Enumeration of Subclasses of ($\underline{2}$+$\underline{2}$)-free Partially Ordered Sets

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Abstract

We investigate avoidance in ($\underline{2}$+$\underline{2}$)-free partially ordered sets, posets that do not contain any induced subposet isomorphic to the union of two disjoint chains of length two. In particular, we are interested in enumerating the number of partially ordered sets of size n avoiding both $\underline{2}$+$\underline{2}$ and some other poset α. For any α of size 3, the results are already well-known. However, out of the 15 such α of size 4, only 2 were previously known. Through the course of this paper, we explicitly enumerate 7 other such α of size 4. Also, we consider the avoidance of three posets simultaneously, $\underline{2}$+$\underline{2}$ along with some pair (α, β); it turns out that this enumeration is often clean, and has sometimes surprising results. Furthermore, we turn to the question of Wilf-equivalences in ($\underline{2}$+$\underline{2}$)-free posets. We show such an equivalence between the Y-shaped and chain posets of size 4 via a direct bijection, and in fact, we extend this to show a Wilf-equivalence between the general chain poset and a general Y-shaped poset of the same size. In this paper, while our focus is on enumeration, we also seek to develop an understanding of the structures of the posets in the subclasses we are studying.

1 Introduction

Partially ordered sets, or posets, are a collection of objects that appear in many areas of mathematics. Essentially, a poset consists of a set of elements along with a binary relation such that, for some pairs of elements in the set, one of the elements dominates the other. The notion of a partial order indicates the idea that not every pair of elements has to be related. Therefore, for example, a poset could be used to model an ecological food web. The study of posets has a history extending back to at least the 1930s (see [9]). Since then, the study has blossomed into a rich field encompassing many areas of mathematics including topology [10], complexity theory [7], and representation theory [5], among many others (see [17] for an exposition).

One of the fundamental combinatorial questions is to determine the number of posets of a certain size that avoid another forbidden poset. Avoidance in this context is a fairly rich study. For example, Stanley [13] enumerated N-free (series-parallel) posets. Furthermore, Lewis and Zhang [8] enumerated graded ($\underline{3}$+$\underline{1}$)-free posets, while Guay-Paquet, Morales, and Rowland [6] recently solved the general case for ($\underline{3}$+$\underline{1}$)-free posets.

Here, we investigate ($\underline{2}$+$\underline{2}$)-avoiding or ($\underline{2}$+$\underline{2}$)-free posets, posets that do not contain an induced subposet isomorphic to the union of two disjoint chains of length two. These posets are related to a number of other combinatorial structures. Particularly striking is the result that ($\underline{2}$+$\underline{2}$)-free posets are precisely *interval orders* (shown in [4]), which have applications in mathematics, computer science, and engineering, particularly in the task distributions of complex manufacturing processes (see [16]). They have also been shown to be in bijection with *ascent sequences* and are related to *pattern avoiding permutations* [2]. These applications motivate the study of ($\underline{2}$+$\underline{2}$)-free partially ordered sets.

In this paper, we study avoidance within ($\underline{2}$+$\underline{2}$)-free posets; in particular, we seek to enumerate the number of posets of size n avoiding both $\underline{2}$+$\underline{2}$ and some other poset α. For any size 3 poset α, the result has already been enumerated and is well known. However, this is not the case for size 4 posets α. When $\alpha = \underline{3} + \underline{1}$ (the disjoint union of a chain of length 3 and a single element), the resulting sequence is known to be the Catalan sequence (see [11], [14]). Similarly, when $\alpha = N$ (the size 4 poset in the shape of an N), the sequence is Catalan as well (see [15], [14]). Yet, these 2 are the only size 4 posets α that have been enumerated out of the 15 such α (discluding $\underline{2}$+$\underline{2}$ of course). Through the course of this paper, we enumerate 7 other such α (see Figure 1). In particular, in Section 3, we enumerate $\alpha = \underline{4}$ and $\alpha = Y$. In Section 4, we consider $\alpha = \bowtie$ and $\alpha = \Diamond$, and in Section 5, we look at $\alpha = \surd$. Additionally, in a few instances in this paper, we enumerate the number of posets avoiding $\underline{2}$+$\underline{2}$ and two other posets α and β, with α, β chosen from the posets below. It turns out that in many cases, these enumerations have surprisingly clean results.

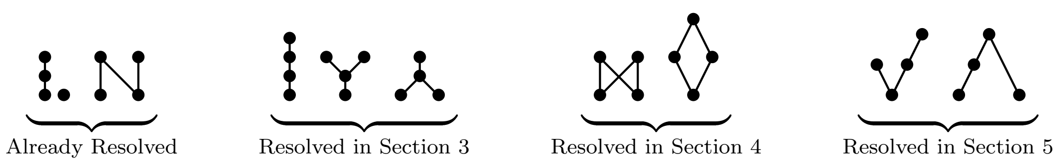

Figure 1: The size 4 posets α that have now been resolved

We also address the question of Wilf-equivalences in ($\underline{2}$+$\underline{2}$)-free posets. Two posets α and β are said to be *Wilf-equivalent in* ($\underline{2} + \underline{2}$)*-free posets* if for all n, the number of posets of size n that avoid $(\underline{2} + \underline{2}, \alpha)$ is equivalent to those that avoid $(\underline{2} + \underline{2}, \beta)$. Since we only look at ($\underline{2}$+$\underline{2}$)-free posets, all Wilf-equivalences in the paper are assumed to be in ($\underline{2}$+$\underline{2}$)-free posets. The size 3 posets, $\vee, \wedge, \underline{2} + \underline{1}, \underline{3}$, are known to be Wilf-equivalent. However for posets of size greater than 3, the only known pair is $\underline{3} + \underline{1}$ and N. In Section 3, we shall show a Wilf-equivalence via a direct bijection. In Section 6, we pose conjectures and a number of open problems.

Note 1.1. *For the sake of space, some proofs are omitted. We give the ideas behind proofs or partial proofs whenever feasible.*

2 Definitions and Notation

A *partially ordered set*, or *poset*, is a set P along with a binary relation $\leq$ over P that satisfies three properties:

- *Reflexivity*: $i \leq i$ for all $i \in P$

- *Antisymmetry*: If $i \leq j$ and $j \leq i$, then $i = j$ for $i, j \in P$
- *Transitivity*: If $i \leq j$ and $j \leq k$, then $i \leq k$ for $i, j, k \in P$.

Note 2.1. *Since the case of comparing an element to itself is fairly trivial, we shall now assume that we are always comparing two distinct elements. Define $x < y$ if $x \leq y$ but $x \neq y$.*

For $i, j \in P$, if $i < j$ or $j < i$ then i and j are said to be *comparable*. Otherwise, they are *incomparable*. We say that y *covers* x if $x < y$ and there exists no z such that $x < z < y$. Notationally, we write $x \lessdot y$. Using this, we define a *chain* of length a as some a elements such that $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_a$.

A *Hasse Diagram* is a way to pictorially or geometrically represent the cover relations in a partially ordered set. For example the poset $(\mathcal{P}(\{x, y, z\}), \subseteq)$ is modeled in Figure 2.

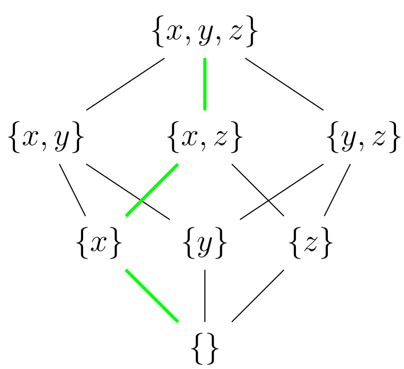

Figure 2: Left is the Hasse Diagram for the poset $(\mathcal{P}(\{x, y, z\}), \subseteq)$, where $\mathcal{P}$ is the powerset function, or set of all subsets of $\{x, y, z\}$. Note that $\{y\}$ and $\{x, z\}$ are incomparable and $\{\}, \{x\}, \{x, z\}$, and $\{x, y, z\}$ form a chain of length 4 (in green).

