	0	0	0	0	0					
6	0	0	0	0	0	0						
7	0	0	0	0	0							
8	0	0	0	0								
9	0	0	0									
10	0	0										
11	0											

Table 6: N_2 : $\mathbb{Z}\langle x_1, x_2\rangle/(x_1^4, x_2^6)$, Time: 911.82 sec, Memory: 769.03MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	0 (2)	0	0	0	0	0	
2	0	1	3	3	2 (4)	0 $(2\cdot 4)$	0	0	0	0		
3	0	1	2 (3)	2 (3)	1 $(3\cdot 4)$	0 (4)	0	0	0			
4	0	0 (3)	0 (3^2)	0 (3^2)	0 (3)	0	0	0				
5	0	0	0	0	0	0	0					
6	0	0	0	0	0	0						
7	0	0	0	0	0							
8	0	0	0	0								
9	0	0	0									
10	0	0										
11	0											

Table 7: N_3 : $\mathbb{Z}\langle x, y\rangle/(x^3, y^4)$, Time: 912.87 sec, Memory: 789.53MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	0	0	0	0	0	0	
2	0	1	3	3	2	0	0	0	0	0		
3	0	1	3	3	2	0	0	0	0			
4	0	1	2	2	1	0	0	0				
5	0	0	0	0	0	0	0					
6	0	0	0	0	0	0						
7	0	0	0	0	0							
8	0	0	0	0								
9	0	0	0									
10	0	0										
11	0											

Table 8: $N_3:\ \mathbb{Z}_3\langle x,y\rangle/(x^3,y^4)$, Time: 97654.05 sec, Memory: 2783.16MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	0 (7)	0	0	
2	0	1	3	3	3	3	3	2 (7)	0 (7^2)	0		
3	0	1	3	3	3	3	3	2 (7)	0 (7^2)			
4	0	1	3	3	3	3	3	2 (7)				
5	0	1	3	3	3	3	3					
6	0	1	3	3	3	3						
7	0	1	2 (7)	2 (7)	2 (7)							
8	0	0 (7)	0 (7^2)	0 (7^2)								
9	0	0	0									
10	0	0										
11	0											

Table 9: $N_3:\ \mathbb{Z}\langle x,y\rangle/(x^7,y^7)$, Time: 879.42 sec, Memory: 754.81MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
$(0,n)$	0	0	0	0	0	0	0	0	0	0	0	0
$(1,n)$	0	0	1	1	1	1	1	1	1	0	0	
$(2,n)$	0	1	3	3	3	3	3	3	2	0		
$(3,n)$	0	1	3	3	3	3	3	3	2			
$(4,n)$	0	1	3	3	3	3	3	3				
$(5,n)$	0	1	3	3	3	3	3					
$(6,n)$	0	1	3	3	3	3						
$(7,n)$	0	1	3	3	3							
$(8,n)$	0	1	2	2								
$(9,n)$	0	0	0									
$(10,n)$	0	0										
$(11,n)$	0											

Table 10: $N_3:\ \mathbb{Z}_7\langle x,y\rangle/(x^7,y^7)$, Time: 15927.51 sec, Memory: 4333.34MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	1	0 (4)	0	
2	0	1	3	3	3	3	3	3	2 (8)	0 $(4\cdot 8)$		
3	0	1	3	3	3	3	3	3	2 (8)			
4	0	1	3	3	3	3	3	3				
5	0	1	3	3	3	3	3					
6	0	1	3	3	3	3						
7	0	1	3	3	3							
8	0	1	2 (8)	2 (8)								
9	0	0 (4)	0 $(4\cdot 8)$									
10	0	0										
11	0											

Table 11: $N_3:\ \mathbb{Z}\langle x,y\rangle/(x^8,y^8)$, Time: 876.37 sec, Memory: 754.19MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	1	1	0 (9)	
2	0	1	3	3	3	3	3	3	3	2 (9)		
3	0	1	3	3	3	3	3	3	3			
4	0	1	3	3	3	3	3	3				
5	0	1	3	3	3	3	3					
6	0	1	3	3	3	3						
7	0	1	3	3	3							
8	0	1	2 (8)	2 (8)								
9	0	0 (4)	0 $(4 \cdot 8)$									
10	0	0										
11	0											

