

COUNTING PERMUTATIONS MODULO PATTERN-REPLACEMENT EQUIVALENCES FOR THREE-LETTER PATTERNS

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ABSTRACT. We study a family of equivalence relations in S_n , the group of permutations on n letters, created in a manner similar to that of the Knuth relation and the forgotten relation. For our purposes, two permutations are in the same equivalence class if one can be reached from the other through a series of pattern-replacements using patterns whose order permutations are in the same part of a predetermined partition of S_c .

When the partition is of S_3 and has one nontrivial part of size greater than two, we provide formulas for the number of classes created in all unresolved cases. When the partition is of S_3 and has two nontrivial parts, each of size two (as do the Knuth and forgotten relations), we enumerate the classes for 13 of the 14 unresolved cases. In two of these cases, enumerations arise which are the same as those yielded by the Knuth and forgotten relations. The reasons for this phenomenon are still largely a mystery.

1. INTRODUCTION

In 1970, Donald Knuth discovered the so-called Knuth relation, using it as a tool for studying the RSK (Robinson-Schensted-Knuth) correspondence. The RSK correspondence is a bijection between permutations in S_n (for fixed n) and pairs of standard Young tableaux of the same shape with n cells. The Knuth relation was designed to connect two permutations exactly when the first tableau that each one is mapped to under the RSK correspondence is the same for both. However, over time, it would find applications not only to combinatorics, but also to abstract algebra. The Knuth relation is particularly well-known for its aid in the proof of the Littlewood-Richardson rule, an identity which can be interpreted as a multiplication rule for Schur polynomials (among many other things) and has applications in particle physics. This proof was historically one of the first proofs to be found for the rule.

In 2008, Novelli and Schilling brought to light the so-called forgotten relation, originally discovered by Lascoux and Schützenberger in 1981, but heretofore ignored in most existing literature (whence the name). The forgotten relation and the Knuth relation share a common structure: Each of them is an equivalence relation on S_n (or, more generally, on some set of words) which connects two permutations obtained from each other by a rearrangement of three consecutive letters according to certain rules. The forgotten relation also shares a common application with its fore-father. When either of the forgotten or Knuth relations is forced on the free associative algebra, we get a structure in which the elementary symmetric functions (appropriately defined) commute. The similar structure and uses of these two relations inspired two research groups to systematically analyze relations in the same family in [LPRW] and [PRW].

There are several ways to set the rules that allow rearranging letters in a permutation. The ones we are going to deal with allow the rearranging of more than three consecutive letters, but the number of letters rearranged is fixed, and the rules which determine when rearrangements are allowed depend only on the relative order of the letters we are rearranging and on the order

in which we want to rearrange them. This is what [LPRW] and [PRW] call the "only indices adjacent" case (or the P^{\parallel} case). We formalize the rules into the notion of a replacement partition (see the definitions below). Both [LPRW] and [PRW] studied only replacement partitions with one non-trivial part using patterns of size three. They considered two problems: First, how many equivalence classes does a given such relation subdivide S_n into? Second, how big is the class containing the identity? Along with solving the second problem, they were able to solve all but seven cases of the first, as long as the nontrivial part of the replacement partition is required to be of size greater than two. We extend their work by solving the remaining seven cases (Section 2). We furthermore study a family of relations more closely related to the Knuth and forgotten relations, by considering replacement partitions of S_3 having two non-trivial parts, each of size two. We enumerate the classes in all but one of the unresolved cases (Section 3). Finally, we study a connection between pattern-avoidance and the enumeration of equivalence classes in Section 4. We conclude with comments on methodology and future work (Section 5).

Before we present our results, we establish some conventions. Most of these formalize conventions made in [LPRW].

Definition 1.1. *A word is a finite, possibly empty sequence of positive integers. The elements of a word are called letters. We denote the i -th letter of word w as w_i , with w_1 as the first letter.*

Definition 1.2. *We regard any permutation of $\{1, 2, \dots, n\}$ as a word (by writing it in one-line notation). Conversely, given a word w with no two letters equal, we define the order permutation of w as the unique permutation $\pi \in S_n$ (where n is the length of w) such that for any i and j , we have $\pi_i < \pi_j$ if and only if $w_i < w_j$. This permutation π is also known as the standardization of w .*

For example, the order permutation of 425 is 213. The notion of an order permutation is more generally defined for any words (possibly with equal letters), which requires a subtler definition, but we are never going to need it in this generality.

Definition 1.3. *If a word w has order permutation w' , then we may simply say that w forms the permutation w' .*

Definition 1.4. *A factor of a permutation is a subsequence of adjacent letters in a permutation.*

Note that we may treat a factor either as a word or as its order permutation.

Definition 1.5. *The position parity of a letter in a permutation is the parity of the position of the letter.*

For example, the first letter in a permutation has odd position parity.

Throughout the paper, may sometimes use notation like " $f(n < 3) = n + 2$ " which is just shorthand for "when $n < 3$, $f(n) = n + 2$ ".

Definition 1.6. *If a and b are two words, then the concatenation of a with b is defined as the word obtained by attaching b to the end of a . It is denoted by ab or $a.b$. (Note: If π is a permutation of size n , then πn denotes the concatenation $\pi.n$, not the number $\pi(n)$.)*

For example, the concatenation of 21 with 72 is 2172.

Definition 1.7. *Let $\pi \in S_n$. Let w be a word with no two letters equal. If we can write w in the form aub for some three (possibly empty) words a , u and b such that u has order permutation π , then we say that u is a π in w (or a π pattern in w). We say that w avoids π if there is no π in w .*

For example, 574 is a 231 in 2657431 because the order permutation of 574 is 231. However, 3124 avoids 231.

We now will define the equivalence relations that we are going to study.

Definition 1.8. Let $k \in \mathbb{N}$. A replacement partition of S_k is a set partition of the symmetric group S_k .

Definition 1.9. Given a replacement partition K of S_k and a positive integer n , the K -equivalence on S_n is defined as the equivalence relation on S_n generated by the following requirement: If $\phi \in S_n$ and $\psi \in S_n$ are such that $\phi = aub$ and $\psi = avb$ for some words a, b, u and v , where u and v have length k , and the order permutation of u lies in the same part of K as the order permutation of v , then we say that ϕ is equivalent to ψ .

Moreover, in this case, we say that ψ results from ϕ by a K -transformation, or more precisely, ψ results from ϕ by a transformation $p \rightarrow q$, where p is the order permutation of u and q is the order permutation of v .

When K is clear from the context, we abbreviate “ K -equivalence” as “equivalence”, and “ K -transformation” as “transformation”.

We also write $\phi \equiv \psi$ for “ ϕ is equivalent to ψ ”.

Note that if two permutations are equivalent, we may say that they are *reachable* from each other. This tends to be used when we are providing the manner through which we can reach one from the other.

If two permutations are connected by a transformation, then we say the letters in the hit used in the transformation are *involved* in the transformation, even if they are static in the transformation.

What we call “ K -equivalence” is denoted as “ K^{\mid} -equivalence” in [LPRW].

Example: Let $n = 5$, $k = 3$, and $K = \{\{123, 321\}, \{132, 231\}, \{213\}, \{312\}\}$. We will later abbreviate this by $K = \{123, 321\} \{132, 231\}$, leaving out the outer brackets and the one-element parts of the partition. Then, the permutation $15324 \in S_5$ is K -equivalent to $12354 \in S_5$ (because $15324 = aub$ and $12354 = avb$ with $a = 1$, $u = 532$, $v = 235$ and $b = 4$, and the order permutation of 532 lies in the same part of K as the order permutation of 235). More precisely, 12354 results from 15324 by a transformation $321 \rightarrow 123$. Similarly, 12354 is equivalent to 12453 (here, $a = 12$, and b is the empty word), and 12453 results from 12354 by a transformation $132 \rightarrow 231$. Combining these, we see that 15324 is equivalent to 12453 , although 12453 does not directly result from 15324 by any transformation.

Definition 1.10. Given a replacement partition K and a positive integer n , the K -equivalence on S_n partitions S_n into equivalence classes. We will briefly refer to these equivalence classes as classes. A class is called trivial if it consists of one element only.

Definition 1.11. Let K be a replacement partition. If w is a word with no two letters equal, then a hit (or more precisely, a K -hit) in w is a word u such that $w = aub$ for some words a and b , and such that the order permutation of u lies in a nontrivial part of K . A word with no two letters equal is said to avoid K if it contains no hit, i.e., if it avoids every permutation in every nontrivial part of K . Otherwise it is said to be non-avoiding, or, equivalently, a non-avoider (with respect to K).

Observe that each permutation that avoids K forms a trivial class (with respect to the K -equivalence), whereas non-avoiders lie in nontrivial classes.

We will occasionally refer to the elements of K as *patterns*, hoping that no confusion with the notation of “ π pattern in w ” can arise.

Figure 1 shows the number of classes created in S_n by the replacement partitions considered in this paper. Each of these results are proven in the following two sections.

replacement partition	number of classes in S_n
$\{213, 231, 132\}$	$2^{n-2} + 2n - 4$
$\{123, 132, 231\}$	2^{n-1}
$\{123, 132, 321\}$	$(n-1)!! + (n-2)!! + n - 2$
$\{123, 132, 312\}$	$f(n \geq 5) = f(n-1) + (n-2) \cdot f(n-2) + 1$
$\{123, 132, 213, 231\}$	n
$\{123, 132, 231, 321\}$	2 for $n > 3$
$\{213, 132, 231, 312\}$	3
$\{123, 132\}\{312, 321\}$	2^{n-1}
$\{123, 132\}\{213, 231\}$	2^{n-1}
$\{123, 231\}\{132, 321\}$	2^{n-1}
$\{132, 312\}\{321, 213\}$	$(n^2 + n)/2 - 2$
$\{123, 231\}\{213, 132\}$	$n^2 - 3n + 4$
$\{123, 321\}\{213, 231\}$	3 for $n > 5$
$\{123, 132\}\{231, 312\}$	$3 \cdot 2^{n-3} + n - 2$ for $n > 5$
$\{123, 132\}\{213, 321\}$	Sum of the first $n-1$ Motzkin numbers
$\{123, 132\}\{213, 312\}$	$f(n \geq 3) = f(n-1) + (n-1) \cdot f(n-2)$
$\{123, 321\}\{132, 213\}$	$\binom{n}{\lfloor n/2 \rfloor} + \binom{n-2}{\lfloor (n-2)/2 \rfloor} + 3$
$\{123, 231\}\{321, 213\}$	$f(n > 5) = \begin{cases} 3n, & \text{if } n \text{ is even} \\ 3n-1, & \text{if } n \text{ is odd} \end{cases}$
$\{123, 321\}\{132, 231\}$	$\sum_{x=1}^l x! \cdot \binom{n-x-1}{h-1} + \sum_{x=1}^h x! \cdot \binom{n-x-1}{l-1},$ where $l = \lfloor n/2 \rfloor, h = \lceil n/2 \rceil$.
$\{123, 231\}\{213, 312\}$	$g(n, k) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n - 2k + 1 = 0, \\ \sum_{j=1}^{\lfloor (n+1)/2 \rfloor} g(n-1, j) & \text{if } k = 1, \\ \sum_{x=k-1}^{n-k} \sum_{j=1}^{n-k-x} \binom{x-1}{k-2} \cdot g(n-2k+1, j) & \text{otherwise.} \end{cases}$ $f(n) = \sum_{k=1}^{\lfloor n/2 + 1 \rfloor} g(n+1, k) + n - 2$

FIGURE 1. The number of classes created in S_n by various replacement partitions of S_3 . Unless otherwise specified, $n \geq 3$. In the table, $f(n)$ equals the number of classes created and is used for recursive formulas.

2. SINGLE REPLACEMENTS

In this section, we consider the number of classes created in S_n by replacement partitions of S_3 with exactly one nontrivial part. This problem was previously addressed in some cases by [LPRW] and [PRW]. We provide formulas for the number of classes in all cases where the nontrivial part of the replacement partition is of size greater than two, apart from those that have been previously solved.

When considering replacement partitions, it is important to note that some relations are equivalent to others. In fact, let w be the permutation $k(k-1)\dots 1 \in S_k$ (where k is the size of the permutations in the replacement partition, and needs not be 3 in this argument). Without changing the structure of the classes, one can either replace each pattern π in the replacement partition by $\pi \circ w = \pi_k \pi_{k-1} \dots \pi_1$, or replace each pattern π in the replacement partition by $w \circ \pi = (k - \pi_1 + 1)(k - \pi_2 + 1) \dots (k - \pi_k + 1)$. By these two operations (which generate a Klein four-group if $k \geq 3$), the set of all replacement partitions of S_k is subdivided into orbits (mostly of size 4, but occasionally smaller), and if we can count the number of classes which are generated in S_n by one partition in each orbit, we automatically obtain the same number for all the other partitions in all of the orbits. Noting this, we will only provide results for the distinct cases.

The following subsections are concerned with one replacement partition each. In each subsection, the replacement partition K is to be understood to be the partition mentioned in the title of the subsection.

2.1. $\{123, 132, 321\}$ -Equivalence.

Lemma 2.1. *Let $n \geq 5$. Let $w \in S_n$ be a permutation that is avoiding except for one hit in the final three letters (i.e., its first $n-1$ letters form an avoiding word, but its final three letters form a hit). Then, w is equivalent to a permutation that is non-avoiding in the left-most $n-1$ letters (i.e., whose first $n-1$ letters form a non-avoider) under the $\{123, 132, 321\}$ -equivalence.*

Proof. The word formed by the first $n-1$ letters of w is avoiding, and hence contains no hit. In particular, it contains no 123s and no 132s. Thus, the letter 1 must be among the final three letters of w . Since these final three letters form a hit, we can rearrange them as a 123. Let w' be what w becomes after this rearrangement; clearly, $w \equiv w'$. Let w'' be the what w' becomes upon swapping 1 with the final letter of w ; then, $w'' \equiv w'$ (since w'' is obtained from w' by a $123 \rightarrow 321$ transformation).

The final 4 letters of w' form either a 2134, a 3124, or a 4123. In the first and the third cases, the permutation w'' contains a hit which ends with the penultimate letter of w'' . In the second case, either w' or w'' contains a hit ending with the antepenultimate letter. Hence, in each case, either w' or w'' is non-avoiding in the left-most $n-1$ letters. Since $w'' \equiv w' \equiv w$, this yields that w is equivalent to a permutation that is non-avoiding in the left-most $n-1$ letters. \square

Note that Lemma 2.1 does not hold for $n = 4$. For example, it would fail in the case of $w = 3124$.

Before continuing, we will first define a set $B_n \subset S_n$.

Definition 2.2. *The permutation $b_n \in S_n$ is $(1, 2, \dots, n-2) \circ (1, 2, \dots, n-4) \circ (1, 2, \dots, n-6) \circ \dots$, where the product ends with (1) if n is odd and with $(1, 2)$ if n is even.*

For example, $b_3 = 123$, $b_4 = 2134$, $b_5 = 23145$, and $b_6 = 324156$.

Definition 2.3. *We say that $B_2 = \{\}$. Then, $B_{n>2}$ is the set containing the following:*

- (1) wn for each $w \in B_{n-1}$;

(2) b_n .

For example, $B_4 = \{1234, 2134\}$, $B_5 = \{12345, 21345, 23145\}$, and $B_6 = \{123456, 213456, 231456, 324156\}$. Note that B_n has $n - 2$ elements.

Three important observations should be made.

- (1) For $w \in B_n$, the final three letters of w are in increasing order.
- (2) For $w \in B_n$, w ends with $(n - 2)1(n - 1)n$.
- (3) For $w \in B_n$, 1 and the two letters following it form a 123 hit. If $w \neq b_n$, this means that w contains a hit in its first $n - 1$ letters ¹.

Lemma 2.4. *Let $n \geq 5$. In S_n , every non-avoiding permutation is equivalent to a permutation in B_n under the $\{123, 132, 321\}$ -equivalence.*

Proof. Assume the lemma holds for all lower $n \geq 5$. The inductive base case of S_5 is easily shown computationally. Let $w \in S_n$ be non-avoiding. By Lemma 2.1, we may assume without loss of generality that the left-most $n - 1$ letters of w are non-avoiding. Applying the inductive hypothesis to the left-most $n - 1$ letters of w , we rearrange them to ensure that they lie in B_{n-1} ². The resulting permutation will be called w' . By Observation (1), w' contains a hit ending with the penultimate letter. So, we can apply the inductive hypothesis to the right-most $n - 1$ letters of w' yielding a permutation w'' . Observation (2) shows that n must be among the right-most $n - 1$ letters of w' , and hence is among the right-most $n - 1$ letters of w'' . Thus, by Observation (2) again, n is the final letter of w'' . If w'' is non-avoiding in the left-most $n - 1$ letters, then we apply the inductive hypothesis to them and we are done. If w'' is avoiding in the left-most $n - 1$ letters, then by Observation (3), the right-most $n - 1$ letters of w'' form b_{n-1} . Thus the final four letters of w'' are of the form 2134 (by Observation (2)) where 1 is the actual letter 1 (because otherwise 1 would be the first letter of a hit in the left-most $n - 1$ letters, a contradiction).

Rearranging those letters to form 2431, we arrive at w''' which is non-avoiding in the left-most $n - 1$ letters and which ends with the letter 1. Performing the inductive hypothesis on the left-most $n - 1$ letters, we arrive at $x \in S_n$ which ends with a 2341 (by Observation (2)). Note that acting as the 1 is the letter 1, and that by Observation (2), acting as the 3 and 4 are $n - 1$ and n respectively. Performing $2341 \rightarrow 2431 \rightarrow 2134$ on the final 4 letters, we get x' .

Let $z \in B_{n-1}$ be the order permutation of the first $n - 1$ letters of x . Let z' be the order permutation of the first $n - 1$ letters of x' . If z' is non-avoiding, then applying the inductive hypothesis to first $n - 1$ letters of x' , we are done. Otherwise, z' is avoiding and so must be the first $n - 3$ letters of z . Hence, by Observation (3), z is either b_{n-1} or $b_{n-2}(n - 1)$.

If $z = b_{n-2}(n - 1)$, then recalling that the final letter of x is the actual 1, we see that x' is obtained from the b_{n-2} by adding 1 to each of its letters, appending $n1$ to the right end of the resulting word, and replacing the final three letters $(n - 1)n1$ by $1(n - 1)n$. It's not hard to see that as a result, $x' = b_n$ and we are done.

Consider the remaining case, when $z = b_{n-1}$. Then, to the right-most 5 letters of x , we apply $32451 \rightarrow 32541 \rightarrow 32145 \rightarrow 12345$ (note that the 5 here is n , due to Observation (2)) and reach a permutation y . We know that y has n in the right-most position and has a hit ending with the penultimate letter. So, we may apply the inductive hypothesis to the first $n - 1$ letters and reach a permutation in B_n . \square

¹This is because 1 is in position $k - 2$ in b_k for each $k > 2$. So, each $w \neq b_n$ in B_n has 1 in position less than $n - 2$.

²This is a slight abuse of notation. What we really mean is that the order permutation of these $n - 1$ letters lies in B_{n-1} .

Lemma 2.5. *No two distinct elements in B_n are equivalent under the $\{123, 132, 321\}$ -equivalence for $n \geq 3$.*

Proof. Let $k \in \{1, 2, \dots, n\}$. We say that a word w satisfies *property A_k* if there exists a strictly increasing sequence (i_1, i_2, \dots, i_k) of integers **of equal parity** satisfying

$$w_{i_1} = k, \quad w_{i_2} = k - 1, \quad w_{i_3} = k - 2, \quad \dots, \quad w_{i_k} = 1$$

(i.e., satisfying $w_{i_j} = k - j + 1$ for every $j \in \{1, 2, \dots, k\}$).

