Modified Farey Sequences

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The Farey Sequence

$$\frac{a}{b}, \frac{c}{d} \rightarrow \frac{a+c}{b+d}$$

$$\frac{0}{1} \quad \frac{1}{1}$$

$$\frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1}$$

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1}$$

$$\frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1}$$

Properties

Let $\frac{a}{b}$, $\frac{c}{d}$ be two consecutive fractions, with their *mediant* equal to $\frac{a+c}{b+d}$. Some useful properties:

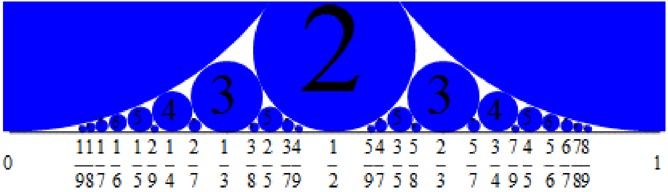
- The mediant never needs to be reduced
- $bc - ad$ equals 1
- The first half of the list of numerators equals the previous
- Denominators are increased by the numerator

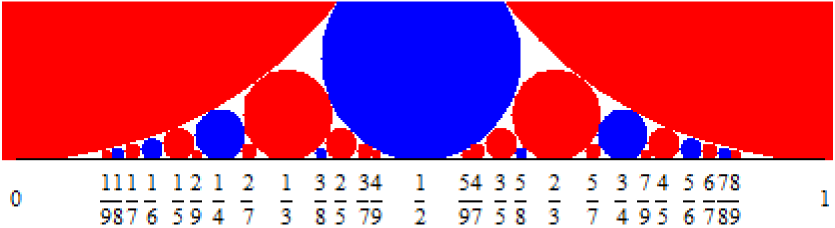
Theorem

(Well Known) Every rational number between 0 *and* 1 *appears somewhere.*

Ford Circles

- Begin with 2 circles, radius $\frac{1}{2}$, placed at $(0, 0)$ and $(1, 0)$
- Between any two tangent circles, insert a circle tangent to the two and the axis
- x-coordinates are given by the Farey Sequence
- Curvatures equal twice denominator squared





- Visual representation of irreducibility
- Odd denominators are red
- Even denominators are even

A Different Farey Sequence

$$\frac{a}{b}, \frac{c}{d} \to \frac{2a+c}{2b+d}, \frac{a+2c}{b+2d}$$

$$\frac{0}{1} \quad \frac{1}{1}$$

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{1}{1}$$

$$\frac{0}{1} \quad \frac{1}{5} \quad \frac{2}{7} \quad \frac{1}{3} \quad \frac{4}{9} \quad \frac{5}{9} \quad \frac{2}{3} \quad \frac{5}{7} \quad \frac{4}{5} \quad \frac{1}{1}$$

Here we insert weighted mediants.

The General Farey Sequence

In general, insert $k-1$ fractions; weights sum to k. The case above is $k=3$, and the original is $k=2$. We pose the following questions:

- For which k do all rational numbers appear?
- Can we categorize which rationals appear based on k?
- How do the properties of the original sequence generalize?

Some lemmas for $k = 3$

For consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$:

Lemma

$bc - ad$ is a power of 3.

Lemma

b and d are odd

Some more lemmas for $k = 3$

Lemma

$\gcd(2a + c, 2b + d) = \gcd(a + 2c, b + 2d) \in \{1, 3\}$.

Lemma

The one with smaller denominator is the closest rational approximation with smaller, odd denominator to the other.

A couple of theorems $k = 3$

$N(r, i)$ is numerator of i^{th} fraction in row r. $D(r, i)$ defined similarly. For any consecutive rows n and $n + 1$:

Theorem

$N(n, i) = N(n + 1, i)$

Theorem

$2N(n, i) + D(n, i) = D(n + 1, i)$

Numerators

$$\frac{0}{1}\ \frac{1}{1}$$

$$\frac{0}{1}\ \frac{1}{3}\ \frac{2}{3}\ \frac{1}{1}$$

$$\frac{0}{1}\ \frac{1}{5}\ \frac{2}{7}\ \frac{1}{3}\ \frac{4}{9}\ \frac{5}{9}\ \frac{2}{3}\ \frac{5}{7}\ \frac{4}{5}\ \frac{1}{1}$$

$$\frac{0}{1}\ \frac{1}{7}\ \frac{2}{11}\ \frac{1}{5}\ \frac{4}{17}\ \frac{5}{19}\ \frac{2}{7}\ \frac{5}{17}\ \frac{4}{13}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{13}{27}\ \frac{14}{27}\ \frac{5}{9}\ \frac{4}{7}\ \frac{3}{5}\ \frac{2}{3}\ \frac{9}{13}\ \frac{12}{17}\ \frac{5}{7}\ \frac{14}{19}\ \frac{13}{17}\ \frac{4}{5}\ \frac{9}{11}\ \frac{6}{7}\ \frac{1}{1}$$

and Denominators

$$\frac{0}{1}\ \frac{1}{1}$$

$$\frac{0}{1}\ \frac{1}{3}\ \frac{2}{3}\ \frac{1}{1}$$

$$\frac{0}{1}\ \frac{1}{5}\ \frac{2}{7}\ \frac{1}{3}\ \frac{4}{9}\ \frac{5}{9}\ \frac{2}{3}\ \frac{5}{7}\ \frac{4}{5}\ \frac{1}{1}$$

$$\frac{0}{1}\ \frac{1}{7}\ \frac{2}{11}\ \frac{1}{5}\ \frac{4}{17}\ \frac{5}{19}\ \frac{2}{7}\ \frac{5}{17}\ \frac{4}{13}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{13}{27}\ \frac{14}{27}\ \frac{5}{9}\ \frac{4}{7}\ \frac{3}{5}\ \frac{2}{3}\ \frac{9}{13}\ \frac{12}{17}\ \frac{5}{7}\ \frac{14}{19}\ \frac{13}{17}\ \frac{4}{5}\ \frac{9}{11}\ \frac{6}{7}\ \frac{1}{1}$$

Determinant

$bc - ad$ is called the *determinant* of $\frac{a}{b}, \frac{c}{d}$

$$\frac{0}{1} \quad \frac{1}{1}$$

$$1$$

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{1}{1}$$

$$1 \quad 3 \quad 1$$

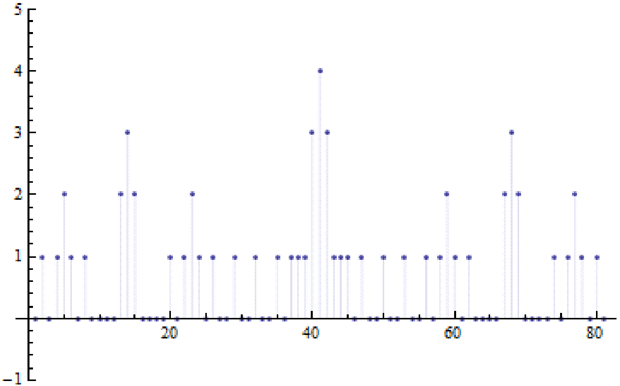
$$\frac{0}{1} \quad \frac{1}{5} \quad \frac{2}{7} \quad \frac{1}{3} \quad \frac{4}{9} \quad \frac{5}{9} \quad \frac{2}{3} \quad \frac{5}{7} \quad \frac{4}{5} \quad \frac{1}{1}$$

$$1 \quad 3 \quad 1 \quad 3 \quad 9 \quad 3 \quad 1 \quad 3 \quad 1$$

Notice the interesting fractal-like behavior.

The Determinant for $k = 3$

The following is a visual representation of the power of three in the determinant in a row:



The recursive rule found for this sequence is quite involved, and was hence omitteed.

The Determinant in General

For prime k:

- Determinant is a power of k
- The list of determinants in row n is the set $\{1, k, \ldots, k^n\}$

In general:

- Determinant divides k^n
- GCD divides determinant

For k odd, every divisor of k^n appears among the list of determinants in row n.

The General Case

- Most interesting for prime k
- Stronger conclusions, non-trivial reduction
- Not all numbers appear

Theorem

For odd prime k, only those numbers with denominators that are $1 \mod k-1$ *can appear.*

For odd prime k, all rational numbers with denominators $1 \mod k-1$ appear For even k, all rational numbers appear somewhere

Future Research

- Proving any unproven conjectures
- Completely finishing up the case $k = 3$
- Inventing a continued fraction variant
- Generalizing results to prime, and then all k
- Finding, for all k, exactly which fractions appear

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