Note 2.2. *From here onwards, for ease of notation, we will always use P to refer to the entire poset. So, P includes both the set and the order relations among the elements in the set.*

A poset P *contains* a poset S if there exists some subposet W of P that is isomorphic to S. If P does not contain S, then P is said to *avoid* S. Define $\underline{k}$ to be the poset chain of length k. So, a poset is said to be $(\underline{2}+\underline{2})$-free if it does not contain an induced subposet that is isomorphic to the poset $\underline{2}+\underline{2}$, the union of two disjoint 2-element chains (see Figure 3).

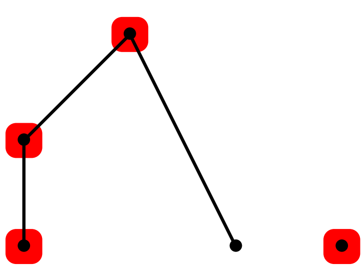

Figure 3: The poset to the left contains no subposet isomorphic to $\underline{2}+\underline{2}$, and so, is $(\underline{2}+\underline{2})$-free. However, the poset contains a subposet isomorphic to $\underline{3}+\underline{1}$ (in red), and so contains $\underline{3}+\underline{1}$.

Define $P_n(\alpha)$ as the set of posets of size n that avoid the poset α. Now, define the *dual* of a poset P, P' to refer to the poset obtained by swapping all order-relations (i.e. if $x < y$ in P, then $y < x$ in P'). In other words, geometrically, the Hasse Diagram of P' is obtained by the inversion or flipping of the Hasse Diagram of the poset P. It is clear that for all n, $|P_n(\alpha)| = |P_n(\alpha')|$, for if a poset P avoids α then its dual P' avoids α'. Similarly, since $\underline{2}+\underline{2}$ is its own dual, $|P_n(\underline{2}+\underline{2}, \alpha)| = |P_n(\underline{2}+\underline{2}, \alpha')|$ (where $P_n(\underline{2}+\underline{2}, \alpha)$ refers to the posets that avoid both $\underline{2}+\underline{2}$ and α).

Define the *down-set* of an element $x \in P$ as $D(x) = \{z \in P : z < x\}$. Similarly, define the *up-set* of x as $U(x) = \{z \in P : x < z\}$. Now, we shall introduce an important notion regarding ($\underline{2}+\underline{2}$)-free posets. It is well known that a poset is ($\underline{2}+\underline{2}$)-free if and only if its down-sets may be linearly ordered by inclusion, i.e. for any $i, j \in P$, $D(i) \subseteq D(j)$ or $D(j) \subseteq D(i)$ (see, for example, [1]). Because of the importance of this result, we now show the proof.

Lemma 2.1. *A poset is ($\underline{2}+\underline{2}$)-free if and only if its down-sets are linearly ordered by inclusion.*

Proof. Assume, for the sake of contradiction, that there is a ($\underline{2}+\underline{2}$)-free poset P such that the down-sets cannot be linearly ordered by inclusion. Pick elements $x, y \in P$ such that there exist $a \in D(x) \setminus D(y)$ and $b \in D(y) \setminus D(x)$. Note that x and y must be incomparable; otherwise if $x < y$, $D(x) \subset D(y)$, and vice-versa in the other case. However, then a, b, x, y form a subposet isomorphic to $\underline{2}+\underline{2}$, a contradiction since P is defined to avoid $\underline{2}+\underline{2}$. Now, we show the other direction. Again, for the sake of contradiction, assume there exists some elements a, b, x, y in P forming a subposet isomorphic to $\underline{2}+\underline{2}$ such that $a < x$ and $b < y$. However, we then have a contradiction as both $D(x) \setminus D(y)$ and $D(y) \setminus D(x)$ are non-empty. ∎

3 $\underline{2}+\underline{2}$,Y(k) and $\underline{2}+\underline{2}$, $\underline{k}$

While our main goal is enumeration, in this section we shall first show a bijection that will help us understand the structures of ($\underline{2}+\underline{2}, Y(k)$)-free and ($\underline{2}+\underline{2}, \underline{k}$)-free posets. We will then use this understanding of these structures to aid us in enumeration of $|P_n(\underline{2}+\underline{2}, Y)|$ and $|P_n(\underline{2}+\underline{2}, \underline{4})|$. Lastly, we consider the simultaneous avoidance of 3 different posets: $\underline{2}+\underline{2}$, Y, and the chain poset $\underline{k}$.

Now, we define a function $Y(n)$, $n \geq 3$ as follows.

- $Y(3) = \vee$ (the poset $\{a, b, c\}$ where $a < b$ and $a < c$).
- $Y(k)$ is the result of adding a minimal element to $Y(k-1)$.

So, the procedure is equivalent to extending the tail on a Y-shaped poset (see Figure 4).

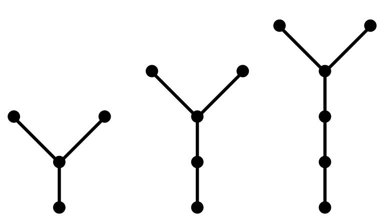

Figure 4: $Y, Y(5)$, and $Y(6)$. Note we use Y to refer to $Y(4)$.

3.1 Bijection between $P_n(\underline{\mathbf{2}}+\underline{\mathbf{2}}, Y(k))$ and $P_n(\underline{\mathbf{2}}+\underline{\mathbf{2}}, \underline{k})$

We shall show that there exists a bijection between $P_n(\underline{2}+\underline{2}, Y(k))$ and $P_n(\underline{2}+\underline{2}, \underline{k})$.

Now, we fix k. Define the set $A(P)$ for any poset P as follows: $A(P) = \{x \in P : D(x) \text{ contains } \underline{k-2}\}$. We now investigate the structure of the set $A(P)$ of a poset P in $P_n(\underline{2}+\underline{2}, Y(k))$ and $P_n(\underline{2}+\underline{2}, \underline{k})$ separately.

Lemma 3.1. *For any poset P in $P_n(\underline{2}+\underline{2}, Y(k))$, $A(P)$ is a chain in P.*

Proof. Assume, for the sake of contradiction, $A(P)$ is not a chain. Rather, assume there exist $x_1, x_2 \in A(P)$ such that x_1 and x_2 are incomparable. Then, by the definition of $A(P)$, the down-sets of x_1 and x_2 must each contain some chain of length at least $k-2$, namely C_1 and C_2 respectively. However, since P avoids $\underline{2}+\underline{2}$, by Lemma 2.1, $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$ (or both). Without loss of generality, let $C_1 \subseteq C_2$. Then x_1, x_2, and C_1 form a subposet isomorphic to $Y(k)$, a contradiction. Thus, x_1 and x_2 must be comparable and so, it follows that $A(P)$ must be a chain. ■

Lemma 3.2. *For any poset P in $P_n(\underline{2}+\underline{2}, \underline{k})$, $A(P)$ is a series of incomparable elements in P.*

Proof. Assume, for the sake of contradiction, there exist two elements $x_1, x_2 \in A(P)$ such that $x_1 < x_2$ (i.e. x_1 and x_2 are comparable). By the definition of $A(P)$, the down-set of x_1 must contain some chain of length at least $k-2$, namely C_1. However, then x_2, x_1, and C_1 form a chain of length k, a contradiction since P is defined to avoid $\underline{k}$. Thus, by contradiction, x_1 and x_2 must be incomparable and so, it follows that $A(P)$ must be a series of incomparable elements. ■

We have now determined the structure of $A(P)$ in both $P_n(\underline{2}+\underline{2}, Y(k))$ and $P_n(\underline{2}+\underline{2}, \underline{k})$. Define $A'(P) = P \setminus A(P)$. The following Lemma captures the rationale behind our definition of $A(P)$ and also highlights the relative unimportance of $A'(P)$, thus its invariance in the bijection we shall soon show.