It is easy to see that if a permutation in S_n satisfies property A_k for some k , then every permutation equivalent to it also satisfies A_k for the same k . Indeed, whenever we perform one of the transformations $123 \leftrightarrow 132$, $132 \leftrightarrow 321$, and $321 \leftrightarrow 123$, the position parity of a letter i does not change as long as every letter smaller than i lies to the right of i and has the same position parity as i . But as long as property A_k holds, this is guaranteed for any of the letters $k, k - 1, \dots, 1$. So, the position parity of letters $\leq k$ cannot change under any of these transformations as long as A_k holds. Moreover, as long as A_k is satisfied, for every $j \in \{1, 2, \dots, k - 1\}$, the letter j is at least two positions to the right of the letter $j + 1$ in our permutation (since i_1, i_2, \dots, i_k are of equal parity and increase), and thus it cannot "jump" over $j + 1$ in a single application of one of the transformations $123 \leftrightarrow 132$, $132 \leftrightarrow 321$, $321 \leftrightarrow 123$ ³, i.e., it stays to the right of $j + 1$. Thus, the order of the letters $k, k - 1, \dots, 1$ in our permutation does not change. Hence, after we apply a transformation, the permutation still satisfies property A_k for the same k .

Thus, for every $k \in \{1, 2, \dots, n\}$, the validity of property A_k is an invariant under our equivalence relation. Since the $2m$ -th and $2m + 1$ -th elements of B_n (where the i -th element of B_n means the element which has 1 in position i) satisfy properties A_1, A_2, \dots, A_m but none of A_{m+1}, A_{m+2}, \dots , they can not be equivalent to any other elements of B_n . However, since the $2m$ -th element and the $2m + 1$ -th elements of B_n have 1 in different position parities (and in fact, each of the letters from 1 to m in different position parities), they can not be equivalent. Thus, no two distinct elements of B_n are equivalent. \square

Remark 2.6. Noting Lemmas 2.4 and 2.5, we see that every non-trivial class in S_n under the equivalence has exactly one representative in B_n . When given a non-avoider $w \in S_n$, the B_n -representative of its class can be easily found algorithmically by checking which of the relations A_k (defined in the proof of Lemma 2.5) are satisfied for w and checking the position parity of the letter 1 in w .

Proposition 2.7. *Let $n \geq 5$. In S_n , there are $n - 2$ non-trivial classes under the $\{123, 132, 321\}$ -equivalence.*

Proof. This follows from Lemma 2.4 and Lemma 2.5. \square

Proposition 2.8. *Let $n \geq 5$. There are $(n - 1)!! + (n - 2)!! + n - 2$ classes in S_n under the $\{123, 132, 321\}$ -equivalence.*

Proof. By Proposition 2.7, there are $n - 2$ non-trivial classes. By Theorem 3 of [Ki], there are $(n - 1)!! + (n - 2)!!$ trivial classes. \square

Remark 2.9. Proposition 2.8 is of particular interest because [LPRW] collected computational data in an attempt to enumerate the $\{123, 132, 321\}$ -equivalence, but was unable to find a formula for the number of classes in S_n . Proposition 2.8 provides this formula.

³In fact, if a letter is at least two places to the right of another letter, then the only way it can jump over that other letter in one single application of one of these transformations is when these two letters are the last and the first letter of a hit. But the letters j and $j + 1$ cannot be the last and the first letter of any hit, respectively.

2.2. $\{123, 132, 231\}$ -Equivalence.

Definition 2.10. Let w be a word with no repeating letters. A left to right minimum of w is a letter which is smaller than each letter to its left in w .

Definition 2.11. Let w be a word with no repeating letters. If x is a letter of w , then the x -min of w is the first letter to x 's right that is a left to right minimum. The x -min needs not exist for all x .

Definition 2.12. We call a letter x of a word w (with no repeating letters) odd-tailed if x is a left to right minimum and the x -min exists and is of different position parity than x .

Lemma 2.13. Let x be an odd-tailed letter of a permutation $z \in S_n$. Then, x is odd-tailed in any permutation equivalent to z under the $\{123, 132, 231\}$ -equivalence.

Proof. Let x be an odd-tailed letter of the permutation z in S_n . Let y be the x -min. Notice that by their definitions, x is less than everything to its left and y is also less than everything to its left (including x). As a consequence, x and y cannot be adjacent letters in any 123 pattern, 132 pattern, or 231 pattern, and because x and y have different position parity, they can not be in the same hit.

In a hit of the form 123, the only possible position for x is the first (since x is a left to right minimum). The same holds for a hit of the form 132. In a hit of the form 231, the first and the third positions are possible candidates for containing left to right minima, but x cannot be in the first (because then, the x -min would be in the third place, but that contradicts the fact that the x -min is of different position parity than x). Hence, if x lies in a hit, then x is acting as the 1 in the hit. So, after rearrangement, x is still a left to right minimum and maintains its position parity.

What remains to be shown is that the position parity of the x -min is unchanged after a transformation. This is trivial if y is not in the hit being transformed. If y is in the hit, then there are three subcases. If y is the final letter in the hit then the hit must be of the form 231, and after any rearrangement, y is still the x -min and has the same position parity. We know that y can not be the second letter in a hit because a left to right minimum can never be the second letter in a hit. The final possibility is that y is the first letter in the hit. Then, any rearrangement of the hit is either $123 \leftrightarrow 132$, in which case y remains the x -min with the same position, or is one of the rearrangements $231 \leftrightarrow 123$ and $231 \leftrightarrow 132$. In the latter scenario, the new x -min is either the first or the final letter in the hit after the rearrangement and thus has the same position parity as y does in z . \square

Definition 2.14. A permutation is a V-permutation if its letters decrease until the letter 1 and then increase until the end of the permutation.

For example, 531246 is a V-permutation in S_6 . Note that there are 2^{n-1} V-permutations in S_n because for each letter other than 1, we can only choose if it is in the decreasing part or the increasing part of the permutation.

Proposition 2.15. Let $n \geq 3$. In S_n , 2^{n-1} classes are created under the $\{123, 132, 231\}$ -equivalence.

Proof. We will first prove that permutations in S_n are each equivalent to some V-permutation by inducting on n with a trivial base case of $n = 3$. Take $w \in S_n$. If n is in the first position, leave it there. Otherwise, slide n to the final position through repeated applications of $132 \rightarrow 123$ and $231 \rightarrow 123$. Applying the inductive hypothesis to the other $n - 1$ letters, we reach a V-permutation. Since there are 2^{n-1} V-permutations, there are at most 2^{n-1} classes.

Note that the set of odd-tailed letters in a V-permutation is exactly the set of letters to the left of the letter 1. (They are odd-tailed because for a letter x to the left of 1, x is a left to right minimum

and the x -min is the letter directly to x 's right and thus is of different position parity than x .) This set is distinct for each V-permutation, so by Lemma 2.13, none of the V-permutations are equivalent to each other. Hence, there are 2^{n-1} classes, each determined by the set of odd-tailed letters in each of its permutations. \square

2.3. $\{123, 132, 231, 321\}$ -Equivalence.

Proposition 2.16. *There are two classes in S_n under the $\{123, 132, 231, 321\}$ -equivalence for $n \geq 4$.*

Proof. We will prove inductively that all permutations are reachable from either the identity or the identity with 1 and 2 swapped. The base case of S_4 can easily be shown computationally. Assume the result holds for S_{n-1} . Let $w \in S_n$. First transforming the first $n-1$ letters of w into either the identity or the identity except with the first two letters swapped, then transforming the final $n-1$ letters (in the same way), and finally transforming the first $n-1$ letters again (in the same way), we arrive at either the identity or the identity with 1 and 2 swapped. This establishes that there is at most two classes. There must be exactly two classes because the position parity of 1 is invariant under the $\{123, 132, 231, 321\}$ -equivalence. \square

2.4. $\{123, 132, 213, 231\}$ -Equivalence.

Proposition 2.17. *There are n classes created in S_n under the $\{123, 132, 213, 231\}$ -equivalence.*

Proof. We will prove this by inducting on n . The base case of S_3 can easily be shown computationally. Assume the result holds for S_{n-1} . Note that if the left-most letter of a permutation is n , then n is stationary under the relation. Hence, the permutations starting with n are clearly broken into $n-1$ classes by the inductive hypothesis.

We will now show that the remaining permutations are reachable from the identity, thus completing the proof. Assume that all permutations not starting with $n-1$ are reachable from the identity in S_{n-1} . The base case for this induction is S_3 and can easily be shown computationally. The letter n , if not in the first position, can be freely moved from any position except for the first to any position except for the first. Hence, we can move it to the second position in a given permutation; then, we may apply the inductive hypothesis to the left-most $n-1$ letters (resulting in a permutation which begins with 12 or 13 or 23), then to the right-most $n-1$ letters (resulting in a permutation which ends with n), and then to the left-most $n-1$ letters again (which is allowed because the first letter is 1 or 2), and obtain the identity. \square

2.5. $\{123, 132, 312\}$ -Equivalence. In this subsection, let $f(n)$ = the number of classes created in S_n under the $\{123, 132, 312\}$ -equivalence.

Lemma 2.18. *Every permutation not ending with 1 is equivalent to a permutation of the form $\dots 1j$ for some j under the $\{123, 132, 312\}$ -equivalence.*

Proof. Such a permutation is reached through repeated applications of $123 \rightarrow 312$ and $132 \rightarrow 312$ using the actual letter 1. \square

Lemma 2.19. *Permutations of the form $\dots 12$ are equivalent under the $\{123, 132, 312\}$ -equivalence.*

Proof. Let $w = 345\dots 12$ where the letters not shown are in increasing order. We will show that w is equivalent to an arbitrary permutation x of the form $\dots 12$. The base cases of S_3 and S_4 for the following induction are easy to show computationally. If the lemma holds for S_{n-1} , then from

x , we can rearrange the right-most $n - 1$ letters however we want, while not moving the final two. We arrange them to be in increasing order, arriving at x' . Then, we rearrange the final three letters of x' with $312 \rightarrow 123$, arriving at x'' where 1 and 2 are the actual letters 1 and 2. Performing the inductive hypothesis on the left-most $n - 1$ letters of x'' , the actual letter 3 is placed in the left-most position and we get w''' (requires $n \geq 5$). Rearranging the final 3 letters of w''' , $123 \rightarrow 312$, we reach y . Finally, applying the inductive hypothesis to the right-most $n - 1$ letters of y , we reach w . \square

Corollary 2.20. *Permutations of the form $1 \dots$ are equivalent to each other under the $\{123, 132, 312\}$ -equivalence.*

Proof. By Lemma 2.18, every permutation of the form $1 \dots$ is equivalent to a permutation of the form $\dots 1j$ for some j . This j must be 2 (since the smallest letter to the right of 1 never changes, and that smallest letter is 2 in the original permutation). Thus, every permutation of the form $1 \dots$ is equivalent to a permutation of the form $\dots 12$. Since any two permutations of the latter kind are equivalent (by Lemma 2.19), so must be any two permutations of the former kind. \square

Lemma 2.21. *No two permutations of the form $\dots 1x$ and $\dots 1y$ where $x \neq y$ are equivalent under the $\{123, 132, 312\}$ -equivalence.*

Proof. This falls easily from the fact that the smallest letter to the right of 1 never changes under the equivalence. \square

Lemma 2.22. *Let $a \neq 2$ be a letter. Let x and y be equivalent permutations ending with $1a$. Let x' and y' be the permutations made from the first $n - 2$ letters of x and y respectively. Then, x' is equivalent to y' under the $\{123, 132, 312\}$ -equivalence.*

Proof. Let x, y, x' , and y' be as described. Consider a sequence of permutations that starts with x and ends with y , with each pair of consecutive permutations in the sequence connected by a single transformation. In such a sequence, going from one permutation to the next, a can never pass 1 and 1 can never pass 2 because the smallest letter to the right of 1 can not change. Note that since 1 and 2 can never be in the same hit because in the hits considered, 1 is always to the right of 2. Thus, neither 1 nor a can be in the same hit as any letter that is to the left of 2. Consider two consecutive permutations in the sequence, t and w . Let t' and w' be t and w respectively with 1 and a slid to the two right-most positions as $1a$. If the transformation between t and w involves either 1 or a , then it involves letters only to the right of the letter 2. However, these letters (excluding 1 and a) can be rearranged freely in t' and w' by Corollary 2.20. Thus t' and w' are reachable from each other using transformations that do not involve 1 or a . This, of course, also holds if the transformation between t and w involves neither 1 nor a (because then, we can simply apply the same transformation to get w' from t'). Thus, there exists a sequence of permutations in S_{n-2} starting with x and ending with y , with each pair of consecutive permutations in the sequence connected by transformations which do not involve the final two letters $1a$. Hence, $x' \equiv y'$. \square

Lemma 2.23. *Let $a \neq 2$. Permutations of the form $\dots 1a$ break into $f(n - 2)$ classes under the $\{123, 132, 312\}$ -equivalence for $n \geq 5$.*

Proof. They clearly fall into at most $f(n - 2)$ classes. By Lemma 2.22, there are at least $f(n - 2)$ classes. \square

Theorem 2.24. $f(n) = f(n - 1) + (n - 2) \cdot f(n - 2) + 1$ when $n \geq 5$. As base cases, $f(3) = 4$, and $f(4) = 9$.

Proof. Assume $n \geq 5$ (The lower cases are computed manually). The permutations ending with 1 clearly fall into $f(n-1)$ classes because the final letter is immobile. By Lemma 2.18, every permutation not ending with 1 is reachable from $\dots 1j$ for some j and by Lemma 2.21, j is unique. Permutations reachable from $\dots 12$ fall into 1 class by Lemma 2.19. Permutations reachable from $\dots 1j$ where $j \neq 2$ fall into $f(n-2)$ classes for a given j by Lemma 2.23. There are $n-2$ possibilities for such an j . Thus, $f(n) = f(n-1) + (n-2) \cdot f(n-2) + 1$ when $n \geq 5$, $f(3) = 4$, and $f(4) = 9$. \square

2.6. $\{213, 132, 231\}$ -Equivalence.

Remark 2.25. In this subsection, let $f(m)$ = the number of non-trivial classes created in S_m under the $\{213, 132, 231\}$ -equivalence. Let $g(m)$ = the number of trivial classes created under the $\{213, 132, 231\}$ -equivalence in S_m .

Definition 2.26. A permutation in S_n is *reductive* if it satisfies the following three conditions.

- it does not start with $n-2$;
- it ends with $(n-1)n$;
- its left-most $n-1$ letters are non-avoiding.

Definition 2.27. A permutation in S_n is *decent* if it satisfies the following two conditions.

- it starts with $n-2$;
- its right-most $n-1$ letters are non-avoiding, do not begin with $n-1$, and do end with n .

Lemma 2.28. Let $n > 3$. All non-avoiding permutations not beginning with $n-1$ but ending with n are equivalent in S_n under the $\{213, 132, 231\}$ -equivalence.

Proof. We will prove this inductively using the easily checked base case of S_4 . Let $n \geq 5$ and assume that the result holds for S_{n-1} . Note that all reductive permutations are reachable from each other by the inductive hypothesis applied to the left-most $n-1$ letters.

Let $w \in S_n$ be a non-avoiding permutation that is not beginning with $n-1$ and that ends with n . We will prove that w is equivalent to a reductive permutation. Since reductive permutations are all equivalent, this will complete the proof. Applying repeatedly either $132 \rightarrow 213$ or $231 \rightarrow 213$ to w , we place $n-1$ in the second to final position (while keeping n in the final position). If the resulting permutation does not begin with $n-2$, then it is reductive (because its left-most $n-1$ letters are clearly non-avoiding, given how we constructed it) and we are done. In the remaining case, the resulting permutation y begins with $n-2$.

Note that y begins with $n-2$ and ends with $(n-1)n$. Since y begins with $n-2$ and is of size ≥ 5 , the first three letters of y form a 312 or 321 and thus do not form a hit. Since y is non-avoiding and cannot begin with a hit, the right-most $n-1$ letters of y must be non-avoiding. Thus, y is decent. Any two decent permutations are equivalent to each other by the inductive hypothesis applied to the right-most $n-1$ letters. Thus, we only need to show that some decent permutation is equivalent to a reductive permutation.

Consider the permutation $w = (n-2)1(n-1)2n\dots$. Note that the letters not shown are those between 1 and $n-2$, and are in increasing order from left to right. Sliding n to the final position through repeated applications of $132 \rightarrow 213$, and then sliding $(n-1)$ to the second to final position in the same manner, we reach $w' = (n-2)\dots 21(n-1)n$. One can easily check that w' is decent. From w , we can obtain $w'' = 21(n-1)n(n-2)\dots$ in the following way.

$$(n-2)1(n-1)2n\dots \rightarrow$$

$$\begin{aligned}
(n-2)1(n-1)n2 \dots &\rightarrow \\
1(n-1)(n-2)n2 \dots &\rightarrow \\
1(n-1)2n(n-2) \dots &\rightarrow \\
21(n-1)n(n-2) \dots &
\end{aligned}$$

From w'' , we can first slide n to the final position through repeated applications of 231 or 132 \rightarrow 213, and then slide $n-1$ to the second to final position in the same manner to get w''' . Note that w''' begins with the letter 2, ends with the letters $(n-1)n$, and has a hit in the penultimate three letters. Hence, w''' is a reductive permutation and we are done. \square

Lemma 2.29. *Let $n > 3$. All non-avoiding permutations of the form $(n-1)j \dots n$ with $j \neq n-2$ are equivalent in S_n under the $\{213, 132, 231\}$ -equivalence. (The letter j is not fixed here.)*

Proof. Let w be such a permutation in $S_{n>3}$. Since w begins with $n-1$, ends with n , and is of size ≥ 4 , the first three letters of w form a 312 or 321 and thus don't form a hit. Since w is non-avoiding and cannot begin with a hit, the right-most $n-1$ letters of w must be non-avoiding. Thus, they can be rearranged as the identity with $n-2$ and $n-3$ swapped (by Lemma 2.28). \square

Lemma 2.30. *Let $n > 3$. All non-avoiding permutations of the form $(n-1) \dots n$ are equivalent in S_n under the $\{213, 132, 231\}$ -equivalence.*

Proof. We will prove this inductively. Assume it is true for S_{n-1} . The base cases of S_4 and S_5 can be easily checked computationally. By Lemma 2.29, all such permutations whose second letters are not $n-2$ are equivalent. Let w be such a permutation whose second letter is $n-2$. Note that w has the form $(n-1)(n-2) \dots n$. By the inductive hypothesis (applied to the right-most $n-1$ letters), w is equivalent to any non-avoiding permutation of the form $(n-1)(n-2) \dots n$.

One such permutation is $y = (n-1)(n-2) \dots 1n$. Here, the letters not shown are the letters between 1 and $n-1$ and are in increasing order from left to right. Through repeated applications of $213 \rightarrow 132$, we can reach $w' = (n-1)(n-2)1n \dots$. From w' , we reach $w'' = (n-1)n1(n-2) \dots$ in the following manner.

$$\begin{aligned}
(n-1)(n-2)1n \dots &\rightarrow \\
(n-1)1n(n-2) \dots &\rightarrow \\
(n-1)n1(n-2) \dots &
\end{aligned}$$

From w'' , we may first slide $n-2$ to the final position through repeated applications of $132 \rightarrow$ 213, and then slide n to the final position through repeated applications of 132 or $231 \rightarrow$ 213. This yields a permutation which is non-avoiding, begins with $n-1$, ends with n , and has its second letter $\neq n-2$. Thus, all non-avoiding permutations of the form $(n-1)(n-2) \dots n$ are equivalent to a non-avoiding permutation of the form $(n-1)j \dots n$ for some $j \neq n-2$. Hence, by Lemma 2.29, all non-avoiding permutations of the form $(n-1) \dots n$ are equivalent. \square

Definition 2.31. *A permutation in S_n will be called fronted if it starts either with $n-1$ or with $jn(n-1)$ for some $j \leq n-2$.*

Lemma 2.32. *Any permutation starting with $n-1$ is only equivalent to permutations where $n-1$ is in the first or third position under the $\{213, 132, 231\}$ -equivalence.*

Proof. It is easy to see that any transformation, applied to a fronted permutation, yields another fronted permutation. Hence, any fronted permutation is equivalent only to fronted permutations. In particular, any permutation starting with $n-1$ is only equivalent to fronted permutations, and these have $n-1$ either in the first or in the third position. \square

Lemma 2.33. *Let $n > 3$. Non-avoiding permutations not beginning with n in S_n fall into 2 non-trivial classes under the $\{213, 132, 231\}$ -equivalence.*

Proof. From an arbitrary non-avoiding permutation not beginning with n , we can move n to the right-most position through repeated applications of 132 or $231 \rightarrow 213$. By Lemmas 2.28 and 2.30, we have established that non-avoiding permutations not beginning with n break into at most two classes: those containing permutations of the form $(n-1) \dots n$, and those containing permutations not beginning with $n-1$ but ending with n . Noting Lemma 2.32, and that there exists a non-avoiding permutation of the form $(n-1) \dots n$ as well as one which is non-fronted and ending with n , they break into exactly two classes in S_n for $n > 3$. \square

Proposition 2.34. $f(n) = f(n-1) + 2$ for $n > 3$.