Table 12: N_3 : $\mathbb{Z}\langle x,y\rangle/(x^8,y^9)$, Time: 877.02 sec, Memory: 753.88MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	0 $(2 \cdot 5)$	0 (2)	0	0	0	0	
2	0	0	1	3	3	0 $(2^2 \cdot 3 \cdot 4 \cdot 5)$	0 (2^2)	0	0	0		
3	0	1	3	6	5(4)	0 $(2^2 \cdot 4^3 \cdot 3^2)$	0 $(2^2 \cdot 4)$	0	0			
4	0	1	3	6	5(4)	0 $(2^2 \cdot 4^3 \cdot 3^2)$	0 $(2^2 \cdot 4)$	0				
5	0	1	3	5(5)	4$(4 \cdot 5)$	0 $(2 \cdot 4^3 \cdot 3^2)$	0 $(2 \cdot 4)$					
6	0	0 $(2 \cdot 5)$	0 $(5^3 \cdot 2 \cdot 4)$	0 $(5^5 \cdot 2 \cdot 4^2)$	0 $(5^4 \cdot 2 \cdot 4^2)$	0 (4^2)						
7	0	0 (5)	0 (5^2)	0 (5^3)	0 (5^2)							
8	0	0	0	0								
9	0	0	0									
10	0	0										
11	0											

Table 13: N_4 : $\mathbb{Z}\langle x,y\rangle/(x^5,y^4)$, Time: 524.7 sec, Memory: 772.22MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	1	1	1	1	1	
2	0	0	1	3	3	3	3	3	3	3		
3	0	1	3	6	6	6	6	6	6			
4	0	1	3	6	6	6	6	6				
5	0	1	3	6	6	6	6					
6	0	1	3	6	6	6						
7	0	1	3	6	6							
8	0	1	3	6								
9	0	1	3									
10	0	1										
11	0											

Table 14: N_4 : $\mathbb{Z}\langle x,y\rangle/(x^{101})$, Time: 878.2 sec, Memory: 753.88MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0 (2)	0	0	0	0	0	0	
2	0	0	1	3	0 $(2^2 \cdot 3^2)$	0 (3)	0	0	0	0		
3	0	1	3	4 (3)	0 $(3^4 \cdot 2^2)$	0 (3^2)	0	0	0			
4	0	0 (2)	0 $(2^2 \cdot 3^2)$	0 $(3^4 \cdot 2^2)$	0 $(3^4 \cdot 2)$	0 (3^2)	0	0				
5	0	0	0 (3)	0 (3^2)	0 (3^2)	0 (3)	0					
6	0	0	0	0	0	0						
7	0	0	0	0	0							
8	0	0	0	0								
9	0	0	0									
10	0	0										
11	0											

Table 15: N_4 : $\mathbb{Z}\langle x,y\rangle/(x^3, y^3)$, Time: 1730.05 sec, Memory: 1582.34MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	o	o	o	o	o	o	o	o	o	o	o	o
1	o	o	o	1	1	o $(2 \cdot 5)$	o (2)	o	o	o	o	
2	o	o	1	3	3	o $(2^2 \cdot 3 \cdot 4 \cdot 5)$	o (2^2)	o	o	o		
3	o	1	3	5 (3)	4 $(3 \cdot 4)$	o $(2 \cdot 4^3 \cdot 3^2)$	o $(2 \cdot 4)$	o	o			
4	o	o (2)	o $(2^2 \cdot 3^2)$	o $(3^4 \cdot 2^3)$	o $(3^4 \cdot 2^3)$	o $(2^2 \cdot 3^2)$	o (2)	o				
5	o	o	o (3)	o (3^2)	o (3^2)	o (3)	o					
6	o	o	o	o	o	o						
7	o	o	o	o	o							
8	o	o	o	o								
9	o	o	o									
10	o	o										
11	o											

Table 16: $N_4 : \mathbb{Z}\langle x, y\rangle/(x^3, y^4)$, Time: 912.87 sec, Memory: 789.53MB

(m,n)	0	1	2	3	4	5	6	7	8	9	10	11
0	o	o	o	o	o	o	o	o	o	o	o	o
1	o	o	o	o	1	1	o (5)	o	o	o	o	
2	o	o	o	2	5	4 $(2 \cdot 5)$	o (2^4)	o (2^2)	o	o		
3	o	o	2	6	9 (3)	5 $(2^2 \cdot 3^2 \cdot 4^2)$	o $(2^6 \cdot 3 \cdot 4 \cdot 5)$	o (2^3)	o			
4	o	1	4 (2)	6 $(2^2 \cdot 3^3)$	6 $(2^4 \cdot 3^6)$	2 $(2^5 \cdot 3^2 \cdot 4^2)$	o $(2^4 \cdot 3 \cdot 4)$	o (2^2)				
5	o	o	o $(3^2 \cdot 2)$	o $(3^5 \cdot 2^2)$	o $(3^7 \cdot 2^3)$	o $(3^5 \cdot 2^3)$	o $(2^2 \cdot 3^2)$					
6	o	o	o (3)	o (3^2)	o (3^3)	o (3^2)						
7	o	o	o	o	o							
8	o	o	o	o								
9	o	o	o									
10	o	o										
11	o											

Table 17: $N_5 : \mathbb{Z}\langle x, y\rangle/(x^3, y^4)$, Time: 912.87 sec, Memory: 789.53MB

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