Lemma 3.3. *For any poset P, $A'(P)$ avoids $\underline{k}$ and $Y(k)$.*

Proof. Note that both the posets $\underline{k}$ and $Y(k)$ contain chains of length $k-1$. Assume, for the sake of contradiction, $A'(P)$ contains one of $\underline{k}$ or $Y(k)$, and consequently contains some chain of size $k-1$, $\{c_1 \lessdot c_2 \lessdot \cdots \lessdot c_{k-1}\}$. So, the down-set of c_{k-1} contains a chain of size $k-2$. However, by the definition of $A(P)$, then $c_{k-1} \in A(P)$. Consequently, this is a contradiction, since $c_{k-1} \in A'(P)$, and so, $A'(P)$ must avoid both $\underline{k}$ and $Y(k)$. ■

Now, we are ready to show the bijection.

Theorem 3.4. *There exists a bijection between $P_n(\underline{2}+\underline{2}, Y(k))$ and $P_n(\underline{2}+\underline{2}, \underline{k})$.*

Proof. Let P be a poset in $P_n(\underline{2}+\underline{2}, Y(k))$. So, from Lemma 3.1, $A(P) = \{a_1 \lessdot a_2 \lessdot \cdots \lessdot a_\ell\}$, a chain of some length ℓ. Then, $D(a_1) \subset D(a_2) \subset \cdots \subset D(a_\ell)$. Now, we show an injection from $P_n(\underline{2}+\underline{2}, Y(k))$ onto $P_n(\underline{2}+\underline{2}, \underline{k})$. Keeping $A'(P)$ (and its order relations with $A(P)$) as it is, we remove all the order relations within $A(P)$ itself, turning P into a new poset P'. Thus, we turn the chain $A(P)$ in $P_n(\underline{2}+\underline{2}, Y(k))$ into a series of incomparable elements.

Note that since each poset P has its own unique set of order relations between $A(P)$ and $A'(P)$ and since we are maintaining these relations, the map is injective. As explained in the discussion in Lemma 3.3, since $A'(P)$ does not contain a chain of length $k-1$, P' must avoid $\underline{k}$. Thus, $P' \in P_n(\underline{2}+\underline{2}, \underline{k})$.

Now we shall show the other direction. Let P be a poset in $P_n(\underline{2}+\underline{2}, \underline{k})$. From Lemma 3.2, $A(P)$ is a series of some ℓ incomparable elements. Similar to the other direction, here we transform the incomparable elements into a chain. Now, it remains to be shown that there is only one unique way to do this. Since the poset P is ($\underline{2}$+$\underline{2}$)-free, from Lemma 2.1 there must exist an arrangement of indices such that $D(a_1) \subset D(a_2) \subset \cdots \subset D(a_\ell)$. So, there must be only one way to transform these elements into a chain, namely $\{a_1 \lessdot a_2 \lessdot \cdots \lessdot a_\ell\}$. Thus, we have established a bijection between $P_n(\underline{2}+\underline{2}, Y(k))$ and $P_n(\underline{2}+\underline{2}, \underline{k})$, completing the proof of the theorem. (See Figure 5 for a graphical depiction of the bijection). ■

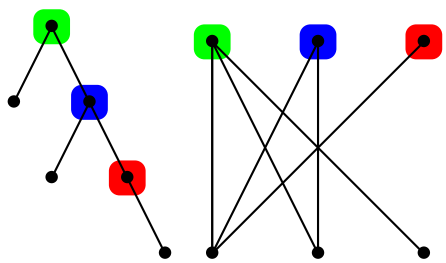

Figure 5: To the left is a representation of the bijection in the case $k = 3$, $P_n(\underline{2}+\underline{2}, \vee)$ and $P_n(\underline{2}+\underline{2}, \underline{3})$. Note that the order relations between $A(P)$ and $A'(P)$ are the same in both. The colored elements correspond to each other; it is easy to see how the chain is easily changed into a series of incomparable elements and vice-versa.

As a corollary to the theorem above, it follows that $|P_n(\underline{2}+\underline{2}, Y(k))| = |P_n(\underline{2}+\underline{2}, \underline{k})|$. Thus, $Y(k)$ and $\underline{k}$ are Wilf-equivalent in ($\underline{2}$+$\underline{2}$)-free posets.

3.2 Enumeration of $|P_n(\underline{2}+\underline{2}, Y)|$

Now, we seek to enumerate $|P_n(\underline{2}+\underline{2}, Y)|$ (and consequently $|P_n(\underline{2}+\underline{2}, \underline{4})|$ by Theorem 3.4), where Y is the poset $Y(4)$.

Theorem 3.5. *The set of $(\underline{2}+\underline{2}, Y)$-free posets of size n containing a $\underline{2}$ is in bijection with the set of triples (a, x, p), $a \in \mathbb{N}$, where $x = (x_0, ..., x_a)$ is a composition of b with $a+1$ parts such that $x_a > 0$, and $p = (\emptyset \neq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_c \subseteq \{0^{x_0}, ..., a^{x_a}\})$ is a flag of the multiset $\{0^{x_0}, ..., a^{x_a}\}$ such that $a \in P_c$, with $a + b + c = n$ and $b, c \geq 1$.*

Proof Idea. Let P be a ($\underline{2}$+$\underline{2}$, Y)-free poset. Decompose P into the following sets of elements.

- $A = \{x \in P : D(x) \text{ contains } \underline{2}\}$
- $B = \{x \in P \setminus A : U(x) \subseteq A\}$
- $C = P \setminus (A \cup B)$

Let a, b, c be the number of elements of A, B, C respectively. From Lemma 3.1, A forms a chain. Now, the first idea is to observe that x corresponds to the order relations between the sets A and B, as we delegate the up-set of each of the elements through the composition of b. The other important idea is to realize that p corresponds to order relations between B and C, as the property that $\emptyset \neq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_c \subseteq \{0^{x_0}, ..., a^{x_a}\}$ follows from Lemma 2.1. (Figure 6 provides a more clear example of the bijection). ■