Proof. We will calculate $f(n)$. Consider the non-avoiding permutations that have n as the first letter. These clearly break into $f(n-1)$ nontrivial classes because the first letter can be ignored. By Lemma 2.33, the remaining non-avoiding permutations fall into two classes. \square

Lemma 2.35. *If the n -th letter of an avoiding permutation in S_n is smaller than the $(n-1)$ -th one, then the permutation must be the decreasing permutation.*

Proof. Let $w \in S_n$ be an avoiding permutation. Whenever, for some $i \in \{2, 3, \dots, n-1\}$, the $(i+1)$ -th letter of w is smaller than the i -th one, it is clear that the i -th letter must be smaller than the $(i-1)$ -th one (since otherwise it would give a 132 or a 231 hit, contradicting the avoidance). By applying this observation iteratively, we see that if the n -th letter of w is smaller than the $(n-1)$ -th one, then the $(n-1)$ -th one must be smaller than the $(n-2)$ -th one, which in turn must be smaller than the $(n-3)$ -th one, etc.. Altogether, this yields that, if the n -th letter of w is smaller than the $(n-1)$ -th one, the permutation w must be decreasing. \square

Proposition 2.36. $g(n) = g(n-1) \cdot 2 - 1$ for $n > 3$.

Proof. We will calculate $g(n)$ for $n > 3$. Consider the avoiding permutations in S_n that have n as the first letter. There are clearly $g(n-1)$ of them because the first letter can be ignored. Now, consider the permutations that have n not as the first letter but are avoiding. Then, n must be the final letter because otherwise there is a 132 or 231 in the permutation. Ignoring the letter n , these permutations are avoiding permutations from S_{n-1} . The converse of this, however, is not true, but instead we have something more delicate: If we take an arbitrary permutation w from S_{n-1} and append n to the end, then the resulting permutation is still avoiding if and only if the $(n-1)$ -th letter of w is greater than the $(n-2)$ -th one. This requirement holds in all but one case (since Lemma 2.35, applied to S_{n-1} , shows that if the $(n-1)$ -th letter of an avoiding permutation in S_{n-1} is smaller than the $(n-2)$ -th one, then this permutation must be $(n-1)(n-2) \dots 1$). Hence, $g(n) = g(n-1) \cdot 2 - 1$ for $n > 3$. \square

Theorem 2.37. *The number of classes created in S_n under the $\{213, 132, 231\}$ -equivalence is $2^{n-2} + 2n - 4$ for $n > 3$.*

Proof. Let $n > 3$. Because $f(3) = 1$, Proposition 2.34 shows that $f(n) = 2 \cdot n - 5$. Because $g(3) = 3$, Proposition 2.36 shows that $g(n) = 2^{n-2} + 1$. Thus, the number of classes created in S_n is $2 \cdot n - 5 + 2^{n-2} + 1 = 2^{n-2} + 2n - 4$ for $n > 3$. \square

2.7. $\{213, 132, 231, 312\}$ -Equivalence.

Proposition 2.38. S_n breaks into three classes under the $\{213, 132, 231, 312\}$ -equivalence.

Proof. There are clearly two trivial classes, one containing the identity and one containing the descending permutation. We will now show that the remaining permutations are equivalent by inducting on n . For the base cases of $n = 3$ and $n = 4$, this is easily shown computationally. Let w be a non-avoiding permutation in S_n for $n > 4$ and assume that we have shown our claim for S_{n-1} . If the first $n - 1$ letters of w are avoiding, then applying the inductive hypothesis to the right-most $n - 1$ letters of w we can clearly reach a permutation that is non-avoiding in the first $n - 1$ letters. So, without loss of generality, w is non-avoiding in the first $n - 1$ letters. Applying the inductive hypothesis to the first $n - 1$ letters of x , we rearrange them as the identity from S_{n-1} except with the second and third to final letters swapped, yielding w' . Note that w' has n among its final $n - 1$ positions (actually in the final two positions) and is non-avoiding in its final $n - 1$ letters. Applying the inductive hypothesis to them, we rearrange them to be the identity from S_{n-1} with the second and third to final letters swapped, yielding w'' . Note that w'' is non-avoiding in the first $n - 1$ letters and has n as its final letter. Thus, applying the inductive hypothesis to the first $n - 1$ letters of w'' , we can reach the identity with 1 and 2 swapped. Hence, all non-avoiding permutations are equivalent and we are done. \square

3. DOUBLE REPLACEMENTS

In this section, we consider the classes created under replacement partitions of S_3 with two nontrivial parts, each of size two. Both the forgotten and Knuth relations are members of this family of relations; they are the main inspiration for this direction of work. We find the number of equivalence classes created in S_n in all but one of the unresolved cases (up to symmetry). In the final case, the $\{231, 132\}\{213, 312\}$ -equivalence, we provide computational data for the use of future authors. When convenient, we also calculate the size of the class containing the identity.

Surprisingly, class enumerations equal to those yielded by each of the Knuth relation and forgotten relation show up in our study of the $\{123, 132\}\{213, 312\}$ -equivalence and $\{123, 231\}\{213, 132\}$ -equivalence respectively. The reason for this is still largely a mystery.

The following subsections are concerned with one replacement partition each. In each subsection, the replacement partition K is to be understood to be the partition mentioned in the title of the subsection.

3.1. $\{123, 132\}\{213, 231\}$ -Equivalence.

Proposition 3.1. There are 2^{n-1} classes in S_n under the $\{123, 132\}\{213, 231\}$ -equivalence.

Proof. We can prove this by inducting on n . For the base case of S_3 , this is trivial. Assume as an inductive hypothesis, that the proposition holds for S_{n-1} . In S_n , the permutations beginning with n clearly fall into 2^{n-2} classes because the first letters of each of them can be ignored and each all are isolated from all other permutations. Permutations not beginning with n are equivalent to a permutation ending with n through repeated applications of $132 \rightarrow 123$ or $231 \rightarrow 213$.

If two permutations in S_n are separated by a transformation, then the two permutations in S_{n-1} obtained by removing the letter n from each of them are either equal or connected by the same transformation. Hence, if two permutations are equivalent, then the permutations obtained by removing n from each must be as well. For permutations ending with n , the converse holds as well; two equivalent permutations in S_{n-1} are still equivalent when n is appended to the end of each.

Thus, by the inductive hypothesis, there are exactly 2^{n-2} classes containing some permutations ending with n .

Hence, there are exactly 2^{n-2} classes containing permutations beginning with n , 2^{n-2} classes containing other permutations, and 2^{n-1} classes in total. \square

Corollary 3.2. *Let w be in $S_{n>1}$ and w' be w with n struck. The size of the equivalence class of w under the $\{123, 132\}\{213, 231\}$ -equivalence is the size of the class containing w' if n is in the first position in w and is the size times $n - 1$ of the class containing w' otherwise.*

Proof. In the proof of Proposition 3.1, we showed the corollary for the case where w begins with n , and that for w not beginning with n , w is equivalent to a w with n struck and then appended to the end. Since, from any permutation ending with n , n can be slid to any of $n - 1$ positions, the case of n not being in the first position of w is solved. \square

Corollary 3.3. *The number of permutations in S_n in the class containing the identity under the $\{123, 132\}\{213, 231\}$ -equivalence is $(n - 1)!$.*

Proof. As a result of Corollary 3.2, this falls from straightforward computation. \square

3.2. $\{312, 321\}\{123, 132\}$ -Equivalence.

Lemma 3.4. (a) *Letters to the right of 1 can be rearranged freely under the $\{312, 321\}\{123, 132\}$ -equivalence.* (b) *Letters to the right of n can be rearranged freely under the $\{312, 321\}\{123, 132\}$ -equivalence.*

Proof. We first prove (a). It is sufficient to prove that in S_n , all permutations of the form $1 \dots$ are equivalent to the identity. We will prove this by induction. The base case of S_3 is trivial. Assume that we have shown the result to hold for S_{n-1} . Let w be a permutation beginning with 1 in S_n . We will prove that w is reachable from the identity. We consider two cases:

- (1) The final letter of w is 2.
- (2) The final letter of w is not 2.

We deal with Case 2 first. In this case, the letter 2 is among the first $n - 1$ positions of w . Applying the inductive hypothesis to the first $n - 1$ letters of w , we rearrange the first $n - 1$ letters as the identity, yielding w' . Note that the first two letters of w' are 12. Thus, applying the inductive hypothesis to the final $n - 1$ letters of w' , we arrive at the identity.

Now, we consider Case 1. In this case, the final letter of w is 2. By the inductive hypothesis, we can rearrange the first $n - 1$ letters of w to be the identity with the final two letters swapped. We apply $321 \rightarrow 312$ to the final three letters, yielding a permutation starting with 1 but not ending with 2. Hence, we can proceed as in Case 2, and conclude that in Case 1, w is equivalent to the identity.

Now, we note that (b) falls from (a) because whenever two permutations x and y are equivalent under the $\{312, 321\}\{123, 132\}$ -equivalence, so are their complements. (The *complement* of a permutation $a_1 a_2 \dots a_n$ is defined as the permutation $(n + 1 - a_1)(n + 1 - a_2) \dots (n + 1 - a_n)$.) \square

Definition 3.5. *The proximum of a word w is the left-most of the largest and smallest letters in w .*

For example, the proximum of 519234 is 1.

Definition 3.6. *Let w be a word consisting of n pairwise distinct letters. We define the set W_w (of letters) in the following way.*

- If w has only one letter, then $W_w = \emptyset$.
- Otherwise, $W_w = \{u\} \cup W_f$, where u is the proximum of w , and f is the factor of w going from the first letter of w to u (inclusive).

For example, if $w = 453216$, then W_w contains 1 as well as the elements of W_{45321} which are 5 as well as the elements of W_{45} , which are just 4. So, W_w contains 1, 5, and 4.

Lemma 3.7. *If w and w' are separated by a single transformation, then $W_w = W_{w'}$.*

Proof. Assume that the lemma holds for smaller n , with an inductive base case of $n = 1$. Let w and w' in S_n be separated by a single transformation using the hit h . We notice that the relative order of 1 and n can never change during a transformation (since this would only be possible if 1 and n are in the same hit, but then they would have to be acting as the 1 and the 3 of that hit). Hence, either 1 is to the left of n in each of w and w' , or n is to the left of 1 in each of w and w' . We will only treat the former of these cases; the latter is completely analogous. Assume that 1 is to the left of n in w and w' . Let f and f' be the factors of w and w' respectively which begin with the first letter of w and w' respectively, and end with 1. If h only involves letters to the left of 1, then $W_w = W_f \cup \{1\} = W_{f'} \cup \{1\} = W_{w'}$ because $W_f = W_{f'}$ by the inductive hypothesis. If h only involves letters to the right of 1, then the lemma is trivial. If h involves 1, then the greatest letter to 1's left does not change under the transformation. Hence, the greatest letter in f and f' is the same. Since the letters to the left of that letter in f and f' clearly are static under the transformation, $W_f = W_{f'}$ and $W_w = W_f \cup \{1\} = W_{f'} \cup \{1\} = W_{w'}$. \square

Definition 3.8. *The origin permutation of a permutation w in S_n is a permutation beginning with the letters of W_w , in the same order that they appear in w , and then continued with the remaining letters in increasing order.*

Lemma 3.9. *Let w and w' in S_n be such that $W_w = W_{w'}$. Then, the origin permutations of w and w' are the same.*

Proof. As a consequence of the definition of W_w , for a given choice of letters to be in the set, there is exactly one possibility for the order of those letters in w . One can find this order in the following way. The right-most letter in W_w is 1 or n . Assume without loss of generality that W_w contains n . Then, the next right-most letter is the smallest letter in W_w , the next right-most letter is the second largest letter in W_w , the next right-most letter is the second smallest letter in W_w and so on. This shows that the order in which the letters of W_w appear in w is uniquely determined by the set W_w . But the origin permutation of w only depends on the set W_w and the order in which the letters of W_w appear in w . Hence, the origin permutation of w is uniquely determined by the set W_w , and the origin permutations of w and w' are the same. \square

Lemma 3.10. *Let $w \in S_n$ and w' be the origin permutation of w . Then, $w \equiv w'$ under the $\{312, 321\}\{123, 132\}$ -equivalence.*

Proof. Inductively assume the result holds in S_{n-1} with a trivial base case of S_1 . Let w be in S_n and w' be the origin permutation of w . Let j_1, j_2, \dots, j_k be the letters of W_w in the order that they appear in w from left to right. If $k = 1$, then we may apply Lemma 3.4 to w to reach w' . Otherwise, we may apply Lemma 3.4 to the factor going from j_1 to j_2 , and slide j_2 to be adjacent to j_1 , reaching x which begins with $j_1 j_2$. Applying the inductive hypothesis to the final $n - 1$ letters of x , we reach w' . \square

Proposition 3.11. *There are 2^{n-1} classes in S_n under the $\{312, 321\}\{123, 132\}$ -equivalence.*

Proof. By Lemma 3.7, each class in S_n gives rise to a set $W \subseteq \{1, 2, \dots, n\}$ such that $W = W_w$ for each w in the class, and such that W contains exactly one of 1 and n . Thus we obtain a map from the set of classes in S_n to $\{W \subseteq \{1, 2, \dots, n\} \mid W \text{ contains exactly one of 1 and } n\}$. This map is injective as a consequence of Lemma 3.9 and Lemma 3.10. We will now show that it is also surjective.

Let W be a set containing letters from 1 to n , including exactly one of 1 and n . Assume without loss of generality that W contains n . Let k be the size of W . We define the word f as the word of size k ending with the largest letter in W , following the smallest letter, following the next largest letter, following the next smallest letter, etc.. Then, we define w to be the permutation in S_n beginning with f and followed by the letters not in f in increasing order. Note that $W_w = W$. Therefore, there is a bijection from the set of classes in S_n to the set

$\{W \subseteq \{1, 2, \dots, n\} \mid W \text{ contains exactly one of 1 and } n\}$, and thus the number of classes in S_n is

$$|\{W \subseteq \{1, 2, \dots, n\} \mid W \text{ contains exactly one of 1 and } n\}| = 2^{n-1}.$$

□

Definition 3.12. Let $w \in S_n$. Let g_1, g_2, \dots, g_k be the letters of W_w in the order in which they appear in w from right to left. Then, g_i is a valley if it is less than each of its adjacent letters in the sequence g_1, g_2, \dots, g_k and is a peak if it is greater than its adjacent letters in the sequence g_1, g_2, \dots, g_k . If $k = 1$, then if $g_1 = 1$ it is a valley and if $g_1 = n$ it is a peak. If $n = 1$, we consider g_1 to be a valley.

Corollary 3.13. Let $w \in S_n$. Let g_1, g_2, \dots, g_k be the letters of W_w in the order that they appear in w from **right to left**. Let $j_i = g_i$ if g_i is a valley and $j_i = n - g_i$ if g_i is a peak. The class containing w is of size

$$\frac{(n-1)!}{\prod_{a=1}^{k-1} (j_a + j_{a+1})}$$

under the $\{312, 321\}\{123, 132\}$ -equivalence.

Proof. We want the number of permutations $w' \in S_n$ with $W_{w'} = W_w$ for a given $w \in S_n$. Recall that in such a situation, w and w' must have the letters of W_w in the same order. We will prove the result by inducting on k ; the base case of $k = 1$ falls from Lemma 3.4. Assume the corollary holds for smaller k . Let $w \in S_n$, g_1, \dots, g_k , and j_1, \dots, j_k be as stated. Note that g_k is the left-most letter of w and of w' . Also note that in this proof, a letter a is said to be *between* two letters b and c if and only if $\min\{b, c\} < a < \max\{b, c\}$. In particular, “between” refers to value, not position, and does not include the boundaries of the interval.

Let E be the set of all $w' \in S_n$ such that $W_{w'} = W_w$. Observe that E is the class of w . Let I be the union of $\{g_k\}$ with the set of all letters between g_{k-1} and g_k . An *I-purged word* is a word which contains every letter from $\{1, 2, \dots, n\} \setminus I$ exactly once (and no other letters). Let S be the set of all *I-purged words* x with $W_x = \{g_1, g_2, \dots, g_{k-1}\}$. Let T be the set

$$\{x \in S_{n-|g_k-g_{k-1}|} \mid W_x = \{g_i \mid 1 \leq i \leq k-1, g_i \text{ is a valley}\} \cup \{g_i - |g_k - g_{k-1}| \mid 1 \leq i \leq k-1, g_i \text{ is a peak}\}\}.$$

Let Y be the set of words of size n consisting of zeros and a single occurrence of each letter between g_k and g_{k-1} as well as g_k which is the first letter.

A bijection between S and T can be created by mapping $s \in S$ to its order permutation. Hence $|S| = |T|$.

Let $e \in E$. Let s be the word obtained by striking the letters between g_k and g_{k-1} as well as g_k from e . Let y be e except that each letter that is neither between g_k and g_{k-1} nor g_k is replaced with a zero. Then, we define $f : E \rightarrow S \times Y$ such that $f(e) = (s, y)$. Note that $s \in S$ because no g_i with $i < k$ is equal to g_k or any letter between g_{k-1} and g_k . Now we go in the other direction; let $s \in S$ and $y \in Y$ be arbitrary. Then, $g : S \times Y \rightarrow E$ is defined such that $g(s, y)$ is y except with each of the zeros of y replaced by the letters of s (in the order that they appear in s). Since f and g are inverses of each other, we have that $|E| = |S \times Y|$. Since $|S| = |T|$, we have $|E| = |T \times Y|$.

It's easy to see that $|Y| = (n-1)(n-2)\dots(n-|g_k - g_{k-1}| + 1)$. By the inductive hypothesis,

$$|T| = \frac{(n-|g_k - g_{k-1}| - 1)!}{\prod_{a=1}^{k-2} (j'_a + j'_{a+1})}$$

where $j'_{i \leq k-1}$ is $g_i = j_i$ if g_i is a valley and is $(n-|g_k - g_{k-1}|) - (g_i - |g_k - g_{k-1}|) = n - g_i = j_i$ if g_i is a peak. Therefore,

$$|E| = (n-1)(n-2)\dots(n-|g_k - g_{k-1}| + 1) \cdot \frac{(n-|g_k - g_{k-1}| - 1)!}{\prod_{a=1}^{k-2} (j_a + j_{a+1})}.$$

This simplifies to

$$|E| = \frac{(n-1)!}{\prod_{a=1}^{k-1} (j_a + j_{a+1})}.$$

□

Proposition 3.14. *The multiset of sizes of classes in S_n under the $\{123, 132\}\{213, 231\}$ -equivalence is the same as the multiset of sizes of classes in S_n under the $\{312, 321\}\{123, 132\}$ -equivalence.*

Proof. Let us consider the multiset of sizes of classes in S_n under the $\{123, 132\}\{213, 231\}$ -equivalence. By the induction in the proof of Proposition 3.1, it is easy to see that in each class in S_n , there is exactly one V-permutation.

