Degrees of Regularity of Colorings of the Integers

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Background

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- (Schur, 1916) Schur's theorem
- (van der Waerden, 1927) van der Waerden's theorem
- (Rado, 1933) Rado's theorem

Definitions

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For example,

$$x + y = 2z$$

is regular under every coloring.

Definitions and Examples

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Examples:

- $x + y = z$ is regular: 1 2 3 4 5/5
- $x + 2y = 4z$ is 2-regular, but not 3-regular.

1 2 3 4 5 6 7 8 9 ...

More Examples

- The equation $x_1 + x_2 + x_3 = 4x_4$ is 3-regular but not 4-regular.

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- The equation $x_1 + x_2 + x_3 = 4x_4$ is 3-regular but not 4-regular.

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  In fact, there are infinitely many colorings with no monochromatic solutions.
- The equation $x_1 + 2x_2 + 3x_3 - 5x_4 = 0$ is completely regular.

Goals

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- Characterize equations that are regular under certain colorings

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- This behavior is expected: replace does not change regularity, especially reducing the number of colors.

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Idea: Find a solution, and since the equation is homogeneous, multiply everything by p.

Shifting Under Periodic Coloring

Shift non-homogeneous equation: $y_i = x_i + \gamma$.

$$a_1 y_1 + a_2 y_2 + \cdots + a_k y_k = n + S\gamma,$$

where S is the sum of the coefficients: $S = a_1 + a_2 + \cdots + a_k$.

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Lemma

With respect to periodic colorings this equation is equivalent to the main equation.

General Equation

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Corollary

The main equation is regular if and only if S divides n.

Discussion

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- This also presents an inexact criterion for regularity under a periodic coloring with less than p colors.

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- This suggests an analogous result for other algebraic structures colored by equivalence class.
- This also presents an inexact criterion for regularity under a periodic coloring with less than p colors.

Conjecture

The main equation is regular under any binary periodic coloring of period $p > 2$.

Degree of Regularity

We now consider the degree of regularity of the general equation.

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Lemma

If k divides S and does not divide n, then the equation is not k-regular.

This gives a measure of how far from regular the equation is.

DOR cont.

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Lemma

When $a_1, a_2, a_3, \ldots$ have the same sign, the equation is not 2-regular when S does not divide n.

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When $a_1, a_2, a_3, \ldots$ have the same sign, the equation is not 2-regular when S does not divide n.

Conjecture

The coloring used in the proof of the previous lemma is the only one that breaks regularity for a binary coloring.

Future Research

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- Find degrees of regularity for specific equations, e.g. $ax + by = z$ for various a, b, $x^2 + y^2 = z^2$.
- Find some structure on colorings and/or equations. A basic example of this was the shifting property for periodic colorings.

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