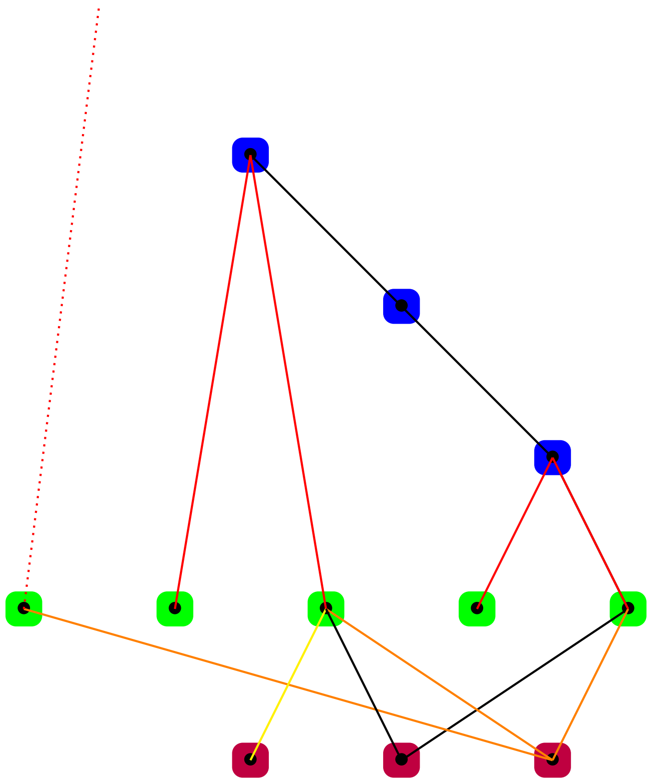

Figure 6: To the left is an example of a $(\underline{2}+\underline{2}, Y)$-free poset. The sets A, B, C are represented by the elements in blue, green, and magenta respectively. Here $x = (1,2,0,2)$, as represented by the red lines showing the order relations between A and B. The remaining lines show the order relationships between the elements in B and C (and A and C), $p = (\emptyset \neq P_1 = \{1\} \subseteq P_2 = \{1,3\} \subseteq P_3 = \{0,1,3\} \subseteq \{0^1, 1^2, 3^2\})$.

Now, we are ready to begin the enumeration.

Corollary 3.6. *The number of $(\underline{2}+\underline{2}, Y)$-free (and so, $(\underline{2}+\underline{2}, \underline{4})$-free) posets of size n is*

$$1 + \sum_{\substack{a+b+c=n \\ b,c\geq 1}} \left[\binom{b+(a+1)(c+1)-1}{b} - \binom{b+a(c+1)}{b} - \binom{b+(a+1)c-1}{b} + \binom{b+ac}{b} \right].$$

Proof. From Theorem 3.2, the number of $(\underline{2}+\underline{2}, Y)$-free posets of size n containing a 2-chain is

$$\sum_{\substack{a+b+c=n \\ b,c\geq 1}} \sum_{\substack{x_0+\ldots+x_a=b \\ x_a>0}} \#\{\emptyset \neq P_1 \subseteq \cdots \subseteq P_c \subseteq \{0^{x_0}, \ldots, a^{x_a}\} : a \in P_c\}.$$

For a fixed composition $x = (x_0, \ldots, x_a)$ of b, the number of such chains p is

$$\left[\binom{x_a+c}{c} - 1 \right] \prod_{i=0}^{a-1} \binom{x_i+c}{c} - \left[\binom{x_a+c-1}{c-1} - 1 \right] \prod_{i=0}^{a-1} \binom{x_i+c-1}{c-1}.$$

By the combinatorial identity,

$$\sum_{x_0+\cdots+x_r=n} \binom{x_0+m}{m} \cdots \binom{x_r+m}{m} = \binom{n+(m+1)(r+1)-1}{n},$$

we have

$$\sum_{\substack{x_0+\ldots+x_a=b \\ x_a>0}} \prod_{i=1}^{a} \binom{x_i+c}{c} = \binom{b+(a+1)(c+1)-1}{b} - \binom{b+a(c+1)-1}{b},$$

$$\begin{aligned} \sum_{\substack{x_0+\ldots+x_a=b \\ x_a>0}} \prod_{i=1}^{a-1} \binom{x_i+c}{c} &= \sum_{x_a=1}^{b} \binom{b-x_a+a(c+1)-1}{b-x_a} \\ &= \binom{b+a(c+1)-1}{b-1}. \end{aligned}$$

So

$$\sum_{\substack{x_0+\ldots+x_a=b \\ x_a>0}} \left[\binom{x_a+c}{c}-1\right] \prod_{i=1}^{a-1} \binom{x_i+c}{c} = \binom{b+(a+1)(c+1)-1}{b} - \binom{b+a(c+1)}{b}.$$

We get the first two terms of the summand in the desired expression in the theorem statement. Replacing c by $c-1$, we get the remaining two terms of the summand. The extra 1 is from the poset consisting of n incomparable elements (since the theorem only considered posets with a 2-chain). ■

Up to this point, we have considered the simultaneous avoidance of only two posets at a time. Now, we look at another intriguing question, investigating the avoidance of three partially ordered sets at once. In particular, we prove the following.

Corollary 3.7. *The number of $(\underline{2}+\underline{2}, Y, \underline{k})$-free posets of size n is*

$$1+\sum_{\substack{a+b+c=n \\ b,c\geq 1, a\leq (k-3)}} \left[\binom{b+(a+1)(c+1)-1}{b} - \binom{b+a(c+1)}{b} - \binom{b+(a+1)c-1}{b} + \binom{b+ac}{b}\right].$$

Proof. Recall from Theorem 3.2 that all a elements form a chain. However, by the definition of A from Theorem 3.2, this chain must also cover some 2-chain. Thus, the size of the largest chain in the poset P is $a+2$. Now, since we are adding the restriction that P avoids $\underline{k}$, then we must have that $a+2<k \Rightarrow a<k-2 \Rightarrow a\leq k-3$. ■

In the case of $k=4$ in the corollary above, substitution yields a nice and somewhat surprising result.

Corollary 3.8. *The number of $(\underline{2}+\underline{2}, Y, \underline{4})$-free posets of size n is F_{2n-1}, where F_k refers to the terms of the Fibonacci sequence with $F_1=F_2=1$.*

4 $\underline{2}+\underline{2}$, $\bowtie$ and $\underline{2}+\underline{2}$, $\Diamond$

Note 4.1. *For ease of writing, we use the symbols $\bowtie$ and $\Diamond$ to refer to their respective size 4 posets in Figure 7 below.*

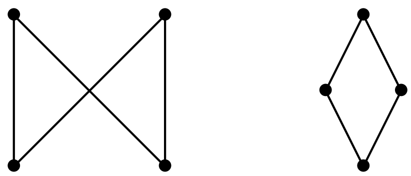

Figure 7: We use $\bowtie$ to denote the bowtie shaped poset to the left and use $\Diamond$ to indicate the diamond shaped poset to its right.

In this section, our focus is on enumeration. First, we seek to enumerate $|P_n(\underline{2}+\underline{2},\bowtie)|$ and $|P_n(\underline{2}+\underline{2},\Diamond)|$. Later, we investigate the simultaneous avoidance of $\underline{2}+\underline{2},\bowtie,$ and $\Diamond$.

4.1 Enumeration of $|P_n(\underline{2}+\underline{2},\bowtie)|$

To enumerate, we first must develop an understanding of the structure of $(\underline{2}+\underline{2},\bowtie)$-free posets. The crux of the structure of these partially ordered sets is captured by the following theorem.

Theorem 4.1. *Let P be a $(\underline{2}+\underline{2},\bowtie)$-free poset. Then there is a chain subposet T of P such that, for every element $x \in P \setminus T$, each up-set $U(x)$ and down-set $D(x)$ of x is contained in T.*

Proof Idea. For the sake of space, we omit the complete proof. However, here we shall present the main ideas behind the proof. Fix a chain subposet T of P. An element $x \in P\setminus T$ is said to be *separate* if either $U(x)$ or $D(x)$ is not contained in T. Define a function $\mathcal{T}(P)$ to return the longest chain T_1 that minimizes the number of separate elements. If T_1 contains no separate elements, then we are done. However, there remains the case when T_1 contains some positive number of separate elements. So, it remains to be shown that this case is impossible and we proceed with proof by contradiction, showing that we may alter the chain T_1 to reduce the number of separate elements. The remainder of the proof is simply extensive casework that results in contradiction within each case (see Figure 8 for an example). ■

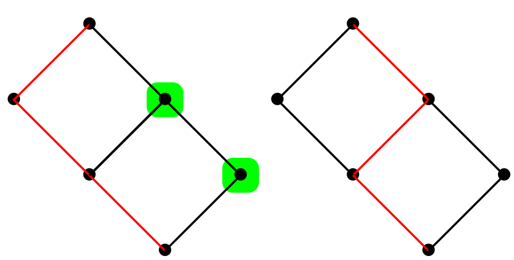

Figure 8: The figure to the far left shows an example of choosing a chain (in red) resulting in exactly 2 separate elements (in green). However, note (in the other figure), that we may choose another chain with 0 separate elements. So, the example on the far left is not valid as it does not minimize the number of separate elements and therefore, is not the result of $\mathcal{T}(P)$.