Let v be a V-permutation. Consider the set L_v containing the letters to the left of 2 in v with the exception of the letter 1. Then, we may define $k-1$ as the size of L_v and $l_1 + 1, \dots, l_{k-1} + 1$ to be the letters in L_v in increasing order (thus from right to left in position in v). Then, by Corollary 3.2, the size of the class containing v is

$$\frac{(n-1)!}{\prod_{i=1}^{k-1} l_i}.$$

For a given choice of letters $l_1 < l_2 < \dots < l_{k-1}$ with $l_{k-1} \leq n-1$ and $l_1 > 1$, there are exactly two classes containing a V-permutation v such that L_v consists of the letters $l_i + 1$. (The two V-permutations are identical except with 1 and 2 swapped.)

Now, we consider the multiset of sizes of classes in S_n under the $\{312, 321\}\{123, 132\}$ -equivalence.

Let j_1, \dots, j_k be distinct letters from 0 to $n-1$ such that j_1 is 1 or 0, $j_2 > 1$ if $k > 1$, $j_i < j_{i-2}$ for $i > 2$, $j_k + j_{k-1} \leq n-1$ if $k > 1$, and $k \geq 1$. Let $g_i = j_i$ if i is odd and $j_1 = 1$ or if i is even and

$j_1 = 0$. Let $g_i = n - j_i$ otherwise. Then, since the set g_1, \dots, g_k contains exactly one of 1 or n , we know that there is exactly one class in S_n such that $W_w = \{g_1, \dots, g_k\}$ and that it is of size

$$\frac{(n-1)!}{\prod_{a=1}^{k-1} (j_a + j_{a+1})}.$$

Given a permutation w in S_n , we can construct g_1, \dots, g_k and j_1, \dots, j_k for some $k \geq 1$ as done in Corollary 3.13 so that they meet the stated restrictions and so that w is in a class of size

$$\frac{(n-1)!}{\prod_{a=1}^{k-1} (j_a + j_{a+1})}.$$

Thus, there is a bijection between sets $\{j_1, \dots\} \subseteq \{1, 2, \dots, n\}$ of size k for some $k \geq 1$ such that j_1 is 1 or 0, $j_2 > 1$ if $k > 1$, $j_i < j_{i-2}$ for $i > 2$, and $j_k + j_{k-1} \leq n - 1$ if $k > 1$ and classes in S_n which are of size

$$\frac{(n-1)!}{\prod_{a=1}^{k-1} (j_a + j_{a+1})}.$$

For such a set J , let L be the set containing $l_i = j_i + j_{i+1}$ for $i < k$. Since $j_i < j_{i+2}$, $l_i < l_{i+1}$ for $i < k - 1$. Since $j_{k-1} + j_k \leq n - 1$, l_i is from 1 to $n - 1$ for $i < k$. If L is not empty, then since $a_2 > 1$, $l_1 > 1$. For any set $L \subseteq \{2, 3, \dots, n - 1\}$ (possibly empty), there are two possible (and valid) J which could yield such a set L , one with $j_1 = 1$ and one with $j_1 = 0$. Thus, for a given choice of letters $l_1 < l_2 < \dots < l_{k-1}$ with $l_{k-1} \leq n - 1$ and $l_1 > 1$, there are exactly two classes of size

$$\frac{(n-1)!}{\prod_{i=1}^{k-1} l_i}.$$

Hence, the multiset of sizes of classes in S_n is the same for both equivalences. \square

3.3. $\{123, 231\}\{132, 321\}$ -Equivalence.

Proposition 3.15. *There are 2^{n-1} classes in S_n under the $\{123, 231\}\{132, 321\}$ -equivalence.*

Proof. We will show that the permutations with the letter 1 in an odd position (type 1) break into 2^{n-2} classes and that those with 1 in an even position (type 2) break into 2^{n-2} classes, thus completing the proof. We will do this by inducting on n with a trivial base case of S_3 . Assume that the result holds in S_{n-1} . We will first make three observations:

- (1) The two types of permutations being considered are clearly not equivalent;
- (2) From a permutation w , we can reach any permutation which is the same as w , except with 1 slid to an arbitrary position of the same position parity as it has in w .
- (3) If 1 takes part in a transformation, then the relative position of any two letters, neither of which are 1, does not change with the transformation.

Using these observations, we can complete our induction. Noting Observation (3), we can conclude that there are at least 2^{n-2} classes that each type breaks into because we can ignore the 1 and apply the inductive hypothesis. Hence, by observation (1), there are at least 2^{n-1} classes in total.

Let $w \in S_n$ and w' be w with 1 struck from the permutation. Let a, b, c be three adjacent letters in w' which form a hit. Then, in w , by observation (2), we can slide 1 so that a, b , and c are adjacent and form a hit⁴. Then, rearranging the hit, we can slide 1 back to its original position (observation (2)). Hence, if 1 is in position k in w , then for each w'' equivalent to w' , w is equivalent to w'' except with 1 inserted in position k . Therefore, by the inductive hypothesis, there are at most 2^{n-2} classes that each type falls into, and we are done. \square

Definition 3.16. A permutation is trivializable if it is equivalent to the identity.

Proposition 3.17. In S_n , under the $\{123, 231\}\{132, 321\}$ -equivalence, the number of trivializable permutations is $\lceil n/2 \rceil!^2$ if n is even and $\lceil n/2 \rceil!^2 / \lceil n/2 \rceil$ if n is odd. By symmetry, the same holds for the number of permutations equivalent to the descending permutation.

Proof. We will prove this by inducting on n , noting that for the base case of S_3 , the statement is trivial. Assume that the result holds for S_{n-1} . Noting observations (2) and (3) from the previous proof, when constructing from the identity a trivializable permutation w , one can always first construct the permutation which is w except with 1 slid to the first position, and then slide the 1 to its proper position in w . Thus, there are at most the number of trivializable permutations in S_{n-1} times $\lceil n/2 \rceil$ trivializable permutations in S_n . However, there are also at least $\lceil n/2 \rceil$ times as many trivializable permutations because from a given rearrangement of the final $n-1$ letters of the identity, the letter 1 can then always be slid to any one of $\lceil n/2 \rceil$ positions, resulting in a unique permutation. Noting an inductive base case of S_3 , the enumeration provided inductively holds. \square

3.4. $\{132, 312\}\{321, 213\}$ -Equivalence.

Lemma 3.18. There are $n-1$ trivial classes under the $\{132, 312\}\{321, 213\}$ -equivalence.

Proof. Because 312 and 213 are hits, we only need to consider Λ -permutations, that is permutations that increase to n and then decrease until the final letter. Since 321 is a hit, we only need to consider Λ -permutations which increase for at least the first $n-1$ letters. There are n such permutations because given an ending letter, the rest of such a permutation can be uniquely determined. Since the final possible hit is a 132, the final letter can be any letter except for $n-1$. Hence, there are $n-1$ avoiders. \square

Definition 3.19. We say that $w \in S_k$ is in T_k if and only if the letters of w other than 1 and 2 are in increasing order, and the letters 1 and 2 are adjacent and in 21 order.

For example, $T_5 = \{21345, 32145, 34215, 34521\}$.

Lemma 3.20. The elements of T_n form an equivalence class under the $\{132, 312\}\{321, 213\}$ -equivalence.

Proof. Let $w \in T_n$. Clearly, the only possible hits in w are $k21$ and $21j$ for some k and j with the actual letters 2 and 1. We can rearrange such a hit as $21k$ and $j21$ respectively. By the definition of T_n , the permutation reached is in T_n . Furthermore, sliding the letters 21 in this way, all of the permutations in T_n are reachable from each other. \square

Note that any three-letter factor abc with $a > b$ must form a hit. This will be useful in the proof of Lemma 3.21.

⁴This is because $a1bc \equiv abc1$ and $ab1c \equiv 1abc$.

Lemma 3.21. *Let $w \in S_n$ be non-avoiding and not in T_n . Then, w is equivalent to a permutation which begins with the letter n under the $\{132, 312\}\{321, 213\}$ -equivalence.*

Proof. Assume inductively that the proposition holds for S_k for $k < n$. Inductive base cases of $n \in \{3, 4, 5\}$ can easily be shown computationally. We can assume that the final $n - 1$ letters of w are non-avoiding. In fact, if the only hit is in the first three letters, then the hit is either 312 or 213 and rearranging it as either a 132 or 321 creates a hit to the right. We can also assume that the final $n - 1$ letters of w do not form a permutation in T_{n-1} . In fact, if they do, then they can be rearranged as the identity in S_{n-1} with 1 and 2 swapped. Then, the first four letters are either 1324, 2314, 3214, or $k213$ where $k > 3$. The third case will never happen because then w would have to be in T_n . In each remaining case there is a series of steps with which one can reach a permutation which is non-avoiding in the final $n - 1$ letters and for which the final $n - 1$ letters do not form a permutation in T_{n-1} : $1324 \rightarrow 1432 \rightarrow 4132$, $2314 \rightarrow 2431 \rightarrow 4231$, and $k213 \rightarrow k321 \rightarrow 32k1$.

So, we can apply the inductive hypothesis to the final $n - 1$ letters of w , placing n either in the first or second position of w , and obtaining w' . Because of the position of n , the first $n - 1$ letters of w' can not form a permutation in T_{n-1} for $n > 5$. Also, because of the position of n , the first 4 letters contain a hit. Thus, since $n > 5$, applying the inductive hypothesis to the first $n - 1$ letters, we can place n in the first position. \square

Proposition 3.22. *The number of classes containing permutations in $S_{n>3}$ that are non-avoiding and not in T_n equals the number of classes created in S_{n-1} under the $\{132, 312\}\{321, 213\}$ -equivalence.*

Proof. Let $f(n)$ be the number of classes created in S_n . By Lemma 3.21, a non-avoiding permutation in S_n that is not in T_n is reachable from a permutation beginning with n . Since $n > 3$, every permutation beginning with n is non-avoiding and not in T_n ; we just need to consider how many classes they fall into. Since we can ignore the first letter, permutations beginning with n fall into at most $f(n - 1)$ classes. In a hit involving n , no two letters that are not n can change relative order. Hence, the permutations beginning with n fall into at least $f(n - 1)$ classes, and we are done. \square

Proposition 3.23. *There are $\frac{n(n+1)}{2} - 2$ classes in $S_{n \geq 3}$ under the $\{132, 312\}\{321, 213\}$ -equivalence.*

Proof. We will show that with the step from S_{n-1} to S_n , n classes are added to the enumeration. Noting the base case of S_3 , this will inductively prove the proposition. Let $f(n)$ be the number of classes created in S_n . By Lemma 3.18 there are $n - 1$ trivial classes. By Lemma 3.20 and Lemma 3.21, there is one nontrivial class containing no permutations beginning with n . By Proposition 3.22, the remaining permutations fall into $f(n - 1)$ classes. So, $f(n > 3) = f(n - 1) + n$ and we are done. \square

3.5. $\{123, 231\}\{213, 132\}$ -Equivalence. Let $f(m)$ be the number of classes in S_m under the $\{123, 231\}\{213, 132\}$ -equivalence.

Lemma 3.24. *The permutations of a given parity which end with n and which do not begin with $n - 1$ are all equivalent under the $\{123, 231\}\{213, 132\}$ -equivalence.*

Proof. We will prove this inductively. The result is simple to show computationally for $n \leq 5$. Assume (as an induction hypothesis) that whenever $x \in S_{n-1}$ ends with n and does not begin with $n - 1$, x is equivalent either to the identity or the identity but with 1 and 2 swapped. Let w be a

permutation in S_n ending with n and not beginning with $n - 1$. Applying $123 \rightarrow 231$ and $213 \rightarrow 132$ repeatedly, we place $n - 1$ in position 2 and then n in position $n - 1$. Then, applying the inductive hypothesis to the left-most $n - 1$ letters, we rearrange them as the identity or the identity with 1 and 2 swapped depending on the parity of the permutation that they form, yielding w' . Note that the first letter of w' is 1, 2, or 3. Sliding n from the second to final position to the final position with an application of either $132 \rightarrow 213$ or $231 \rightarrow 123$, we then apply the inductive hypothesis to the right-most $n - 1$ letters of w' and get w'' . Note that w'' is the identity in S_n except with the first three letters possibly in any order. Finally, applying the inductive hypothesis to the first $n - 1$ letters of w'' , we reach the identity or the identity with 1 and 2 swapped in S_n . Hence, all such w of a given parity are equivalent. \square

Lemma 3.25. *A permutation beginning with a decreasing subsequence of consecutive values starting with n (that is, a permutation beginning with $n(n - 1)(n - 2) \dots (n - k + 1)$ for some $k \geq 0$) will always have that same decreasing subsequence at its start after transformations. In particular, the longest such subsequence never changes in length under the $\{123, 231\}\{213, 132\}$ -equivalence.*

Proof. It is sufficient to note that no hits begin with the largest letter in the hit. \square

Definition 3.26. *We define a k -hill to be k consecutively positioned and consecutively valued letters in a permutation in S_n , each of which is greater than all of the letters to its right in the permutation which are not n . A k -hill may also be called a hill if k is unknown.*

Definition 3.27. *In this subsection, we consider a permutation in S_n to satisfy the property C_k if either*

- *the permutation begins with $(n - 1)$ and begins with a k -hill not followed directly by n , or*
- *the permutation begins with a g -hill followed by the letters jn and then a $(k - g)$ -hill for some integer g between 0 and k inclusive and for some letter j , where $n - 1$ is the first letter of those in the mentioned hills (meaning that $n - 1$ is the first letter of the first hill if $g > 0$, and of the second hill if $g = 0$).*

Lemma 3.28. *If a permutation satisfies C_k , then any equivalent permutation will as well under the $\{123, 231\}\{213, 132\}$ -equivalence.*

Proof. Let w be a permutation satisfying property C_k . We will show that after a transformation, the resulting permutation will still satisfy C_k . There are two cases for w :

- (1) In the first case, w begins with a k -hill not directly followed by n . If a transformation does not involve any letters in the k -hill, then it is easy to see that the resulting permutation still starts with a k -hill not directly followed by n . If a transformation does involve letters in the k -hill, the transformation must be $(n - k)jn \rightarrow jn(n - k)$ for some j . This results in a permutation which starts with a g -hill followed by the letters jn and then a $(k - g)$ -hill for $g = k - 1$. Hence, the yielded permutation still satisfies C_k .
- (2) In the second case, w begins with a g -hill followed by the letters jn and then a $(k - g)$ -hill for some integer g and letter j . There are three subcases for a hit involved in a transformation. The hit can contain only letters to the right of n (subcase 1); contain letters only to the left of n (subcase 2); contain n (subcase 3). In subcase 1, by Lemma 3.25, after a transformation using the hit, the resulting permutation will still satisfy C_k . Subcase 2 can never happen because the letters to the left of n are in strictly decreasing order. In subcase 3, the hit is either 213 or 132 where 3 is the actual n , 1 is the letter referred to as j in

the definition of C_k , and 2 is a letter in one of the two hills mentioned in the definition of C_k . If the hit is used in an application of $213 \rightarrow 132$, the resulting permutation must begin with a g -hill followed by the letters jn and then a $(k - g)$ -hill for one lesser g than before and the same j . If the hit is used in an application of $132 \rightarrow 213$, then either the resulting permutation begins with a g -hill followed by the letters jn and then a $(k - g)$ -hill for one greater g than before and the same j , or it begins with a k -hill not followed directly by the letter n . Hence, the resulting permutation satisfies C_k . □

Proposition 3.29. *There are $n^2 - 3n + 4$ classes in S_n under the $\{123, 231\}\{213, 132\}$ -equivalence.*

Proof. Consider the permutations that begin with n for $n > 3$. They clearly fall into $f(n - 1)$ classes because we can ignore the n . From a permutation not beginning with n , n can be moved to the final position through repeated applications of $231 \rightarrow 123$ and $132 \rightarrow 213$. Hence, each of the remaining uncounted equivalence classes contains a permutation which ends with n . By Lemma 3.24, permutations ending with n and not beginning with $n - 1$ fall into two classes, divided by parity. Hence, we need only consider permutations of the form $(n - 1) \dots n$ which are not reachable from any permutation ending with n but not beginning with $n - 1$. By Lemma 3.28, these are exactly the permutations of the form $(n - 1) \dots n$ since such permutations each satisfy C_1 and permutations not beginning with $n - 1$ and ending with n do not.

Let x and y in S_n both begin with $n - 1$ and end with n . If x and y are of different parity, then clearly they are not equivalent. Assume that x and y are the same parity. Let k be the largest k such that x begins with a k -hill and k is less than $n - 1$ (when $k = n - 1$, the k -hill is followed by n). Let k' be the largest k' such that y begins with a k' -hill and k' is less than $n - 1$.

If $k \neq k'$, then by Lemma 3.28, x and y are not equivalent.

Otherwise, assume that $k' = k$. If $k = n - 2$ or $k = n - 3$, then it is easy to check that x must equal y . Thus, there is a total of two classes for the cases of $k = n - 2$ and $k = n - 3$.

Otherwise, assume that $k < n - 3$. Then, x and y begin with the same first k letters, the remaining letters do not begin with $n - k$ (because then x or y would begin with a $(k + 1)$ -hill), and they do end with n . Hence, we can apply Lemma 3.24 to those remaining letters to conclude that x and y are equivalent.

Thus, x is equivalent to y exactly when they are of the same parity and they satisfy C_j for the same j . When the highest such j is between 1 and $n - 4$ inclusive, there are $2(n - 4)$ resulting classes. As mentioned previously, there is one additional class for each of the cases where the highest such j is $n - 2$ and $n - 3$. Note also that it is impossible for a permutation to satisfy C_{n-1} . This means that permutations beginning with $n - 1$ and ending with n fall into $2(n - 4) + 2$ classes. Recall that permutations beginning with n fall into $f(n - 1)$ classes, and that the permutations ending with n but not beginning with $n - 1$ fall into two more classes. So, $f(n > 3) = f(n - 1) + 2(n - 4) + 2 + 2$, $f(3) = 4$. This recursion inductively implies that there are $n^2 - 3n + 4$ classes. □

Proposition 3.30. *The number of elements in the class containing the identity in S_n is $((n - 2)(n - 1)!)/2$ for $n > 3$ under the $\{123, 231\}\{213, 132\}$ -equivalence.*

Proof. Consider permutations in $S_{n>3}$. By Lemma 3.28, no permutations equivalent to the identity satisfy C_1 . A permutation w fails to satisfy C_1 exactly when w begins with neither $n - 1$ nor $jn(n - 1)$ for any j , or when w begins with $(n - 1)n$. If a permutation is equivalent to the identity, then it also can not begin with n . There are $(n - 2)(n - 1)! - (n - 2)!$ permutations not beginning with $n - 1$, $jn(n - 1)$, or n . There are $(n - 2)!$ permutations beginning with $(n - 1)n$. It is easy to

see that half of the permutations counted in each of these enumerations are odd and half are even. If we consider only the even ones (noting that transformations do not change parity), there are at most $(n-2)(n-1)!/2$ permutations that can be equivalent to the identity. From an even permutation not starting with n or $n-1$ and not starting with three letters of the form $jn(n-1)$ for some j , through repeated applications $132 \rightarrow 213$ and $231 \rightarrow 123$, one can reach an even permutation ending with n and not starting with $n-1$ which by Lemma 3.24 is equivalent to the identity. Similarly, all even permutations of the form $(n-1)n\dots$ are equivalent to the identity. Thus, the number of elements in the class containing the identity is $(n-2)(n-1)!/2$ for $n > 3$. \square

3.6. $\{123, 132\}\{231, 312\}$ -Equivalence.

Lemma 3.31. *The class containing the identity contains exactly the permutations beginning with 1 under the $\{123, 132\}\{231, 312\}$ -equivalence.*

Proof. We will prove this inductively. In S_3 , the lemma is trivial. Assume the lemma holds in S_{n-1} . Let w be a permutation beginning with 1 in S_n . We rearrange the first $n-1$ letters of w as the identity (using the inductive hypothesis), resulting in w' . If w' ends with 2 (i.e., does not have 2 in the second position), then through repeated applications of $231 \rightarrow 312$ (where the actual two is the lowest letter in the hit), we move it to the third position; then, applying $132 \rightarrow 123$ to the first three letters, we arrive at a permutation starting with 12. Thus, w is equivalent to a permutation starting with 12, and applying the inductive hypothesis to the final $n-1$ letters of such a permutation, we reach the identity. \square

Lemma 3.32. *Every permutation is reachable from a permutation of the form $w1\dots$ where w is a word and $w1$ is avoiding. Note that this means $w1$ is decreasing. Thus, by Lemma 3.31, all permutations are reachable from a V-permutation under the $\{123, 132\}\{231, 312\}$ -equivalence.*

Proof. This is trivial for $n = 3$. Assume that the result holds for $n-1$. If 1 is not in the final position of a permutation, then using the inductive hypothesis, we are done. Let x be a permutation ending with 1. By the inductive hypothesis (on the first $n-1$ letters), x is reachable from a permutation which is a V-permutation in the first $n-1$ letters and ends with 1. Let y be such a permutation. If 2 is not the second to final letter, then the final three letters can be rearranged as $231 \rightarrow 312$ yielding a permutation not ending with 1 (a case for which we have already shown that the lemma holds). If 2 is the second to final letter, then the entire permutation is decreasing and we have thus reached a V-permutation. \square

Lemma 3.33. *A V-permutation of the form $j\dots$ and a V-permutation of the form $k\dots$ where $k \neq j$ can not be equivalent under the $\{123, 132\}\{231, 312\}$ -equivalence.*

Proof. The largest letter to the left of the 1 does not change under the transformations considered. \square

Definition 3.34. *Let E_n be the set of permutations in S_n which are the descending permutation except with two consecutive letters swapped.*

For example, $E_4 = \{3421, 4231, 4312\}$.

Lemma 3.35. *The permutations in E_n form an equivalence class under the $\{123, 132\}\{231, 312\}$ -equivalence.*

Proof. We will first show that the set E_n is closed under the transformations considered. Let $w \in E_n$. There are at most two hits in w , a 231 hit and a 312 hit, each using the pair of letters in increasing order in w . After a $231 \leftrightarrow 312$ rearrangement, we have simply swapped which pair of consecutive letters is in increasing order, resulting in a permutation in E_n .

What remains to be shown is that all of the permutations in E_n are equivalent. Through repeated applications of the rearrangement $312 \rightarrow 231$, we can go from the descending permutation with 1 and 2 swapped to each of the permutations in E_n . Hence, the permutations in E_n form an equivalence class. \square

Lemma 3.36. *Each V-permutation of the form $\dots 21 \dots$ is not equivalent to any other V-permutations under the $\{123, 132\}\{231, 312\}$ -equivalence.*

Proof. Let w be a V-permutation of the form $j \dots 21 \dots$. Since no hit can begin with 21, the letters to the left of 1 are in decreasing order, and hits beginning with 1 can only be rearranged to form other hits beginning with 1, we have that all permutations equivalent to w must have the letters leading up to 1 exactly the same as in w . So, w can not be equivalent to any V-permutations other than itself. \square

Lemma 3.37. *Each V-permutation of the form $\dots 31 \dots$ is not equivalent to any other V-permutations under the $\{123, 132\}\{231, 312\}$ -equivalence.*

Proof. Let w be a V-permutation of the form $\dots 31 \dots$ with k letters to the left of 1. We posit that the letters to the left of 1 in each permutation equivalent to w are either exactly the same as in w and are in decreasing order, or form a permutation in E_{k+1} using the same letters as are to the left of 1 in w as well as the letter 2. Assume inductively that this is true for permutations which are j transformations away from w , with an inductive base case of $j = 0$. We will show that the claim holds for permutations $j + 1$ transformations away from w , completing the proof. Let $x \in S_n$ be j transformations away from w and $x' \in S_n$ be one transformation away from x . There are two cases.

- The letters to the left of 1 in x are in decreasing order and are the same letters as in w . If x' is reached using a hit to the right of 1, then the claim is trivial. Otherwise, the rearrangement must be $312 \rightarrow 231$ using the actual letters 1, 2, 3. Then, the letters to the left of 1 in x' form a permutation in E_{k+1} and are exactly the letters to the left of 1 in w in addition to the letter 2.
- The letters to the left of 1 in x form a permutation in E_{k+1} using the same letters as are to the left of 1 in w in addition to the letter 2. If x' is reached by a transformation using letters only to the right of 1, then the claim is trivial. If x' is reached by a transformation using letters only to the left of 1, then by Lemma 3.35, the claim holds. If x' is reached using a transformation using the letter 1, then the transformation is $231 \rightarrow 312$ using the actual letters 1, 2, 3, and the letters before 1 in x' are in decreasing order and are the same as in w . \square

Lemma 3.38. *V-permutations of the form $j \dots k1 \dots$ as well as the V-permutation of the form $j1 \dots$ where $j \geq 4$ is fixed and k is any letter ≥ 4 are equivalent under the $\{123, 132\}\{231, 312\}$ -equivalence.*

Proof. By computation, one can show that $51 \dots$ is reachable from $541 \dots$ (we will refer to this as the hypothesis): $51234 \rightarrow 51324 \rightarrow 35124 \rightarrow 35142 \rightarrow 34512 \rightarrow 35412 \rightarrow 35241 \rightarrow 52341 \rightarrow 52413 \rightarrow 54123$. Consider an arbitrary V-permutation w of the form $j \dots k1 \dots$ for some j and

for some $k \geq 4$. By the hypothesis, this is reachable from a permutation starting with the same decreasing factor which ends with 1 except with k excluded. By Lemma 3.31, this is equivalent to the V-permutation starting with the same factor which terminates with 1 as w does except with k excluded. Thus, all such V-permutations are equivalent to the V-permutation $j1 \dots$ and we are done. \square

Proposition 3.39. *In S_n , there are $3 \cdot 2^{n-3} + n - 2$ classes for $n > 5$ under the $\{123, 132\}\{231, 312\}$ -equivalence.*

Proof. Consider the classes in S_n where $n > 5$. By Lemma 3.32, we simply have to count the number of classes that the V-permutations break into. By Lemma 3.37 there are 2^{n-2} classes with only V-permutations of the form $j \dots 21 \dots$ and 2^{n-3} classes with only V-permutations of the form $j \dots 31 \dots$. By Lemma 3.38 there are $n - 3$ classes with V-permutations of the form $j \dots k1 \dots$ where $k > 3$. Finally, by Lemma 3.31, there is 1 class containing only the remaining permutations, those of the form $1 \dots$. So, there are $2^{n-2} + 2^{n-3} + n - 3 + 1 = 3 \cdot 2^{n-3} + n - 2$ classes in S_n . \square

3.7. $\{123, 132\}\{213, 321\}$ -Equivalence.

Definition 3.40. *We call a permutation w bushy-tailed if the following is true. Let x be the factor of w containing the letters in w to the left of 1 as well as 1. In x , the letters in odd positions are in decreasing order; the letters in even positions are in decreasing order; and each of the letters of the same position parity as 1 is less than the adjacent letter to its right and the one to its left (if there is such a letter).*

Note that bushy-tailed is a property of the letters up to 1 only.

Lemma 3.41. *Permutations of the form $1 \dots$ are equivalent. Thus, there are $(n - 1)!$ elements in the class containing the identity under the $\{123, 132\}\{213, 321\}$ -equivalence.*

Proof. We will prove this inductively. In S_3 , the lemma is trivial. Assume the lemma holds in S_{n-1} and that any permutation starting with 1 is equivalent to the identity in S_{n-1} . Let x be a permutation starting with 1 in S_n . We rearrange the first $n - 1$ letters of x as the identity using the inductive hypothesis, and arrive at x' . If x' ends with 2, then through repeated $321 \rightarrow 213$ rearrangements (where 2 is the lowest letter in the hit), we move it to the third position, and then applying $132 \rightarrow 123$ to the first three letters, we arrive at a permutation starting with 12. Thus, x is reachable from a permutation starting with 12, and applying the inductive hypothesis to the final $n - 1$ letters of such a permutation, we reach the identity. \square

Lemma 3.42. *Every permutation is equivalent to a bushy-tailed permutation of the form $w1 \dots$ where w is a word under the $\{123, 132\}\{213, 321\}$ -equivalence.*

Proof. Note that for $w1$ to form a bushy-tailed permutation is equivalent to $w1$ avoiding 123, 132, 321, and 213. (This is not actually very useful, but is interesting to note.) The lemma is trivial in S_3 . Assume inductively that the result holds in S_{n-1} . If 1 is not in the final position of a permutation in S_n , then using the inductive hypothesis, we are done. Let x be a permutation ending with 1. By the inductive hypothesis (applied to the first $n - 1$ letters), x is equivalent to a permutation which is a bushy-tailed permutation in the first $n - 1$ letters and ends with 1. Let y be such a permutation. There are three cases for y , as follows.

- The second to final letter of y is 2. Applying $321 \rightarrow 213$ to the final three letters, we reach a permutation where 1 is not in the final position, an instance of a case we have already covered.

- The third to final letter of y is 2. Then, the final four letters of y either form the permutation 3241 or the permutation 4231. In the first subcase, we apply $3241 \rightarrow 4321 \rightarrow 4213$ to reach a permutation where 1 is not in the final position. In the second subcase, y is a bushy-tailed permutation.
- The letter 2 is before the third to final letter in y . Applying Lemma 3.41 to the letters to the right of 2 and left of 1 along with 2, we rearrange the two letters to the immediate left of 1 such that the final three letters form the hit 321. Applying $321 \rightarrow 213$, we reach a permutation where 1 is not in the final position.

□

Definition 3.43. *Let w be a bushy-tailed permutation. A letter in w which is to the left of 1 or is 1 is a w -starter.*

Lemma 3.44. *Let y be a bushy-tailed permutation. Let x be a permutation equivalent to y under the $\{123, 132\}\{213, 321\}$ -equivalence. In x , the y -starters are in the same relative order as they are in y (property 1). Furthermore, ignoring each other, they are all left to right minima (property 2). Finally, in x , each y -starter other than 1 (the final y -starter) is less than each letter between it and the next in position y -starter (property 3).*

Proof. Let x be a permutation equivalent to a bushy-tailed permutation y which satisfies properties 1, 2, and 3. We will show that any single rearrangement using a hit in x will result in a permutation for which the properties still are satisfied. Assume a rearrangement exists which swaps two y -starter letters j and k (where k is without loss of generality, to the right of j in x). There are three cases for such a rearrangement.

- The rearrangement is $213 \rightarrow 321$. Due to property 1, not all three letters can be y -starters. So, in order for j and k to be swapped, j must play the role of 3 in the hit, and k must play the role of 2 or 1. Because the remaining letter is not a y -starter, property 2 is not satisfied in x , a contradiction.
- The rearrangement is $321 \rightarrow 213$. Clearly, not all three letters can be y -starters. So, j must play the role of 3 and k must play the role of either 2 or 1. If k plays the role of 1, then property 3 is not satisfied in x , a contradiction. If k plays the role of 2, then property 3 is again not satisfied in x , a contradiction.
- The rearrangement is either $123 \rightarrow 132$ or $132 \rightarrow 123$. Since not all three letters can be y -starters (property 1), in both cases, property 2 is not satisfied in x , a contradiction.

Assume a rearrangement exists which results in a permutation not satisfying property 2. There are two possible cases for the rearrangement.

- The rearrangement is $132 \rightarrow 123$ where the letters playing the roles of 1 and 3 are y -starters and the letter playing the role of 2 is not (the letter playing the role of 1 is a y -starter by property 2; by property 1, not all three letters can be y -starters, and for property 2 to be broken by the transformation, 3 must be a y -starter). However, because 2 is not a y -starter, property 3 is not satisfied in x , a contradiction.
- The rearrangement is $321 \rightarrow 213$ where the letter playing the 3 is a y -starter and one of the remaining letters is not. However, then property 3 is not satisfied in x , a contradiction.

Assume a rearrangement exists which results in a permutation not satisfying property 3. Let j and k be the first and second y -starter respectively which have a letter less than j being brought between them by the rearrangement. Such a letter can not come from the left of j by property 2.

Such a letter can not come from the right of k because then k would be playing the role of 1 in $213 \rightarrow 321$ and property 3 could not be satisfied in x , a contradiction. Thus, such a rearrangement can not exist. \square

Note that by Lemma 3.44, no two distinct bushy-tailed permutations with the letters to the right of 1 in increasing order are equivalent.

Proposition 3.45. *The number of classes in S_n is the sum of the first $n - 1$ Motzkin numbers under the $\{123, 132\}\{213, 321\}$ -equivalence.*

Proof. By Lemmas 3.44 and 3.42, the number of classes in S_n is exactly the number of bushy-tailed permutations with the letters to the right of 1 in increasing order. We will call a bushy-tailed permutation with the letters to the right of 1 in increasing order a *beaver-permutation*. What remains is to count the number of beaver-permutations. Let $f(n)$ be the number of classes in S_n (and thus the number of beaver-permutations). Then, the number of beaver-permutations in S_n with n not to the left of 1 is clearly $f(n - 1)$ because we can just append n to the end of each beaver-permutation from S_{n-1} . Consider a beaver-permutation w with n to the left of 1. Let k be the number of letters to the left of 1 in w . Note that k can be any value greater than 0 and less than n . There are $\binom{n-2}{k-1}$ ways to choose these letters without yet fixing their order.

Let $g(k)$ be the number of possible arrangements for a given set of letters to the left of 1 in a beaver-permutation. Assume that k is even. Then, each arrangement of the letters corresponds with a $2 \times k/2$ standard Young tableau (in English notation); reading from **right to left** in the permutation (starting with the letter preceding 1), we fill the column of the tableau, then the second, and so on⁵. The number of $2 \times k/2$ standard Young tableaux is $C_{k/2}$ where C_n denotes the n -th Catalan number⁶. Hence, $g(k) = C_{k/2}$ for even k . If k is odd, then n is forced in the first position, and ignoring it, we see that $g(k) = g(k - 1) = C_{(k-1)/2}$. So,

$$\begin{aligned} f(n) &= f(n-1) + \sum_{k=1}^{n-1} \binom{n-2}{k-1} \cdot C_{\lfloor k/2 \rfloor} \\ &= f(n-1) + \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot C_{\lfloor k/2 \rfloor} \\ &= f(n-1) + \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-2}{2k} \cdot C_k + \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-2}{2k-1} \cdot C_k. \end{aligned}$$

Recall that M_n is the number of Motzkin n -paths, paths from $(0, 0)$ to $(n, 0)$ in the grid $\mathbb{N} \times \mathbb{N}$ using only steps $U = (1, 1)$, $F = (1, 0)$ and $D = (1, -1)$. We will enumerate M_n . Consider the paths where F is used exactly $n - j$ times for a given j . The uses can be distributed in any of $\binom{n}{n-j} = \binom{n}{j}$ ways. The remaining j steps must consist of uses of U and D . If k is even, then there are clearly $C_{k/2}$ arrangements for these steps. Note that k can not be odd. So,

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k.$$

⁵i.e., if the letter 1 is the g -th letter, then the i -th column contains the $g - (2i - 1)$ -th and $g - 2i$ -th letter.

⁶This is well known. It is also easy to prove by bijecting such standard Young tableaux with Catalan paths by means of traversing the cells of the Young tableau by increasing letter and using each letter in the bottom row to represent a side step and each letter in the top row to represent a down step.

Hence,

$$M_{n-2} = \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-2}{2k} C_k.$$

So,

$$f(n) = f(n-1) + M_{n-2} + \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-2}{2k-1} \cdot C_k.$$

Consider $\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-2}{2k-1} \cdot C_k$. We will show that this is the difference between M_{n-2} and M_{n-1} .

Note that M_{n-2} is the number of Motzkin $n-1$ -paths that go through point $(n-2, 0)$. We will now enumerate those Motzkin $n-1$ -paths which do **not** go through $(n-1, 0)$. (There are $M_{n-1} - M_{n-2}$ such paths.) In such a path, excluding the final step, there must be one more use of U than of D . So, the number of uses of both combined must be odd. Let that value be $2h-1$. We pick $2h-1$ of the first $n-2$ steps to be the steps which are not F . These $2h-1$ steps, with a D appended to the end must form a Catalan path of length $2h$. Thus, there are C_h possibilities for the arrangements of these steps. Hence,

$$M_{n-1} - M_{n-2} = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-2}{2k-1} \cdot C_k.$$

So, $f(n) = f(n-1) + M_{n-1}$. Inductively, $f(n)$ is the sum of the first $n-1$ Motzkin numbers. \square

3.8. $\{123, 132\}\{213, 312\}$ -Equivalence. In this subsection, $f(n)$ be the number of classes that S_n breaks into under the $\{123, 132\}\{213, 312\}$ -equivalence.

Definition 3.46. We call the root-permutation (or root) of a permutation w the permutation obtained from w by applying $123 \rightarrow 132$ and $213 \rightarrow 312$ repeatedly to the hit ending with n in order to bring n to the first or second position (We will call this sliding n).

We first make two observations. When sliding n in x , each transformation only uses letters that were to the left of n before the transformation (observation 1). Additionally, the letters to the left of n after the transformation were static in the transformation (observation 2).

Lemma 3.47. Let x and y be two permutations reachable from each other through a single transformation under the $\{123, 132\}\{213, 312\}$ -equivalence. Let x' and y' be the root-permutations of x and y respectively. Then, either x' and y' are each of the form $n \dots$, or they are each of the form $jn \dots$ for the same j .

Proof. Let x, y, x', y' be such permutations. Consider the transformation connecting x to y . There are the following cases.

- The transformation involves a hit using letters only to the right of n . In this case, the series of transformations from x to x' is the same as the series from y to y' .
- The transformation uses a hit containing n . In this case, x' and y' are clearly the same; the series of transformations used to obtain from x' from x is the same as the one used to obtain y' from y except plus or minus an extra transformation.
- The transformation uses a hit involving only letters to the left of n . There are two subcases.

- (1) The rearrangement connecting y to x is $123 \rightarrow 132$ (without loss of generality, in that order). Let j, k , and r be the letters playing the roles of 1, 2, and 3 respectively, and w be the factor containing the letters to the left of the hit. In x , we may slide n within the factor going from k to the end of x to reach a permutation of the form either $wjkn\dots$ or $wjn\dots$ (observation 2). In the first case, if we continue sliding n , we reach a permutation of the form $wjn\dots$. In y , we may slide n within the factor going from r to the end of y to reach a permutation of the form either $wjrn\dots$ or $wjn\dots$ (observation 2). In the first of the two cases, if we continue sliding n , we reach a permutation of the form $wjn\dots$. Hence, noting observation 1, completing the sliding of n in x and y will result in either a permutation beginning with n in both cases or a permutation of the form $jn\dots$ for the same j in both cases.
- (2) The rearrangement connecting y to x is $213 \rightarrow 312$ (without loss of generality, in that order). Let j, k , and r be the letters playing the roles of 1, 2, and 3 respectively, and w be the factor containing the letters to the left of the hit. In x , we may slide n within the factor going from r to the end of x to reach a permutation of the form either $wkjr n\dots$ or $wk jn\dots$ (observation 2). In each case, we may continue sliding n to reach a permutation of the form $wn\dots$ (we go $wkjr n\dots \rightarrow wk jn\dots \rightarrow wn\dots$ and $wk jn\dots \rightarrow wn\dots$ respectively). In y , we may slide n within the factor going from k to the end of y to reach a permutation of the form either $wr jkn\dots$ or $wr jn\dots$ (observation 2). In each case, we may continue sliding n to reach a permutation of the form $wn\dots$ (we go $wr jkn\dots \rightarrow wr jn\dots \rightarrow wn\dots$ and $wr jn\dots \rightarrow wn\dots$ respectively). Hence, noting observation 1, completing the sliding of n in x and y will result in either a permutation beginning with n in both cases or a permutation of the form $jn\dots$ for the same j in both cases.