Thus, we have established the existence of such a chain T. It may also be easily verified that if a poset contains multiple chains T_1, T_2 satisfying the property, then these chains must be structurally/geometrically identical, i.e. $T_1 = T_2$ with respect to the rest of the poset. We are now ready to calculate $|P_n(\underline{2}+\underline{2},\bowtie)|$.

Theorem 4.2. *The number of $(\underline{2}+\underline{2},\bowtie)$-free posets of size n is given by*

$$|P_n(\underline{2}+\underline{2},\bowtie)| = 2^{n-1} + \sum_{m=2}^{n}\sum_{k=1}^{m-1}\left[\binom{n-1+k(m-k)}{n-m} - \binom{n-1+(k-1)(m-k)}{n-m}\right].$$

Proof. Theorem 4.1 shows the existence of a chain T that encompasses $U(x)$ and $D(x)$ for every $x \in P \setminus T$. Let $T = \{x_1 \lessdot \cdots \lessdot x_\ell\}$ and $T \setminus P = \{w_1, \cdots, w_m\}$, such that $\ell + m = n$. As a direct result of the theorem, it follows that any w_1, $w_2 \in P \setminus T$ are incomparable. So, $D(w_1) \cap U(w_2) = \emptyset$, and similarly, it holds that $(D(w_1) \cup D(w_2) \cup \cdots \cup D(w_m)) \cap (U(w_1) \cup U(w_2) \cup \cdots \cup U(w_m)) = \emptyset$. Thus, there must exist some x_k in T such that for all w_i, $D(w_i) \subseteq \{x_1, \cdots, x_k\}$ and $U(w_i) \subseteq \{x_{k+1}, \cdots, x_\ell\}$. Let there be s elements in $P \setminus T$ whose down-set is $\{x_1, \cdots, x_k\}$ (see Figure 9). Now, we proceed by casework based on the value of s.

Case a, $s = 0$. In this case, P must avoid $\vee$ as well. As has already been enumerated $|P_n(\underline{2}+\underline{2},\vee)| = 2^{n-1}$, and so, there are 2^{n-1} such posets in this case.

Case b, $s > 0$. From this, it also follows that $\ell > k > 0$. Now, for each of the s elements, we must construct its up-set. Since T is a chain of maximal length in P, the up-set for each of these elements cannot include x_{k+1}, and so, there are $\ell - k$ choices for the up-set of each of the s elements. So, the number of unique ways to assign the up-sets to the s elements is $\binom{s+\ell-k-1}{\ell-k-1} = \binom{s+\ell-k-1}{s}$. Now, we construct the up-set and down-set for each of the remaining $m-s$ elements. There are a total of $(\ell-k+1)k$ up-set, down-set combination possibilities for each of these elements. So, the number of distinct ways to assign these combinations to the $m-s$ elements is $\binom{m-s+(\ell-k+1)k-1}{m-s}$.

Thus, combining these cases,

$$|P_n(\underline{2}+\underline{2},\bowtie)| = 2^{n-1} + \sum_{\substack{\ell,m,k,s:\\ \ell+m=n,\ell>k>0,m>s>0}} \binom{s+\ell-k-1}{s}\binom{m-s+(\ell-k+1)k-1}{m-s},$$

which by further algebra may be simplified to

$$2^{n-1} + \sum_{m=2}^{n}\sum_{k=1}^{m-1}\left[\binom{n-1+k(m-k)}{n-m} - \binom{n-1+(k-1)(m-k)}{n-m}\right].$$

■

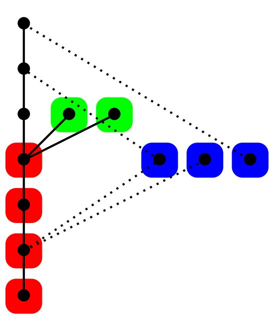

Figure 9: The diagram shows an example of such a poset with size $n = 12$. There are $\ell = 7$ elements in the chain. The element x_k is the 4th element in the chain (i.e. the uppermost element in red). There are 5 elements outside of the chain, indicating $m = 5$ and there are $s = 2$ elements whose down-set includes x_k.

4.2 Enumeration of $|P_n(\underline{2}+\underline{2},\Diamond)|$

As in the previous section, we first seek an understanding of the structure of $(\underline{2}+\underline{2},\Diamond)$-free posets to aid us in enumeration.

Let P be a $(\underline{2}+\underline{2},\Diamond)$-free poset. Define M to be the set of *maximal* (empty up-set) elements. Choose the element m of M with the largest down-set. Note that the $D(m) = P \setminus M$. Let the subposet W of P refer to $\{m\} \cup D(m)$. It is easy to see that W avoids $\vee$, and so W consists of a chain C of some length r, $c_1 \lessdot c_2 \lessdot \cdots \lessdot c_r$, along with some s elements that are each covered by some element in the chain. In particular, we utilize the composition of s, $x = (x_1, ..., x_{r-1})$ where every x_i denotes the number of the s elements that are covered by c_{i+1}. Define each X_i to be the set of the x_i nodes covered by c_{i+1}.

Now, we consider the remaining t elements in $P \setminus W$. Since P avoids $\Diamond$, these t elements cannot be covered by any element in W. The following lemma follows directly from the fact that P is $(\underline{2}+\underline{2})$-free.

Lemma 4.3. *For every element a in $P \setminus W$, all $b \in W \setminus C$ that are covered by a must be contained in the same set X_i.*

Now, using Lemma 4.3 consider the composition of t, $y = (y_1, ..., y_{r-1}, y_r)$ where y_r refers to the number of *free elements* (empty up-set and down-set). Every other y_i refers to the number of elements that cover some element in X_i or, if the element does not cover anything outside of the chain C, that cover c_i. Define each Y_i to be the set of the y_i nodes. We proceed by the following lemma.

Lemma 4.4. *Consider an element e in Y_i that covers some element in X_i. Then e must cover either c_{i-1} or c_i.*

Proof. Let e cover some $a \in X_i$. So, $a \lessdot e$. Since $a \lessdot c_{i+1}$, it follows that c_{i+1} and e must be incomparable. Thus, e must cover either c_{i-1} or c_i, or else $\{c_{i-1}, c_i, a, e\}$ would form a subposet isomorphic to $\underline{2}+\underline{2}$. ∎

From the lemma above, we may see that there exist two distinct types of elements in Y_i. For the ease of writing throughout the remainder of this subsection, we shall refer to them as *type 1* and *type 2* crossings (see Figure 10). Type 1 crossings refer to those elements that cover c_i and may possibly cover elements in y_i. Type 2 crossings, on the other hand, refer to those elements that cover c_{i-1} and therefore must cover some element in y_i.

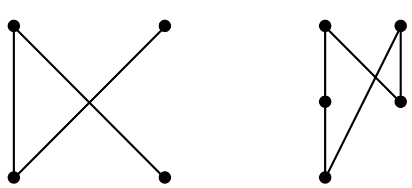

Figure 10: An example of type 1 (left) and type 2 crossings (right).