□

Lemma 3.48. *Let $x, y \in S_n$ be permutations separated by a single transformation under the $\{123, 132\}\{213, 312\}$ -equivalence. Let x', y' be the root-permutations of x and y . Then, x' is reachable from y' using hits only to the right of n under the $\{123, 132\}\{213, 312\}$ -equivalence.*

Proof. We use as a base case (the “hypothesis”) that this holds in S_5 . One can also easily show that the result holds in $S_{\leq 5}$. Assume that $n > 5$. Let x, y, x', y' be as described in S_n . There are three cases for the hit used to reach x from y :

- (1) The hit uses only letters to the right of n . Then x' and y' are reachable from each other with the same transformation (observation 1).
- (2) The hit contains n . Then $x' = y'$ because the transformation is undone in sliding n to the left in one of x or y .
- (3) The hit uses only letters to the left of n . Then, n can be slid within the letters to the right of the hit to reach a and b from x and y respectively. In both a and b , n is either immediately to the right of the hit, or to the right of the hit and separated from it by one letter. Consider the factors of each a and b containing the hit and the two letters to its right, w and w' respectively. By the hypothesis, the root-permutations of w and w' are reachable from each other using only hits containing letters to the right of n . Hence, sliding n within w and w' of a and b respectively, we reach a' and b' such that the letters to the left of n form exactly the same factor, and the letters to the right of n form permutations that are equivalent in a' and b' . So, the root-permutations of a' and b' are reachable from each other by using hits

only containing letters to the right of n (observation 1). Noting that the root-permutations of a and b are x' and y' , x' is reachable from y' using hits only containing letters to the right of n . □

Proposition 3.49. $f(n \geq 3) = f(n-1) + (n-1) \cdot f(n-2)$.

Proof. Sliding n to the left in a given permutation, we can reach a permutation either of the form $jn \dots$ or $n \dots$ for any permutation. By Lemma 3.47, permutations of those forms are not equivalent for distinct j . By Lemma 3.48, permutations of the form $jn \dots$ fall into $f(n-2)$ classes for a given j and permutations of the form $n \dots$ fall into $f(n-1)$ classes. So, $f(n \geq 3) = f(n-1) + (n-1) \cdot f(n-2)$. □

3.9. $\{123, 321\}\{213, 231\}$ -Equivalence.

Lemma 3.50. *All non-avoiding permutations are equivalent in S_n for $n > 5$ under the $\{123, 321\}\{213, 231\}$ -equivalence.*

Proof. We will prove by inducting on n that every non-avoiding permutation is equivalent to the identity, noting that the base case of S_6 can be checked computationally. Let w be a permutation in S_n . Without loss of generality, the first $n-1$ letters of w are non-avoiding (otherwise, the final $n-1$ letters are non-avoiding and the proof goes similarly). Applying the inductive hypothesis to the first $n-1$ letters of w , and then to the final $n-1$ letters of w (noting that they are then non-avoiding), we reach a permutation which is non-avoiding in its first $n-1$ letters and ends with n . Applying the inductive hypothesis to the first $n-1$ letters of such a permutation, we are done. □

Proposition 3.51. *There are 3 classes in S_n under the $\{123, 321\}\{213, 231\}$ -equivalence for $n > 5$.*

Proof. By Lemma 3.50, there is one class containing the non-avoiders. Any avoiders must consist of only 312 and 132 patterns. Let w be an avoiding permutation. Clearly the letters of one position parity in w are in decreasing order and the letters of the other position parity are in increasing order. Furthermore, the position parity which has its letters in decreasing order also has all of its letters greater than those of the other position parity. So, w can be one of exactly two permutations (determined by which position parity contains letters in decreasing order). Hence, there are three equivalence classes. □

Corollary 3.52. *The class containing the identity in S_n contains $n! - 2$ elements for $n > 5$ under the $\{123, 321\}\{213, 231\}$ -equivalence.*

Proof. This is because all $n! - 2$ of the non-trivial permutations are reachable from the identity. □

3.10. $\{123, 321\}\{132, 231\}$ -Equivalence.

Definition 3.53. *The fall of a permutation $x \in S_n$ is the set $\{k \in \{1, 2, \dots, n\} \mid \text{each letter greater than } k \text{ in } x \text{ has the same position parity as } k\}$.*

Proposition 3.54. *Let x be a permutation containing letters a and b separated by one letter. Let k be the greatest letter of different position parity than a and b . Assume that a and b are not both in the fall of x . Then, x is equivalent to the permutation which is identical to x except with a and b swapped under the $\{123, 321\}\{132, 231\}$ -equivalence.*

Proof. Let x be such a permutation with letters a , b , and k as described. Note that $k > a$ or $k > b$ because a and b are not both in the fall of x . We will prove the proposition by inducting on n . Assume, inductively, that the result holds for lesser n with a trivial base case of $n = 3$. Due to symmetries of the relation, we can assume without loss of generality that a and k are to the left of b . If k is between a and b , then we simply swap them. Otherwise, consider the factor of x beginning with k and ending with b . If it is of length less than n , then applying the inductive hypothesis to it, we are done. If it is of length n , then n must be even. Thus, we can slide the letter 1 to the first or final position by repeatedly swapping it with the letter two positions to its left or right respectively. Applying the inductive hypothesis to the remaining $n - 1$ letters, we can swap a and b . Then, sliding 1 back to its previous position by repeatedly swapping it with the letter two positions to its right or left respectively, we reach the desired permutation. \square

Proposition 3.55. *The set of permutations equivalent to a permutation x under the $\{123, 321\}\{132, 231\}$ -equivalence is exactly the set of all permutations with the letters in their fall being in the same relative order as in x and with all letters having the same position parity as in x .*

Proof. By Proposition 3.54, every permutation in the latter set is equivalent to x . Observe that under the relation, any permutation equivalent to x must have each letter be of the same position parity as in x . Therefore, such a permutation must also have the same fall as x has. The relation never allows two letters in the fall of a permutation to swap relative positions. Hence, any permutation equivalent to x is in the set. \square

Corollary 3.56. *Let x be a permutation with a fall of size j . Then, the size of the class containing x is $\frac{[n/2]! \cdot [n/2]!}{j!}$ under the $\{123, 321\}\{132, 231\}$ -equivalence.*

Proof. Any two letters of the same position parity not both in the fall of x can be swapped. There would be $[n/2]! \cdot [n/2]!$ elements in the class if letters both in the fall could be swapped as well. However, keeping in mind that they can not, we must divide by the number of ways they can be sorted, $j!$. \square

Corollary 3.57. *Let l be $[n/2]$ and h be $\lceil n/2 \rceil$. There are $\sum_{j=1}^l j! \cdot \binom{n-j-1}{h-1} + \sum_{j=1}^h j! \cdot \binom{n-j-1}{l-1}$ classes in S_n under the $\{123, 321\}\{132, 231\}$ -equivalence.*

Proof. Each class is determined by its fall, the order of the letters in its fall in each permutation in the class, the position parity of the letters in its fall, and the position parity of each of the remaining letters. So, if the letters in its fall are of a given position parity and the fall is of size j (each possibility is summed over in the final equation), then there are $j!$ possible orderings for the letters in the fall. Then, out of the $n - j$ remaining letters, the position parity of the letter $n - j$ is already determined as the opposite of that of n , but the other $n - j - 1$ letters can be of either position parity, thus yielding the binomial coefficient portion of the equation. \square

3.11. **$\{123, 231\}\{321, 213\}$ -Equivalence.** Let i_k be the identity in S_k and u_k be the identity in S_k except with 1 and 2 swapped.

Definition 3.58. *A permutation is layered if each letter of one position parity is less than each letter of the other.*

Remark 3.59. Note that when studying the $\{123, 231\}\{321, 213\}$ -equivalence, we will often use the symmetry (specific to the relation) between a permutation and the corresponding permutation which has every letter k mapped to $n - k + 1$.

Lemma 3.60. *Let x be a permutation in $S_{n \geq 5}$ containing a factor which forms either i_5 or u_5 . Then, x is equivalent to either i_n or u_n under the $\{123, 231\}\{321, 213\}$ -equivalence.*

Proof. We will prove this by inducting on n , using S_5 and S_6 as our base cases. (These can easily be checked computationally.) Assume the lemma holds in S_{n-1} . Let x be a permutation as described. Without loss of generality, x contains a factor forming i_5 or u_5 in its first $n-1$ letters. Applying the inductive hypothesis to the first $n-1$ letters, and then the final $n-1$ letters (reaching a permutation ending with n), and then the first $n-1$ letters again, we reach either i_n or u_n . \square

Lemma 3.61. *Let $n > 5$. Let w be a permutation in S_n not equivalent to i_n or u_n under the $\{123, 231\}\{321, 213\}$ -equivalence. For symmetry reasons, we may assume without loss of generality that w begins with a descent. Under the transformations considered, all permutations equivalent to w begin with a descent as well under the $\{123, 231\}\{321, 213\}$ -equivalence.*

Proof. One can check computationally that the lemma holds in S_6 . Let $w \in S_{n > 6}$ not be equivalent to i_n or u_n , begin with a descent, and be equivalent to a permutation not beginning with a descent. Then, there is some transformation which connects two permutations in w 's equivalence class, one of which begins with a descent and one of which begins with an ascent. Applying the inductive hypothesis to the first six letters of such a permutation, we rearrange them to form i_6 or u_6 , and by Lemma 3.60, we can reach either i_n or u_n . \square

Lemma 3.62. *Let a be the decreasing permutation and b be the decreasing permutation with its first two letters swapped, both in $S_{n > 5}$. Then, a and b are each equivalent to a different one of i_n and u_n under the $\{123, 231\}\{321, 213\}$ -equivalence.*

Proof. Computationally, one can easily check that this holds for $n = 6$. For $n > 6$, we may rearrange the first 6 letters of a and b to form either i_6 or u_6 . Then, applying Lemma 3.60, we reach either i_n or u_n from each of a and b . Noting that that parity is invariant under the relation, we conclude that a and b are each equivalent to a different one of i_n and u_n . \square

For the rest of this subsection, we ask for the reader to remember that Lemma 3.62 makes a permutation being equivalent to i_k or u_k the same as a permutation being equivalent to a or b (as defined in the proof of the lemma). This means that when a permutation is equivalent to one of u_k or i_k , the permutation which is the same but with each letter j mapped to $n-j+1$ is as well; this argument of symmetry, although nontrivial, will be assumed to work for the rest of the subsection.

Lemma 3.63. *Let $x \in S_{n > 5}$ be a layered permutation not beginning with n or $n-1$, but starting with a decrease. Then, x is equivalent to i_n or u_n under the $\{123, 231\}\{321, 213\}$ -equivalence.*

Proof. Inductively assume the lemma holds in S_{n-2} , using S_6 and S_7 as base cases (these can be checked computationally). Let x be a permutation as described. Assume x is not equivalent to i_n or u_n . We must not be able to apply the inductive hypothesis to the first $n-2$ letters, or else x would be equivalent to i_n or u_n by Lemma 3.60. Hence, x must have n or $n-1$ in the final two positions and start with $n-2$. Since we must also not be able to apply the inductive hypothesis to the final $n-2$ letters, the third letter in x must be either $n-1$ or n . So, we know that we can slide the rightmost of $n-1$ and n two positions to the left through a $123 \rightarrow 231$ or $213 \rightarrow 321$ rearrangement without moving any other of $n, n-1$, or $n-2$. Then, applying the inductive hypothesis to the first $n-2$ letters, and noting Lemma 3.60, x is equivalent to i_n or u_n , a contradiction. \square

Lemma 3.64. *Let $x \in S_{n > 5}$. For symmetry reasons, we assume without loss of generality that x begins with a decrease. Then, x is equivalent to a layered permutation starting with n or $n-1$*

which starts with a decrease or x is equivalent to either i_n or u_n under the $\{123, 231\}\{321, 213\}$ -equivalence.

Proof. Let $x \in S_n$ be not equivalent to the i_n or u_n and begin with a decrease. We assume inductively that the result holds in S_{n-1} with an inductive base case of S_6 . Applying the inductive hypothesis to the first $n-1$ letters of x , we reach a permutation which is layered in all of its letters but possibly its final letter (otherwise, they could be rearranged as i_{n-1} or u_{n-1} , we would be able to apply Lemma 3.60, and x would be equivalent to the identity). Without loss of generality, this permutation begins with $n, n-1$, or $n-2$ because its first $n-1$ letters do not form a permutation equivalent to i_{n-1} or u_{n-1} (Lemma 3.63). This creates the following three cases.

- (1) The first letter is n . Applying the inductive hypothesis to the final $n-1$ letters (Using symmetry, they may be rearranged to be either layered or to form i_{n-1} or u_{n-1} ; by Lemma 3.60, they must form a layered permutation; by Lemma 3.61, this layered permutation begins with an increase.), we reach a layered permutation starting with n .
- (2) The first letter is $n-1$. Applying the inductive hypothesis to the final $n-1$ letters (Using symmetry, they may be rearranged to be either layered or to form i_{n-1} or u_{n-1} ; by Lemma 3.60, they must form a layered permutation; by Lemma 3.61, this layered permutation begins with an increase.), we reach a layered permutation starting with a $n-1$.
- (3) The first letter is $n-2$. Applying the inductive hypothesis to the final $n-1$ letters (Using symmetry, they may be rearranged to be either layered or to form i_{n-1} or u_{n-1} ; by Lemma 3.60, they must form a layered permutation; by Lemma 3.61, this layered permutation begins with an increase.), we reach a layered permutation starting with a descent and starting with $n-2$. By a Lemma 3.63, we reach a contradiction, as such permutation is equivalent to i_n or u_n .

□

Lemma 3.65. *Let $x \in S_{n>5}$ be a layered permutation beginning with $(n-1)$ and not equivalent to i_n or u_n under the $\{123, 231\}\{321, 213\}$ -equivalence. Then, n is in the final position of odd parity.*

Proof. As a base case, one can show computationally that this holds in S_6 and S_7 . Inductively, it holds in S_{n-2} . Let w be a layered permutation in $S_{n>6}$ beginning with $(n-1)$ and not equivalent to i_n or u_n . Assume that n is not in the final position of odd parity in w . Then, so that we can not apply the inductive hypothesis to the first $n-2$ letters and then apply Lemma 3.60 to reach i_n or u_n , n must be in the second to final position of odd parity. Furthermore, so that we can not apply Lemma 3.63 to the final $n-2$ letters and then Lemma 3.60 to reach i_n or u_n , the third letter of x must be $n-2$. However, applying the inductive hypothesis to the final $n-2$ letters and then using Lemma 3.60, this means we can reach i_n or u_n from x , a contradiction. □

Lemma 3.66. *Excluding the class containing i_n and the class containing u_n , the classes containing some permutation starting with n in $S_{n>6}$ are exactly the classes with permutations not equivalent to i_n or u_n and beginning with an ascent in S_{n-1} , except with n appended to the beginning of each permutation in each class, under the $\{123, 231\}\{321, 213\}$ -equivalence.*

Proof. Let $n > 6$. Consider a layered permutation in S_{n-1} which is not equivalent to i_n or u_n and which (without loss of generality) starts with an ascent. Appending n to the beginning, we reach a layered permutation $x \in S_n$. In permutations equivalent to x , n can never be involved in a hit because by Lemma 3.61, the two letters to the right of it will always be in increasing order under the relation considered. Thus, x is a layered permutation which is not equivalent to i_n or

u_n . Now, consider y , an arbitrary layered permutation not equivalent to i_n or u_n . By Lemma 3.60 and symmetry, the final $n - 1$ letters of y must form a layered permutation not equivalent to i_{n-1} or u_{n-1} . Hence, if y starts with n , then y is one of the permutations found in the manner that x is found, completing the proof. \square

Let $s \in S_n$ for odd n be the permutation beginning with 2 and then followed by consecutive values every two positions until the third to final position, ending with 1, and then going from right to left, increasing every two positions from the second to final letter. Let s' be s except with each letter j mapped to $n - j + 1$.

Let $d \in S_n$ for even n be the permutation starting with 2 and then followed by consecutively increasing letters every two positions until the second to final position, ending with 1, and increasing from right to left in the remaining positions. Let f be d except with the final two letters swapped. Let d' and f' be f and d respectively except with each letter j mapped to $n - j + 1$.

Under the transformations considered, parity is maintained. Hence, it is worth noting that if $n \equiv 1 \pmod{4}$, then s and s' are equivalent only to even permutations; if $n \equiv 1 \pmod{4}$ then s and s' are only equivalent to even and odd permutations respectively; if $n \equiv 0 \pmod{4}$, then d and d' are only equivalent to even permutations while f and f' are only equivalent to odd ones; if $n \equiv 2 \pmod{4}$, then d and f' are only equivalent to odd permutations while d' and f are only equivalent to even ones.

Lemma 3.67. *Consider the layered permutations not equivalent to i_n or u_n under the $\{123, 231\}\{321, 213\}$ -equivalence which do not begin with 1 or n for $n \geq 6$. For odd n , there are two of them, s and s' , each in a class of size $(n + 1)/2$. For even n , there are four of them, d , f , d' and f' , each in classes of size $n + 1$, $n/2$, $n + 1$, and $n/2$ respectively.*

Proof. Assume the lemma holds in S_{n-1} and S_{n-2} (with inductive base cases of S_6 and S_7). For simplification, we will consider **only** the permutations beginning with a decrease and note that this describes the others as well by symmetry (and are separate by Lemma 3.61). By the inductive hypothesis for S_{n-1} , and the inductive hypothesis for S_{n-2} working along with Lemma 3.66 (applied using symmetry so that we append 1 rather than $n - 1$), in S_{n-1} there is exactly one layered permutation ending with $(n - 1)$ and not equivalent to i_{n-1} or u_{n-1} , and two layered permutations with $n - 1$ in the second to final position which are not equivalent to i_{n-1} or u_{n-1} . Recall that any layered permutation not equivalent to i_n or u_n and not beginning with n must begin with $n - 1$ by Lemma 3.63. Further noting that the final $n - 1$ letters of such a permutation must form a layered permutation⁷ with $n - 1$ in the final position of odd parity (Lemma 3.65), we may conclude that there are at most 2 layered permutations with n in the second to final position and 1 layered permutation with n in the final position in S_n which are not equivalent to i_n or u_n . By explicitly describing the classes they determine, we will complete the proof.

Keeping symmetry in mind, it is sufficient to show that s is in a class of size $(n + 1)/2$, d is in a class of size $n + 1$, f is in a class of size $n/2$, and none of these classes overlap. We will do this by characterizing each class.

The class containing s clearly just contains copies of s except with 1 slid an even number of positions to its left. So, the class containing s has $(n + 1)/2$ elements. For example, in S_5 the class contains 25341, 25134, and 12534.

From d , the only transformation is to slide the third to final letter to the right-most position ($321 \rightarrow 213$), bringing us to e . From e , 1 can then be slid two positions to the left ($231 \rightarrow 123$),

⁷Every factor of a layered permutation forms a layered permutation.

resulting in e' (without going back to d , this is the only transformation possible). From e' , one can rearrange the first three letters as a 231 and then slide 1 two positions to the right. Other than that, without returning to e , from e' , one can only slide the 1 an even number of positions to the left (repeated 231 \rightarrow 123 rearrangements), and choose if the last hit will form a 123 or 231 pattern for each of slide of 1. This results in $5 + 2(n-4)/2 = n+1$ permutations being in the class. As an example, in S_8 , the class containing d contains 28374651, 28374516, 28371456, 28371564, 28475614, 28137456, 28137564, 12837456, and 1283756.

From f , one can only slide 1 to any position of the same position parity, resulting in $n/2$ permutations being in its class. For example, when $n = 6$, the class contains 263541, 263154, and 216354. \square

Proposition 3.68. *For $n > 5$ odd, there are $3n - 1$ classes in S_n under the $\{123, 231\}\{321, 213\}$ -equivalence. For $n > 5$ even, there are $3n$ classes.*

Proof. Let $n > 6$. Permutations equivalent to u_n or i_n (or to u_n or i_n except with each letter j mapped to $n - j + 1$ by Lemma 3.62) fall into two classes, separated by parity. The remaining permutations fall into the same number of classes as they fall into in S_{n-1} (Lemma 3.66), with the exception of, when n is even, four additional classes, and when n is odd, two additional classes (Lemma 3.67). So, noting the number of classes in S_6 , for $n > 5$ odd, there are $3n - 1$ classes, and for $n > 5$ even, there are $3n$ classes. \square

Proposition 3.69. *For $n > 5$, one can easily count the number of permutations in the class containing i_n or in the class containing u_n under the $\{123, 231\}\{321, 213\}$ -equivalence. For the sake of brevity, enumerations are provided inside of the proof.*

Proof. Recall that there are two large classes containing permutations equivalent to i_n or u_n in S_n . For $n > 6$, these classes each consist of the permutations of a given parity not in the smaller classes which were constructed in the proofs of Lemma 3.67 and Lemma 3.66. To find how many permutations are in each of the two, it is sufficient to count how many odd and even permutations are in the smaller classes. Let oi_n, od_n, ei_n, ed_n be the number of permutations in these smaller classes that are odd/even and beginning with an increase/decrease as indicated by the variables. We will keep track of these in a matrix (as shown below). Bearing in mind the construction of the classes provided in Lemmas 3.67 and 3.66, these variables can be kept track of in the following manner. Let

$$M_n = \begin{pmatrix} oi_n & od_n \\ ei_n & ed_n \end{pmatrix}.$$

If $n \equiv 0 \pmod{4}$, then

$$M_n = \begin{pmatrix} od_{n-1} + n/2 & ei_{n-1} + n/2 \\ ed_{n-1} + n + 1 & oi_{n-1} + n + 1 \end{pmatrix}.$$

If $n \equiv 1 \pmod{4}$, then

$$M_n = \begin{pmatrix} od_{n-1} & oi_{n-1} \\ ed_{n-1} + (n+1)/2 & ei_{n-1} + (n+1)/2 \end{pmatrix}.$$

If $n \equiv 2 \pmod{4}$, then

$$M_n = \begin{pmatrix} od_{n-1} + n + 1 & ei_{n-1} + n/2 \\ ed_{n-1} + n/2 & oi_{n-1} + n + 1 \end{pmatrix}.$$

If $n \equiv 3 \pmod{4}$, then

$$M_n = \begin{pmatrix} od_{n-1} & oi_{n-1} + (n+1)/2 \\ ed_{n-1} + (n+1)/2 & ei_{n-1} \end{pmatrix}.$$

Keeping these identities in mind, enumerations for the variables fall inductively. Assume as the inductive hypothesis that

$$M_{4k+0} = \begin{pmatrix} 4k^2 + 3k + 4 & 4k^2 + 3k + 4 \\ 4k^2 + 3k - 1 & 4k^2 + 3k - 1 \end{pmatrix}.$$

Note that the hypothesis holds for a base case of $k = 2$. Using the identities, we get that

$$M_{4k+1} = \begin{pmatrix} 4k^2 + 3k + 4 & 4k^2 + 2k + 4 \\ 4k^2 + 5k & 4k^2 + 5k \end{pmatrix},$$

and that

$$M_{4k+2} = \begin{pmatrix} 4k^2 + 7k + 7 & 4k^2 + 7k + 1 \\ 4k^2 + 7k + 1 & 4k^2 + 7k + 7 \end{pmatrix},$$

and that

$$M_{4k+3} = \begin{pmatrix} 4k^2 + 7k + 1 & 4k^2 + 9k + 9 \\ 4k^2 + 9k + 9 & 4k^2 + 7k + 1 \end{pmatrix},$$

and that

$$\begin{aligned} M_{4k+4} &= \begin{pmatrix} 4k^2 + 11k + 11 & 4k^2 + 11k + 11 \\ 4k^2 + 11k + 6 & 4k^2 + 3k + 6 \end{pmatrix} \\ &= \begin{pmatrix} 4(k+1)^2 + 3(k+1) + 4 & 4(k+1)^2 + 3(k+1) + 4 \\ 4(k+1)^2 + 3(k+1) - 1 & 4(k+1)^2 + 3(k+1) - 1 \end{pmatrix}. \end{aligned}$$

Hence, the inductive hypothesis holds for $k + 1$. Although we use S_8 as our base case, we only do this for aesthetic reasons. We could instead use S_6 (for which the proposition holds) as a base case, and it is easy to see that the inductive step would still work. \square

3.12. $\{123, 321\}\{132, 213\}$ -Equivalence.

Definition 3.70. Let j and k be two letters in a permutation w . A letter s is a j, k -extreme if s is either less than j or greater than k and s is positioned between j and k .

Definition 3.71. Let j and k be two letters in a permutation. We say that j and k are a dangerous pair of letters when the following requirements are satisfied.

- We require that j is to the left of k , $j < k$, and both j and k are of the same position parity.
- We require that there are more j, k -extremes which are of different position parity than j and k than there are of the same.

Lemma 3.72. Let j and k be two letters in a permutation w . Let w' be a permutation reached from w through a single $321 \rightarrow 123$ transformation using the hit h in w . If j and k do not form a dangerous pair of letters in w , then they are not a dangerous pair of letters in w' .

Proof. Let j, k, w, w' , and h be as stated. Because of the definition of dangerous, we only need to consider the case where j is to the left of k , $j < k$, and j and k are of the same position parity. If the hit h does not contain j or k , then the lemma is trivial. The hit cannot contain both j and k because $j < k$ and j is to the left of k . If the hit contains only one of j and k , then by symmetry, we may assume without loss of generality that it contains j . Hence, there are three cases.

- (1) In this case, j is the first letter of h . Then, in w' , there are two less j, k -extremes (acting as 2 and 1 in h), one of each position parity. Hence, j and k are not a dangerous pair of letters in w' .
- (2) In this case, j is the second letter of h . Then, in w' , a j, k -extreme of different position parity than j and k (acting as 1 in h) is replaced with a letter which may or may not be a j, k -extreme (acting as 3 in h). Hence, there are still not more j, k -extremes of the same position parity as j and k than there are of different position parity in w' , and j and k are not a dangerous pair of letters in w' .
- (3) In this case, j is the final letter of h . Since, the transformation using h only adds letters to the set which are between j and k , the number of j, k -extremes of each position parity can only increase. If a j, k -extreme of different position parity than j and k is added (acting as the 2 in h), then so is one of the same position parity (acting as the 3 in h). Hence, j and k are not a dangerous pair of letters in w' .

□

Definition 3.73. A permutation is *zipped* if for each letter except for the final letter, the letter two positions to its right is smaller.

Definition 3.74. A permutation is k -downed for $k > 0$ if every permutation which can be reached through a single $123 \rightarrow 321$ rearrangement is $k - 1$ -downed. A permutation is 0-downed if it is zipped.

Proposition 3.75. Let d be a downed permutation and w be a permutation equivalent to d under the $\{123, 321\}\{132, 213\}$ -equivalence. Let k be the number of transformations needed for d to be reached from w . Then, w is k -downed and both 132 and 213 avoiding.

Proof. We will prove this by inducting on k , with a trivial base case of $k = 0$. Assume as the inductive hypothesis, that the proposition holds for all smaller k . Let d and w be as described. Since w is reachable from d through some k transformations, let us pick such a sequence of transformations and let r be w after the first transformation which uses the hit h . By the inductive hypothesis, r is $k - 1$ -downed. If w is reached from r through a $123 \rightarrow 321$ transformation, then w is $k - 2$ -downed, which cannot be. If w is reached from r through a $132 \leftrightarrow 213$ transformation, then r is a $k - 1$ -downed permutation which is not 132 and 213 avoiding, a contradiction. Hence, w is reached from r through a $321 \rightarrow 123$ rearrangement.

Since w can be reached from d through a sequence of $321 \rightarrow 123$ transformations, by Lemma 3.72, w contains no pair of dangerous letters. Hence, w is 231 and 213 avoiding since both patterns contain a pair of dangerous letters.

It remains to show that w is k -downed. Because of the inductive hypothesis, it is sufficient to show that any $123 \rightarrow 321$ transformation applied to w brings us to a permutation r' which can be reached from d through $k - 1$ transformations⁸. Let r' be a permutation reached from w through a single $123 \rightarrow 321$ rearrangement using the hit h' . If $h' = h$ then $r' = r$ which clearly can be reached from d in k transformations. If h includes just some of the letters of h' , then the letters around and including h must form a 1234 pattern; in r , this becomes a pattern which contains a 132 or 213 pattern, which cannot be. In the final case, h' and h do not use any of the same letters. Let x be the permutation reached from r by rearranging h and from r' by rearranging h' . Since x can be reached

⁸Note that r' cannot be i -downed for $i < k - 1$ because then w would not require k transformations to be reached from d .

from r through a $123 \rightarrow 321$ rearrangement, it is $k - 2$ -downed, and hence reachable from d in $k - 2$ transformations. Thus, r' is reachable from d in $k - 1$ transformations, and we are done. \square

Definition 3.76. A permutation is partially zipped if for each letter but the final three, the letter two positions to its right is smaller, and the final two letters are $1n$.

Definition 3.77. A permutation is k -pdowned for $k > 0$ if every permutation which can be reached through a single $123 \rightarrow 321$ rearrangement is $k - 1$ -pdowned and each permutation reached through a single $1nl \rightarrow l1n$ rearrangement is k -pdowned. A permutation is 0-pdowned if it is zipped.

Note that the preceding definition is both explicitly and implicitly recursive, with two actual recursion parameters. Given the shortest path of transformations from a k -downed permutation and a 0-downed permutation, we use both k , the parameter for the number of $123 \rightarrow 321$ rearrangements in this path, and the number of $1nl \rightarrow l1n$ rearrangements in the path as parameters.

Definition 3.78. A pair of letters in a permutation is pdangerous if the pair is dangerous and one of the following is true.

- Neither letter is 1 or n .
- Exactly one of the two letters is 1 or n and the other of 1 or n is not between the two letters.

Lemma 3.79. Let w be a permutation which contains no pair of pdangerous letters, which has 1 and n with different position parities and which has 1 to the left of n , and for which the letters positioned between 1 and n can be paired up so that each letter with the same position parity as 1 is paired with a smaller letter of the same position parity as n . Let w' be a permutation reached from w with a single $321 \rightarrow 123$ rearrangement or a $l1n \rightarrow 1nl$ rearrangement. Then, w' satisfies the properties noted for w .

Proof. If the rearrangement going from w to w' does not involve n or 1, then this falls from Lemma 3.72. If the rearrangement involves both n and 1, then it must be $ln1 \rightarrow 1nl$. In this case, the set of letters between 1 and n does not change; no pair of letters with either 1 or n can be pdangerous because 1 and n are adjacent; and any pair of pdangerous letters, j and k , remains pdangerous after the transformation because if either 1 or n changes its state of being a j, k -extreme, so does the other which is of different position parity.

In the remaining case, the transformation from w to w' uses exactly one of n or 1. Let h be the hit in w involved in the transformation. Because of symmetry, we can assume without loss of generality that h uses n as its first letter and is of the form nab in w and ban in w' . By Lemma 3.72, no pair of letters in w' which does not include one of n or 1 can be pdangerous in w' which was not pdangerous in w . Pairing up a with b , we see that the claim of “the letters positioned between 1 and n can be paired up so that each letter with the same position parity as 1 is paired with a smaller letter of the same position parity as n ” still holds after the transformation. As a consequence, a and 1 cannot form a pdangerous pair in w' because there are at least as many $1, a$ -extremes with the same position parity as 1 and a as there are with different position parity; this means there are no pdangerous pairs of letters including 1 in w' . As an additional consequence, n cannot be in a pdangerous pair in w' because the pair of letters would have the same letters between them as in w except with the addition of a and b ; since a being a j, n -extreme implies b is as well, there are still at least as many j, n -extremes with the same position parity as n and j as there are with different position parity. So, w' has no pdangerous pairs of letters. \square

Lemma 3.80. *Let w be a k -pdowned permutation with 1 and n adjacent. Let w' be w after a $j1n \rightarrow 1nj$ rearrangement. Then, w' is k -pdowned.*

Proof. Let w and w' be as specified. Let h and h' be the hit w and w' respectively which can be rearranged to obtain w' and w respectively. It is trivial to note that rearranging h' in w' yields a k -downed permutation. We need to show that any $123 \rightarrow 321$ rearrangement in w' using the hit g results in a $k - 1$ -pdowned permutation. Assume inductively that the lemma holds for lower k , with a trivial inductive base case of $k = 0$. Since w is k -pdowned, w with $1n$ slid to the right-most position, yielding x , is k -pdowned as well. Note that x is also equal to w' with $1n$ slid to the right-most position. Since g cannot involve 1 or n , g is still a hit in x using the same letters as in w' . Let x' be x but with that hit rearranged. Since x is k -pdowned, x' is $k - 1$ -pdowned. Note that x' can be reached from w' with g rearranged be sliding $1n$ to the end of w' with g rearranged. Therefore, by repeated applications of the inductive hypothesis, we see that w' with g rearranged is $k - 1$ -pdowned as well, and we are done. \square

Proposition 3.81. *Let d be a partially zipped permutation and w be a permutation equivalent to d under the $\{123, 321\}\{132, 213\}$ -equivalence. Of the transformations needed for d to be reached from w in the fewest number of transformations possible, let k be the least number of necessary transformations which are not $1nj \rightarrow j1n$. Then, w is k -pdowned. Furthermore, w is 132 and 213 avoiding except for when 1 and n act as 1 and 3 respectively.*

Proof. Let d and w be as stated. If 1 and n are adjacent in w , then by Lemma 3.80, we may assume that w ends with $1n$. We will assume inductively that the lemma holds for smaller k , with a trivial base case of $k = 0$. Since w is reachable from d through a series of transformations, k of which are not $1nj \rightarrow j1n$, let us pick such a sequence of transformations and let r be w after the first transformation which uses the hit h . If h is a 132 or 213 pattern that does not use 1 and n as 1 and 3, then r is a $k - 1$ -pdowned permutation containing a 132 or 213 pattern of that form, a contradiction. Note that h cannot be $1nj$ for any j since we already assumed that if 1 and n are adjacent, they are at the end of w . If w is reached from r through a $j1n \rightarrow 1nj$ transformation, then by the inductive hypothesis, w is $k - 1$ -pdowned, a contradiction. If w is reached from r through a $123 \rightarrow 321$ transformation, then by the inductive hypothesis, w is $k - 2$ -pdowned, a contradiction. Hence, w is reached from r through a $321 \rightarrow 123$ rearrangement and by the inductive hypothesis, r is $k - 1$ -pdowned.

Since r is $k - 1$ -pdowned and r is reached from w by a $123 \rightarrow 321$ rearrangement, by repeated applications of Lemma 3.79 (noting that the lemma can be applied to any partially zipped permutation), w must meet the requirements set by the lemma. As a consequence, w is 132 and 213 avoiding except for when 1 and n act as 1 and 3 respectively.

It remains to show that w is k -pdowned. Because of the inductive hypothesis, it is also sufficient to show that any $123 \rightarrow 321$ or $1nj \rightarrow j1n$ transformation applied to w brings us to a permutation r' from which d can be reached through $1nj \rightarrow j1n$ and $123 \rightarrow 321$ transformations, $k - 1$ of which are $123 \rightarrow 321$. Recall that we already assumed no $1nj$ hit exists, so we do not need to consider that case. Let r' be a permutation reached from w through a single $123 \rightarrow 321$ rearrangement using the hit h' . If $h' = h$ then $r' = r$ which is $k - 1$ -pdowned. If h and h' do not use any of the same letters, then let x be the permutation reached from r by transforming h' and from r' by transforming h ; noting the inductive hypothesis, x is $k - 2$ -pdowned. Hence, d is reachable from r' through $1nj \rightarrow j1n$ and $123 \rightarrow 321$ transformations, $k - 1$ of which are $123 \rightarrow 321$. In the final case, h and h' share some but not all of their letters. Since, h and h' share some letters, the letters

surrounding h must form the permutation 1234 where h is the final or first three letters of the permutation. In order for r to be 132 and 213 avoiding except for cases where 1 and n play the role of 1 and 3, 1 and n must play the role of 1 and 4 in the 1234. Hence, in order for h' to exist in w , it must contain the other of either the first three or final three letters in the 1234. Let x be w except with the 1234 rearranged as 3214. Either r is x or r with a $1nj \rightarrow j1n$ rearrangement is x . Hence, x is $k-1$ -pdowned. Either r' is x or r' with a $1nj \rightarrow j1n$ rearrangement is x . Hence, d is reachable from r' through $1nj \rightarrow j1n$ and $123 \rightarrow 321$ transformations, $k-1$ of which are $123 \rightarrow 321$ and we are done. \square

Proposition 3.82. *Let $f(n)$ be the number of classes created in S_n under the $\{123, 321\}\{132, 213\}$ -equivalence. Then,*

$$f(n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n-2}{\lfloor (n-2)/2 \rfloor} + 3.$$

Proof. For $n < 5$, this can be shown computationally. Let n be ≥ 5 . By Proposition 3.75 and Proposition 3.81, zipped and partially zipped permutations are each in their own classes. This accounts for the first two terms of the formula. We will show that the remaining permutations fall into 3 classes.

There must be at least three remaining classes because the number of inversions in a permutation modulo 3 is an invariant in the relation considered. We will now show that all permutations in the remaining classes (for $n \geq 5$) are equivalent to one of 12543678..., 21435678..., and 12435678.... It is not hard to see that these permutations have 3, 2, and 1 inversions respectively, putting them into each of the three prospective classes. Our reason for choosing them is because each has a 132 hit starting in the second position. This will be useful to us shortly.

Let w be a permutation not equivalent to a zipped or partially zipped permutation. Through repeated applications of $123 \rightarrow 321$, one can reach a 123 avoiding permutation. Because the permutation is not zipped or partially zipped, it must contain a 132 or 213 pattern. If 1 and n are in the said pattern, we can slide them to the right-most positions and repeat the process on the remaining letters, bringing us to a permutation which contains a 132 or 213 pattern which does not use both 1 and n . Hence w is equivalent to some w' which contains a 132 or 213 pattern not using 1 and n as the letters 1 and 3.

If w' has a 132 or 213 hit in the first $n-1$ letters such that it does not use both the smallest and largest of the $n-1$ letters or in the final $n-1$ letters such that it does not use both the smallest and largest of the $n-1$ letters, we define w'' as w . Otherwise, if w' has only one 132 or 213 hit which does not use both n and 1, the said hit is in the right-most position and contains both 1 and n , n is in the first position of w' , then arranging the hit as 132, we reach a w' such that its first $n-1$ letters form a permutation containing a 213 hit that does not use both the highest and lowest letter. Similarly, if 1 is in the final position of w' and the only 132 or 213 hit not using 1 and n in w' instead uses n and 2 and is in the final three positions, then arranging the hit as 213, we reach a w'' such that the final $n-1$ letters contain a 132 pattern does not use both the highest and smallest letter of those $n-1$ letters.

If w' has a 132 or 213 hit in the first $n-1$ letters such that it does not use both the smallest and largest of the $n-1$ letters or in the final $n-1$ letters such that it does not use both the smallest and largest of the $n-1$ letters, we define w'' as w' . Otherwise, (noting symmetry) w' either begins with 1 and ends with $2jn$ or $j2n$ or ends with 1 and begins with $2jn$ or $j2n$ for $2 < j < n$. In each of these cases, if we were not able to already define w'' simply as w , it is not hard to see that a rearrangement of the hit creates a new 132 or 213 hit either starting with the second letter or ending

with the second to final letter of the new permutation. This yields a w'' which has a 132 or 213 hit in the first $n - 1$ letters such that it does not use both the smallest and largest of the $n - 1$ letters or in the final $n - 1$ letters such that it does not use both the smallest and largest of the $n - 1$ letters.