From our recognition of these two types of crossings, we are ready to begin the enumeration.

Theorem 4.5. *The number of $(\underline{2}+\underline{2},\Diamond)$-free posets of size n is*

$$1+\sum_{\substack{r+s+t=n\\ n\geq 2}}\sum_{x_1+\ldots+x_{r-1}=s}\sum_{y_1+\ldots+y_r=t}\binom{x_1+y_1}{y_1}\prod_{i=2}^{r-1}\left[y_i\binom{x_i+y_i-1}{y_i}+\binom{x_i+y_i}{y_i}\right].$$

It is equal to the coefficient of x^{n-2} of the expression

$$\frac{1-2x}{(1-x)(1-5x+6x^2-x^3)}.$$

Proof. All that remains to be done is to assign the order relations of each y_i onto W. In the case of y_1, there are only type 1 crossings, and so this is equivalent to constructing a multiset of size y_1 from $\{0,1,2,\ldots,x_1\}$. This simplifies to $\binom{x_1+y_1}{y_1}$ such ways to assign these relations. However, for all the other case, we must take type 2 crossings into account. In the case when there are only type 1 crossings, the number of ways to assign the relations is $\binom{x_i+y_i}{y_i}$, for reasons similar to the case of y_1 above. For type 2 crossings, we first choose the number of such crossings, for a total of y_i choices. Then, there are $\binom{x_i+y_i-1}{y_i}$ ways to assign the order relations, as it is equivalent to constructing a multiset of size y_i from $\{1,2,\ldots,x_i\}$. Thus there are $y_i\binom{x_i+y_i-1}{y_i}+\binom{x_i+y_i}{y_i}$ combinations in the general case.

Now, we continue by showing the simplification.

$$\begin{aligned}
&\sum_{x_1+\ldots+x_{r-1}=s}\sum_{y_1+\ldots+y_r=t}\binom{x_1+y_1}{y_1}\prod_{i=2}^{r-1}\left[y_i\binom{x_i+y_i-1}{y_i}+\binom{x_i+y_i}{y_i}\right]\\
&=\sum_{x_1+\ldots+x_r=s+t}\sum_{y_1,\ldots,y_{r-1}:0\leq y_i\leq x_i}\binom{x_1}{y_1}\prod_{i=2}^{r-1}\left[y_i\binom{x_i-1}{y_i}+\binom{x_i}{y_i}\right]\\
&=\sum_{x_1+\ldots+x_r=s+t}\left[\sum_{y_1=0}^{x_1}\binom{x_1}{y_1}\right]\prod_{i=2}^{r-1}\left\{\sum_{y_i=0}^{x_i}\left[y_i\binom{x_i-1}{y_i}+\binom{x_i}{y_i}\right]\right\}\\
&=\sum_{x_1+\ldots+x_r=s+t}2^{x_1}\prod_{i=2}^{r-1}\left\{\sum_{y_i=0}^{x_i}\left[y_i\binom{x_i-1}{y_i}+\binom{x_i}{y_i}\right]\right\}.
\end{aligned}$$

Note that

$$\sum_{m=0}^{n}m\binom{n-1}{m}=\begin{cases}0 & \text{if } n=0,\\ (n-1)2^{n-2} & \text{otherwise}\end{cases}$$

is the coefficient of the term x^n of $\frac{x^2}{(1-2x)^2}$. Let the notation $[x^k]Q(x)$, where $Q(x)$ is a power series, refer to the coefficient of the x^k term in $Q(x)$. So

$$\sum_{m=0}^{n}\left[m\binom{n-1}{m}+\binom{n}{m}\right]=[x^n]\left(\frac{x^2}{(1-2x)^2}+\frac{1}{1-2x}\right)=[x^n]\left(\frac{1-x}{1-2x}\right)^2.$$

Thus

$$\begin{aligned}
&\sum_{x_1+\ldots+x_r=s+t}2^{x_1}\prod_{i=2}^{r-1}\left\{\sum_{y_i=0}^{x_i}\left[y_i\binom{x_i-1}{y_i}+\binom{x_i}{y_i}\right]\right\}\\
&=\sum_{x_1+\ldots+x_r=s+t}[x^{x_1}]\frac{1}{1-2x}\cdot[x^{x_r}]\frac{1}{1-x}\prod_{i=2}^{r-1}[x^{x_i}]\left(\frac{1-x}{1-2x}\right)^2\\
&=[x^{s+t}]\frac{1}{1-2x}\cdot\frac{1}{1-x}\cdot\left(\frac{1-x}{1-2x}\right)^{2r-4}.
\end{aligned}$$

So

$$\begin{aligned}
|P_n(\underline{2}+\underline{2},\Diamond)|&=1+\sum_{\substack{r+s=n\\r\geq 2}}[x^s]\frac{1}{1-2x}\cdot\frac{1}{1-x}\cdot\left(\frac{1-x}{1-2x}\right)^{2r-4}\\
&=1+\sum_{r=0}^{n-2}[x^{n-2}]\frac{1}{(1-2x)(1-x)}(xy)^r\\
&=1+[x^{n-2}]\frac{1-(xy)^{n-1}}{(1-x)(1-2x)(1-xy)}\\
&=1+[x^{n-2}]\frac{1}{(1-x)(1-2x)(1-xy)}\\
&=1+[x^{n-2}]\frac{(1-2x)}{(1-x)(1-5x+6x^2-x^3)}
\end{aligned}$$

where $y=\left(\frac{1-x}{1-2x}\right)^2$. ■

4.3 Enumeration of $|P_n(\underline{2}+\underline{2},\bowtie,\Diamond)|$

We utilize the structure of the $(\underline{2}+\underline{2},\bowtie)$-free poset we analyzed in Theorem 4.2. However, since the poset now also avoids $\Diamond$, for $w\in P\setminus T$ (where T once again refers to the chain), if $D(w)\neq\emptyset\in T$ then $U(w)=\emptyset$ and similarly if $U(w)\neq\emptyset\in T$ then $D(w)=\emptyset$. A graphical representation of this type of poset is shown in Figure 11 below.

In this case, then, the x_k term from Theorem 4.2 literally divides the poset into two halves. We use this fact in establishing the following bijection.

Theorem 4.6. *There exists a bijection between $(\underline{2}+\underline{2},\bowtie,\Diamond)$-free posets containing a $\underline{2}$ and sequences $\{0,1,*\}^{n-1}$ with exactly one $*$.*

Proof. The proof is geometric in nature. Begin with a 2-chain. Start at the upper element. Read the sequence from left to right. For every 0, we add on to the chain. For every 1,

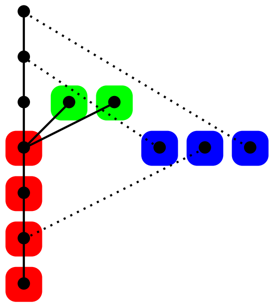

Figure 11: Note the key difference between this figure and the previous one. Each of the elements in blue (the ones not in the chain) can have either its down-set or its up-set but *not both* contained in the chain. Thus, at least one of its up-set or down-set must be empty.

we create a branch from the chain; in this case since we are looking at the upper part of the poset, the new element's up-set is in the chain. We continue this until we reach the $*$, signaling that we have completed constructing the top half of the poset. Now, we begin constructing the bottom half of the poset, beginning with the lower element of the initial 2-chain. We wish to treat this element as the x_k term in Theorem 4.2. So, as we continue reading the sequence after the $*$, the 0's we encounter before the first 1 will be free elements. As soon as we reach the first 1, we continue similar to our construction of the top half; we add to the chain with every 0 and create a branch with every 1 (however, these element's down-sets are in the chain). Clearly, this process can be reversed, and so, we have established the desired bijection. Figure 12 provides an example of the bijection. ■