Assume as an inductive hypothesis for the rest of the proof, that any permutation containing a 132 or 213 pattern not using both 1 and n in S_k for $k < n$ is equivalent to one of 12543678..., 21435678..., and 12435678.... The base cases of $k = 5$ and $k = 6$ for this induction are easy to check computationally. We will now show that w'' is equivalent to one of 12543678..., 21435678..., and 12435678..., completing the proof. If w'' has a hit in the first $n - 1$ letters not using the highest and smallest letter of those first $n - 1$ letters, we apply the inductive hypothesis to them, reaching x . Because of the hit that x has in the final $n - 1$ letters (starting with the second letter), we can apply the inductive hypothesis to the final $n - 1$ letters to reach x' which has n in the final position. Finally, applying the inductive hypothesis to the first $n - 1$ letters (using the hit starting in the third position), we are done. If instead, w'' initially has a 132 or 213 pattern in the final $n - 1$ letters not using both of the highest and lowest letters in the final $n - 1$ letters, we can apply the inductive hypothesis to the final $n - 1$ letters to reach an instance of the case we have already covered. \square

3.13. $\{123, 231\}\{213, 312\}$ -Equivalence. We will first count the number of classes which contain only 123 and 231 avoiding permutations.

Definition 3.83. A peak in a word is a letter that is greater than each of its adjacent letters in the word. A dip in a word is a letter that is less than each of its adjacent letters in the word.

Definition 3.84. The 321-leading subword of a permutation w is the factor containing the letters before the first occurrence of 321 in w as well as the first letter of that occurrence. If there is no occurrence, then it is simply w .

Definition 3.85. We consider the k -length of a permutation to be the number of peaks in its 321-leading subword.

Definition 3.86. We define the 321-leading segments of a permutation w to be the factors beginning with the final two letters of a 321 hit and going to the first letter of the next 321 hit. The 321-leading subword as well as the factor going from the second letter of the final 321 hit to the end of a permutation are considered 321-leading segments as well.

Definition 3.87. The k -length of a 321-leading segment is the number of peaks in the 321-leading segment.

Definition 3.88. We say that a permutation is compact if it satisfies one of the following two conditions.

- (condition 1) w begins with a decrease. The 321-leading subword of w is decreasing in the odd positions and increasing in the even positions from left to right. The letters in the odd positions of the 321-leading subword of w are the largest letters in w . Also, The portion of w not included in the 321-leading subword, w' either satisfies this condition (recursively) or is of length 0. Finally, if it is not of length 0, then where k is the k -length of w , k' is the k -length of w' , and j is the value of the letter in the final dip of 321-leading subword of w , we have $k + k' \leq n - j$.
- (condition 2) w begins with an increase. The final $n - 1$ letters of w form a permutation satisfying condition 1.

Proposition 3.89. *Each compact permutation is only equivalent to permutations avoiding both 123 and 231 under the $\{123, 231\}\{213, 312\}$ -equivalence. Each class containing only 123 and 231 avoiding permutations contains exactly one compact permutation.*

Proof. We will first show that each compact permutation determines a unique class containing only 123 and 231 avoiding permutations. We claim that the class containing a compact permutation w contains exactly the permutations which are w except with the peaks in each 321-leading segment rearranged arbitrarily. Assume as the inductive hypothesis that the claim holds for permutations with r or less 321-leading segments (with the base cases of $r = 1$ and $r = 0$ being fairly obvious).

Let w be a compact permutation with $r + 1$ 321-leading segments. Let w' be w except excluding the 321-leading subword. Let l be the 321-leading subword of w and l' be the 321-leading subword of w' . Let k be the k -length of w , k' be the k' -length of w' , and j be the value of final dip l (if it does not exist, then the 321-leading subword of w is of size 1 and the claim easily falls from the inductive hypothesis). Note that l ends with a peak since otherwise, there would be a 321-leading subword prior to the first one in w . Recall that w is compact. Since all of the dips in l are less than or equal to j , all of the peaks in l are among the greatest k letters of w , all of the peaks of l' are among the greatest k' letters of w' , and $k + k' \leq n - j$, each peak in l' is greater than j (observation 1). By the inductive hypothesis, the permutations contained in the equivalence class of w' are simply copies of w' where the peaks in each 321-leading segment are possibly scrambled (observation 2). Also, by the inductive hypothesis, the permutations contained in the equivalence class of l are simply copies of l except with its peaks arbitrarily scrambled (observation 3). By observations 1, 2, and 3, for any permutation x reachable from w by rearrangements of hits in l and rearrangements of hits in w' , the letter immediately following l is a peak in w' which is greater than j . We also know the said letter is less than the peak immediately following j . Hence, j and the two letters following it form a 132 pattern in x . Since the three letters following j form a 321 pattern in x , there is no hit in x containing both letters from l and w' . Therefore, the class containing w contains only versions of w where the peaks within each 321-leading segment have been scrambled arbitrarily and only permutations avoiding 123 and 231. Thus, w uniquely determines such a class.

We will now show that every class C containing only 123 and 231 avoiding permutations contains a compact permutation w , thus completing the proof. Let x be a permutation in such a class C . Applying repeated $213 \rightarrow 312$ rearrangements to x , we reach a permutation w consisting only of 321, 132, and 312 patterns. We will prove that this permutation either meets condition 1 or condition 2. Assume as the inductive hypothesis that this claim is true if w were to have m or less 321-leading segments (with a base case of $m = 0$). Assume w has $m + 1$ 321-leading segments. The 321-leading subword of w is alternating, has peaks decreasing from left to right, and dips increasing from left to right. The peaks of the 321-leading subword of w must be the largest letters in the w because otherwise a 123 or 213 would occur somewhere. By the inductive hypothesis, the portion of w not included in its 321-leading subword satisfies condition 1. Finally, if there are more than one 321-leading segments in w , then where k is the k -length of w , k' is the number of peaks in the second to left-most 321-leading segment in w , and j is the value of the final dip in the 321-leading subword of x , $k + k' \leq n - j$. This inequality must hold because otherwise a peak in the second 321-leading segment in w which has value less than j could be moved to the position two to the right of j , forming a 231 pattern, a contradiction. Thus, w is compact. \square

Proposition 3.90. *Let $g(n, k)$ count the number of permutations meeting condition 1 for compactness and with a 321-leading subword containing $k - 1$ dips and k peaks. We assume that k is such*

that at least one such permutation exists. Then,

$$g(n, k) = \begin{cases} 1, & \text{if } n = 1 \text{ or } n - 2k + 1 = 0 \\ \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} g(n-1, j), & \text{if } k = 1 \\ \sum_{x=k-1}^{n-k} \sum_{j=1}^{n-k-x} \binom{x-1}{k-2} \cdot g(n-2k+1, j), & \text{otherwise.} \end{cases}$$

Proof. If $n = 1$, then it is trivial that $k = 1$ and $g(n, k) = 1$. If $n - 2k + 1 = 0$, then there is only one permutation which $g(n, k)$ counts, having peaks in decreasing order and containing the larger half of the letters, and the dips increasing from left to right and containing the remaining letters.

Otherwise, if $k = 1$, then we need the number of possible permutations of size $n - 1$ satisfying condition 1. (We know $n - 1 > 0$; we can simply append n to each of these permutations.) We define j as the number of peaks in the 321-leading subword of a permutation in S_{n-1} . Then, j can be anywhere from 1 to $\lfloor (n-1)/2 \rfloor$. So,

$$g(n, k) = \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} g(n-1, j).$$

Otherwise, we know that $n > 1$, $k > 1$, and $n - 2k - 1 \neq 0$. Let w be a permutation counted by $g(n, k)$ in this case. We define x to be the value of the final dip in the 321-leading subword of w . Because the dips in the 321-leading subword increase from left to right, x is at least $k - 1$. Because the largest k letters in w are used in peaks of the 321-leading subword, x is at most $n - k$. However, x can be any value in-between inclusive. We define j to be the number of peaks in the second 321-leading segment of w . Note that j can be anywhere between 1 and $n - k - x$ inclusive (by the inequality in condition 1). Now, given x and j , we choose $k - 2$ dips in the 321-leading segment of w (x is already chosen). They can have any values less than x . So, there are $\binom{x-1}{k-2}$ choices. Then, we choose w' , the permutation created by the letters to the right of the 321-leading subword of w . There are $g(n - 2k + 1, j)$ choices for w' . So, we get

$$g(n, k) = \sum_{x=k-1}^{n-k} \sum_{j=1}^{n-k-x} \binom{x-1}{k-2} \cdot g(n-2k+1, j).$$

□

Proposition 3.91. *Let the function g be defined as in the previous proposition. Let $f(n)$ be the number of classes in S_n under the $\{123, 231\} \{213, 312\}$ -equivalence. Then,*

$$f(n) = \sum_{k=1}^{\lfloor n/2+1 \rfloor} g(n+1, k) + n - 2.$$

Proof. First, we calculate the number of classes containing only 123 and 231 avoiding permutations. The number of these containing permutations starting with a decrease is

$$\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} g(n+1, k).$$

This is also the same as the number of such classes in S_{n+1} containing only permutations starting with a 321 pattern (we create a bijection by just appending $n+1$ to the left). The number of such

classes S_n that contain permutations starting with an increase is the same as the number of such classes in S_{n+1} containing permutations starting with a decrease but not with a 321. This falls from the definition of condition 2. Adding these together, we get that the number of such classes in S_n is the number of such classes in S_{n+1} starting with a 321 added to the number of such classes in S_{n+1} starting with a decrease but not a 321, which is just the number of classes containing only 123 and 231 avoiding permutations starting with a decrease in S_{n+1} . We know that this is $\sum_{k=1}^{\lfloor n/2+1 \rfloor} g(n+1, k)$.

Now, will calculate the number of classes containing at least one permutation which contains a 123 or 231 pattern. First we note an invariant. Let w be a permutation in S_n . If 1 and 2 are of the same position parity as each other and 1 is to the left of 2, then they cannot be involved in a hit with each other until one of them changes position parity under the transformations considered. However, unless they are in the same hit, they always both act as 1 in any hit, thus maintaining their position parity. So, in this case, 1 and 2 can never swap relative order and can never change position parity under the transformations considered. Then, if 3 is to the right of 2 and of the same position parity, then 3 and 2 can never swap relative order or change position parity. Similarly, if the k smallest letters are ordered increasing from left to right and are of the same position parity, then none of their position parities can ever change and none of them can ever swap relative order. So, the highest k for which this is true and the parity of the lowest k letters for a permutation does not change under the transformations considered. Let k be that highest such k for a given permutation w in S_n . If 1 has an odd position, k can have any of $\lceil n/2 \rceil$ values. If 1 has an even position, k can have any of $\lfloor n/2 \rfloor$ values. However, in the case where every odd position is less than every even position or vice versa, w is in one of the 123 and 231 avoiding classes. Hence, there are at least $n-2$ classes which contain a permutation containing 123 or 231 patterns fall into.

We will now show that permutations containing a 123 or 231 fall into exactly $n-2$ classes. Let $w \in S_n$ contain either a 123 or 231 pattern. Let k be the largest k such that the lowest k letters are in increasing order from left to right and each of the same position parity in w . Let p be the position parity of 1 in w . We will show that w is equivalent to w' , the permutation which increases from left to right except with 1 inserted in the left-most position with parity p and the next $k-1$ letters in value inserted every two positions to its right. (e.g., if $k=3$, $n=8$, and 1 is of odd position parity, then $w' = 14253678$.) Assume as an inductive hypothesis that this claim holds in S_{n-1} (with the base cases of $n \leq 5$ easy to check). Note that all such w' have their final three letters in increasing order because we are not considering the cases where all the letters of one position parity are greater than those of the other.

If a 123 or 231 pattern occurs only in the final three letters of w , then applying the inductive hypothesis to the final $n-1$ letters, we reach a permutation where that is not the case. Hence, we may assume that w has a 123 or 231 pattern in its first $n-1$ letters. We apply the inductive hypothesis to them, reaching a permutation x which has the three letters preceding its final letter in increasing order. If 1 is in the first position of x , then we apply the inductive hypothesis to the right-most $n-1$ letters of x and reach w' . If 1 is in the second position in x , then we apply the inductive hypothesis to the right-most $n-1$ letters of x to reach x' which has its final three letters in increasing order. If x' is not w' , then through an application of $312 \rightarrow 213$, we place the first letter in w' in the first position in x' (note that this letter was previously in the third position in x'). Then, applying the inductive hypothesis to the right-most $n-1$ letters, we reach w' . If 1 is in the final position of x , then we apply the inductive hypothesis to the final $n-1$ letters, reaching a permutation which starts with some letter t and increases afterwards, m . Applying the inductive

hypothesis to the first $n - 1$ letters of m and then the to the final $n - 1$ letters, we are done (and we reach the identity). Hence, w and w' are always equivalent and the proposition holds. \square

3.14. **$\{132, 231\}\{213, 312\}$ -Equivalence experimental data.** Figure 2 provides computational data for the number of classes created in S_n by the $\{132, 231\}\{213, 312\}$ -equivalence for $n \leq 12$. This equivalence is the only replacement partition of S_3 with two nontrivial parts, each of size two, which has yet to be enumerated.

FIGURE 2. The number of classes created in S_n by the $\{132, 231\}\{213, 312\}$ -equivalence.

n	3	4	5	6	7	8	9	10	11	12
# classes	4	10	26	76	234	782	2804	10972	47246	224648

4. CONNECTING PATTERN-AVOIDANCE AND THE ENUMERATION OF EQUIVALENCE CLASSES

In this section, we study the connection between pattern-avoidance and the enumeration of equivalence classes.

Let P be a partition of S_c such that for each part of P , there is a permutation in the part with less inversions than every other permutation in the part. Let D be the set of permutations which have less inversions than every other permutation in their part. Let $U = S_c \setminus D$.

We define N_n as the number of classes in S_n under the P -equivalence. We define A_n as the number of permutations in S_n which avoid each permutation in U .

In Theorem 4.7, we show that for any $k \geq 2c - 1$, $N_k = A_k \implies N_n = A_n$ for all $n \geq k$. Although the result is quite simple, its effects are wide-ranging. In fact, it can be used to provide an alternative proof for the number of classes created by both the $\{123, 132, 231\}$ -equivalence and the $\{123, 132\}\{213, 231\}$ -equivalence which we previously enumerated. [LPRW] studied pattern-replacement equivalences where one does not require that letters in patterns be consecutive in a permutation; this modified concept of an equivalence was referred to as P^{\vdash} -equivalence for the replacement partition P . It is trivial to modify the proof of Theorem 4.7 so that it holds for these equivalence relations as well⁹. Thus, the result can also be used to provide an alternative proof of the enumeration of the classes for each of the $\{123, 132\}^{\vdash}$ -equivalence and the $\{123, 132, 213\}^{\vdash}$ -equivalence. Although the theorem serves as an alternative proof for each of these results, one should not consider it to substitute their proofs; each of the already known proofs provide insight into the characterization of the equivalence classes in S_n , while the theorem does little more than to enumerate them.

Definition 4.1. We say that a rearrangement of a hit from U to be a hit from D is a down jump.

Definition 4.2. We call w an avoider if w avoids each permutation in U .

Note that this notion of an avoider is slightly different from the one used in the other sections of the paper.

Definition 4.3. We say that the height of a permutation w is the largest k such that through k down jumps, we may go from w to an avoider (Lemma 4.6 shows that such a k exists).

⁹For the sake of brevity, we do not provide the modified proof here. However, one only needs to redefine N_n as the number of classes in S_n under the P^{\vdash} -equivalence, redefine the notion of avoidance to not require that letters in patterns be adjacent, and to redefine f in the proof of Theorem 4.7 as a subword rather than a factor.

Definition 4.4. We say that a permutation is secure if after repeatedly performing down jumps to it, we always reach the same avoider.

Definition 4.5. A pair of permutations is matched if each permutation in the pair is secure, and repeated down jumps applied to one of the permutations in the pair brings us to the same avoider as repeated down jumps applied to the other.

Lemma 4.6. Let $w \in S_n$. If we repeatedly perform down jumps on w , we will eventually reach an avoider.

Proof. Each down jump brings us to a permutation with fewer inversions. Since there are a finite number of permutations in S_n , repeated down jumps must always eventually bring us to an avoider. \square

Theorem 4.7. Let $k \geq 2c - 1$. If $N_k = A_k$, then $N_n = A_n$ for all $n \geq k$.

Proof. Since $N_k = A_k$, we may conclude that each class in S_k contains exactly one avoider (by Lemma 4.6). Hence, each permutation in S_k is secure.

Assume that each of the permutations in $S_{n \geq k}$ of height $\leq h - 1$ are secure. We will show that so are each of the permutations in S_n of height h . The base case for this induction of $h = 0$ is trivial.

Let w be a permutation in S_n of height h . Let a and b be permutations which can be reached from w through a down jump using hits x and y respectively. By the inductive hypothesis, a and b are secure. By showing that a and b are matched, we will establish that w is secure.

If x and y are the same, then $a = b$ and a and b are trivially matched.

If x and y are disjoint, then let w' be w except with each hit x and y rearranged in the same manner as in a and b respectively. Then, w' is matched to each of a and b since it can be reached from either by a down jump and because of the inductive hypothesis. Hence, a and b are matched.

If x and y overlap but are not equal, then we consider a factor¹⁰ f of w containing both x and y which is of size k . We define f' as the avoider reached from f through repeated down jumps. (f is secure since it is in S_k .) Rearranging f as f' in w , we reach a secure permutation w' by the inductive hypothesis. Furthermore, since f is secure, f' and f with either of x or y rearranged in a down jump are matched. This implies that a is matched to w' and b is matched to w' . Hence, a and b are matched.

Thus, each permutation in $S_{n > k}$ is secure. Hence, no two avoiders in S_n are equivalent. Since every permutation in S_n is equivalent to an avoider (Lemma 4.6), $N_n = A_n$. \square

5. CONCLUSION AND FUTURE WORK

In our study of pattern-replacement relations, several new recurring ideas came to light. First, in proofs by induction, we found that modified versions of stooge sort could often be used. Second, we found that sometimes it is easier to treat nontrivial and trivial classes separately. In addition, several previously known tools played an important role in our research, for example the search for invariants and systems of representatives (sets of permutations such that each class has exactly one element of the set).

While the proofs of our results use a common set of tools, they are creatures of different ilks: they vary in difficulty and structure. This makes it all the more surprising that the results show some unexpected similarities, like the 2^{n-1} occurring four times in our enumerations of the classes

¹⁰As noted previously, to modify the proof to hold for the P° -equivalence, we simply consider a subword rather than a factor.

in S_n (and the multiset of class sizes in S_n being the same in two cases). If our results are combined with those of previous works, [PRW] and [LPRW], even more similarities occur. Analyzing the sources of these similarities, as well as viewing our equivalence relations from the more advanced viewpoint of algebra, is a promising direction for future study. It would be interesting to find formulas for the number of classes created by several replacement partitions not yet well-understood, specifically $\{132, 231\}\{213, 312\}$. In addition, there are the following directions of future work:

- (1) In Section 3, we only provide the size of the class containing the identity for relations when the result is convenient. Future authors might further study the sizes of the classes created under those relations.
- (2) Are there connections between equivalence relations having the same number of classes. Is there a reason why the enumerations for each of the Knuth relation and the forgotten relation show up again in our study of the $\{123, 132\}\{213, 312\}$ -equivalence and $\{123, 231\}\{213, 132\}$ -equivalence respectively?
- (3) [LPRW] deals with relations that allow re-ordering only adjacently valued letters or, alternatively, re-ordering any subword (rather than only a contiguous set of adjacent letters). This is an important direction of research to continue.
- (4) Some equivalence classes have additional structure. Can one classify permutations in a given equivalence class based on characteristics such as inversions, length (number of inversions), the locations of hits, ascents, Major index, etc.?
- (5) We can consider the K -equivalence not only on permutations, but on arbitrary words (cf. the Knuth relations). By linearization, this corresponds to studying binomial ideals in noncommutative polynomial rings, and the quotient rings modulo these ideals. These have revealed interesting properties in the cases of the Knuth and forgotten relations. Do similar properties arise in the relations that we have studied?

I would like to thank Professor Richard Stanley as well as the MIT PRIMES program for providing me with this research project. I would like to thank Sergei Bernstein, Darij Grinberg, and Ziling Zhou for many useful conversations throughout the research, as well as for helping with the editing process of this paper.

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