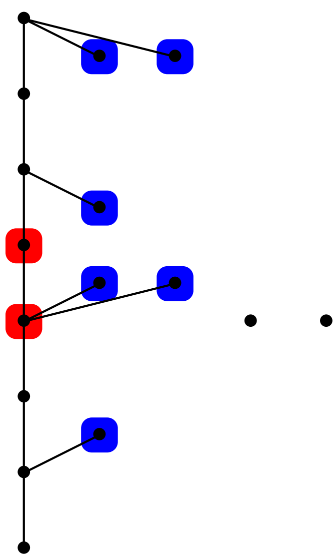

Figure 12: An example of the bijection above. The sequence corresponding to this poset is $(0, 1, 0, 0, 1, 1, *, 0, 0, 1, 1, 1, 0, 0, 1, 0)$. The elements in red correspond to the initial 2-chain and the elements in blue correspond to the 1's in the sequence. Note that the top half of the poset corresponds to 010011 while the bottom half corresponds to 00110010 (the two initial 0's are free elements).

From Theorem 4.6 (and including the poset not containing a 2-chain), it follows that

Corollary 4.7. *The number of* $(\underline{2}+\underline{2}, \bowtie, \Diamond)$*-free posets of size* n *is* $(n-1)2^{n-2}+1$.

5 $\underline{2}+\underline{2}$, $\surd$

Note 5.1. *For ease of writing, we use the symbol* $\surd$ *to refer to the size* 4 *poset in Figure 13.*

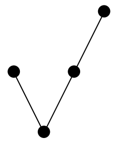

Figure 13: We use $\surd$ to denote the checkmark shaped poset to the left.

In this section, we shall enumerate the number of ($\underline{2}$+$\underline{2}$, $\surd$)-free posets. We accomplish this by both investigating the structure of these posets and also by establishing a bijection to a certain class of combinatorial objects.

Define an element x in poset P to have *level* ℓ if the length of the largest chain in its down-set is ℓ. Similarly, define the function $\ell(x)$ to be the associated level of the element $x \in P$. Note that if $D(x) = \emptyset$, $\ell(x) = 0$.

Now, define $M(P) = \{x \in P : \ell(x) > 0\}$. In other words, $M(P)$ is the set of all elements in P with a non-empty down-set. We proceed with the following lemma.

Lemma 5.1. *Let P be a $(\underline{2} + \underline{2}, \surd)$-free poset. If $a, b \in M(P)$ such that $\ell(a) = \ell(b)$, then $U(a) = U(b)$.*

Proof. Let $\ell(a) = \ell(b) = v$. First we show that a and b both cover some c with $\ell(c) = v - 1$. Since $v \geq 1$ (from the definition of $M(P)$), both a and b each cover at least one other element. However, since the poset is $(\underline{2} + \underline{2})$-free (from Lemma 2.1), $D(a) \cap D(b) \neq \emptyset$. So, there indeed exists some c that is covered by both a and b. Now we shall turn to looking at the up-sets. For the sake of contradiction, assume $U(a)$ contains some element $k \notin U(b)$. But then $\{c, a, b, k\}$ forms a poset isomorphic to $\surd$, a contradiction as the poset P is defined to avoid $\surd$. Thus, $U(b)$ must contain k, and so it follows that $U(a) = U(b)$, completing the proof of the lemma. ∎

From Lemma 5.1, the next lemma follows directly.

Lemma 5.2. *Let P be a $(\underline{2} + \underline{2}, \surd)$-free poset. If a and b are two elements in $M(P)$, such that $\ell(a) < \ell(b)$. Then $a < b$.*

Now, we have a sufficient understanding of structure to enumerate $|P_n(\underline{2} + \underline{2}, \surd)|$.

Theorem 5.3. *The number of $(\underline{2} + \underline{2}, \surd)$-free posets of size n is $\dfrac{3^{n-1}+1}{2}$.*

Proof. From Lemma 5.2, the order relations between elements in $M(P)$ are purely dictated by the levels of the elements in the poset. So, we simply need to assign the elements in $M(P)$ to particular levels. Let $|M(P)| = m$. Since $M(P)$ may have anywhere between 1 and m distinct levels, the total number of ways to assign the m elements in $M(P)$ is $\displaystyle\sum_{\ell=1}^{m} \binom{m-1}{\ell-1} = 2^{m-1}$. We now proceed by casework on m.

When $m = 0$, $M(P) = \emptyset$, and so, P is just a series of free elements; there is only 1 such poset in this case.

When $m > 0$, we must now take the elements with level 0 in P into account. Let $M'(P) = P \setminus M(P)$. Define $m(x_i) = \{x_j \in M'(P) : x_j \in D(x_i)\}$. Utilizing Lemma 5.2 and from Lemma 2.1, it follows that there exists some assignment of the indices such that $m(x_1) \subseteq$

$m(x_2) \subseteq \cdots \subseteq m(x_m)$, and so $1 \leq |m(x_1)| \leq |m(x_2)| \leq \cdots \leq |m(x_m)| \leq n - m$. Thus, this simplifies the computation to constructing an m-element mutliset on $(1, 2, \ldots, n - m)$.

So, $|P_n(\underline{2} + \underline{2}, \surd)| = 1 + \sum_{m=1}^{n-1} 2^{m-1} \left(\!\!\binom{n-m}{m}\!\!\right) = 1 + \sum_{m=1}^{n-1} 2^{m-1} \binom{n-1}{m}$, which simplifies to $\dfrac{3^{n-1}+1}{2}$ after application of the Binomial Theorem. ■

This result sparks a natural question; does there exist some relationship between $(\underline{2} + \underline{2}, \surd)$-free posets and a class of the ternary strings? We answer this question with the following bijection.

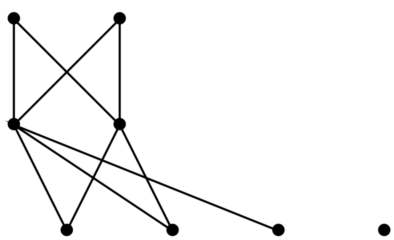

Figure 14: The string 20120210 corresponds to the poset. Note the initial 2 corresponds to the free element on level 0. Each subsequent 2 is covered by all the elements that precede it on the particular level. For example, the second 2 is only covered by a single element on level 1 (only preceded by the 1), while the third 2 is covered by two elements on level 1 (preceded by the 1 and 0).

Theorem 5.4. *There exists a bijection between $(\underline{2}+\underline{2}, \surd)$-free posets containing a $\underline{2}$ and the ternary strings of the form* $\overbrace{22\cdots22}^{k\ 2's}01\overbrace{\cdots\cdots\cdots}^{n\text{-}k\text{-}2\ 0,1,2's}$

Proof. The bijection here is very elementary and straight-forward, so we will only briefly touch upon it (See Figure 14 above for further clarification in an example). The initial 2's refer to the free elements. The 01 refers to our starting 2-chain. After this point, every 0 refers to adding another element to the current level while every 1 refers to going to the next higher level. The 2's refer to the elements in $T'(P)$ from Theorem 5.3. Note that summing across all $0 \leq k \leq n-2$ (and including the poset not containing a 2-chain) yields the expression in Theorem 5.3. ■

6 Future Directions and Open Problems

Throughout this paper, we have enumerated $|P_n(\underline{2} + \underline{2}, \alpha)|$ for a number of posets α. Yet, there is one observation that is particularly striking when we look at those α that we and previous work have resolved; all these posets α avoid $\underline{1} + \underline{1} + \underline{1}$. In fact, upon this paper, the enumeration of all size 4 posets α that avoid $\underline{1} + \underline{1} + \underline{1}$ has been resolved. So, the natural question that arises for further research is

Open Problem 6.1. *Can one enumerate the number of posets that avoid $\underline{2}+\underline{2}$ and another poset α, where α contains $\underline{1}+\underline{1}+\underline{1}$?*

It appears that enumerating $|P_n(\underline{2}+\underline{2}, \alpha)|$ for α containing $\underline{1}+\underline{1}+\underline{1}$ is considerably more difficult. A partial reason for this may be ascertained by looking at the size 3 posets. When

$\alpha = \vee, \wedge, \underline{3}$, or $\underline{2} + \underline{1}$, the enumeration is very straightforward, however for $\alpha = \underline{1} + \underline{1} + \underline{1}$, the enumeration is more intractable and the explicit form is messier (see [12]).

Additionally, we also calculated $|P_n(\underline{2} + \underline{2}, \alpha, \beta)|$ for a few pairs of patterns α and β, notably $|P_n(\underline{2} + \underline{2}, Y, \underline{4})|$ and $|P_n(\underline{2} + \underline{2}, \bowtie, \Diamond)|$. For these ones, the explicit form was fairly nice, suggesting that enumeration for other pairs α, β may be possible.

Open Problem 6.2. *Can one enumerate the number of posets that avoid $\underline{2} + \underline{2}$ and the pair of posets α and β?*

This is likely straightforward, as the first step may be to choose α, β to be posets from this paper, since we have already established an understanding of the structures of each of these individually.

We also return to the question of Wilf-equivalences in $(\underline{2} + \underline{2})$-free posets. In particular, from empirical data, we conjecture that

Conjecture 6.3. *$\underline{2} + \underline{1} + \underline{1}$ and $\Diamond$ are Wilf-equivalent in $(\underline{2} + \underline{2})$-free posets.*

From data (see Data Table), $Y, \underline{4}$ (shown in this paper) and $\underline{3} + \underline{1}, N$ appear to be the only other non-trivial Wilf-equivalence pairs of size 4. In general, we have shown that $Y(k), \underline{k}$ are a Wilf-equivalence pair; however, it appears that Wilf-equivalences in $(\underline{2} + \underline{2})$-free posets are rare.

Open Problem 6.4. *Are there other non-trivial Wilf-equivalences in $(\underline{2} + \underline{2})$-free posets? Do there exist posets α, β such that $|P_n(\underline{2} + \underline{2}, \alpha)| = |P_n(\underline{2} + \underline{2}, \beta)|$ for all n?*

Lastly, we consider ascent sequences. Define a function $a(x)$ to return the ascent sequence associated with a poset x, based on the bijection outlined in [2]. Let $A_n(y)$ refer to the set of posets of size n whose ascent sequences avoid the ascent sequence y. It has been shown that $P_n(\underline{2} + \underline{2}, N) = A_n(0101)$ where $a(N) = 0101$, and $P_n(\underline{2} + \underline{2}, \surd') = A_n(0102)$ where $\surd'$ is the dual of the poset $\surd$ and $a(\surd') = 0102$ (see [3] and [15]).

Open Problem 6.5. *For what posets α is it true that $P_n(\underline{2} + \underline{2}, \alpha) = A_n(a(\alpha))$?*

7 Data Table

Below is a table of $|P_n(\underline{2} + \underline{2}, \alpha)|$ for varying posets α and sizes n. Note that we have data up to size 7 because for some unsolved posets below, we have only generated data up to size 7. The horizontal lines in the table show the Wilf-equivalences ($\bowtie$ and Y give different values for size 8)

	1	2	3	4	5	6	7
$\underline{3}$	1	2	4	8	16	32	64
$\underline{2}+\underline{1}$	1	2	4	8	16	32	64
$\vee$	1	2	4	8	16	32	64
$\wedge$	1	2	4	8	16	32	64
$\underline{1}+\underline{1}+\underline{1}$	1	2	4	9	21	50	120
$\vee + \underline{1}$	1	2	5	14	41	121	255
$\wedge + \underline{1}$	1	2	5	14	41	121	255
$\surd$	1	2	5	14	41	122	365
$/\!\!\wedge$	1	2	5	14	41	122	365
$\underline{2}+\underline{1}+\underline{1}$	1	2	5	14	42	131	417
$\Diamond$	1	2	5	14	42	131	417
$\underline{3}+\underline{1}$	1	2	5	14	42	132	429
N	1	2	5	14	42	132	429
$\underline{4}$	1	2	5	14	42	132	430
Y	1	2	5	14	42	132	430
λ	1	2	5	14	42	132	430
$\bowtie$	1	2	5	14	42	132	430
$\uparrow$	1	2	5	14	43	140	471
$\downarrow$	1	2	5	14	43	140	471
$\underline{1}+\underline{1}+\underline{1}+\underline{1}$	1	2	5	14	45	158	586

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References

[1] K.P. Bogart. An obvious proof of Fishburn's interval order theorem. *Discrete Mathematics* 118.1 (1993): 239–242.

[2] M. Bousquet-Mélou, A. Claesson, M. Dukes, and S. Kitaev. (2+2)-free posets, ascent sequences and pattern avoiding permutations. *Journal of Combinatorial Theory, Series A* 117.7 (2010): 884–909.

[3] P. Duncan and E Steingrimsson. Pattern avoidance in ascent sequences. *The Electronic Journal of Combinatorics* 18.P226 (2011): 1.

[4] P.C. Fishburn. Intransitive indifference with unequal indifference intervals. *Journal of Mathematical Psychology* 7.1 (1970): 144–149.

[5] M. Gerstenhaber and M. Schaps. Finite posets and their representation algebras. *International Journal of Algebra and Computation* 20.01 (2010): 27–38.

[6] M. Guay-Paquet, A. Morales, and E. Rowland. Structure and enumeration of (3+1)-free posets. *arXiv preprint arXiv:1303.3652* (2013).

[7] J. Kahn, M. Saks, and D. Sturtevant. A topological approach to evasiveness. *Combinatorica* 4.4 (1984): 297–306.

[8] J. Lewis and Y. Zhang. Enumeration of graded (3+1)-avoiding posets. *Journal of Combinatorial Theory, Series A* 120.6 (2013): 1305–1327.

[9] H.M. MacNeille. Partially ordered sets. *Trans. Amer. Math. Soc* 42.3 (1937): 416–460.

[10] P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplanes. *Inventiones mathematicae* 56.2 (1980): 167–189.

[11] M. Skandera. A characterization of (3+1)-free posets. *Journal of Combinatorial Theory, Series A* 93.2 (2001): 231–241.

[12] R.P. Stanley. An equivalence relation on the symmetric group and multiplicity-free flag h-vectors. *Journal of Combinatorics* 3.3 (2012): 277–298.

[13] R.P. Stanley. Enumeration of posets generated by disjoint unions and ordinal sums. *Proceedings of the American Mathematical Society* 45.2 (1974): 295–299.

[14] R.P. Stanley. *Enumerative combinatorics, Vol. 1*. Cambridge university press, 2011.

[15] W. Trongsiriwat. *In preparation*.

[16] W.T. Trotter. New perspectives on interval orders and interval graphs. *London Mathematical Society Lecture Note Series* 241 (1997): 237–286.

[17] M.L. Wachs. Poset topology: tools and applications. *arXiv preprint math/*0602226 (2006).