Representations of Infinitesimal Cherednik Algebras

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Abstract

Infinitesimal Cherednik algebras, first introduced in [EGG], are continuous analogues of rational Cherednik algebras, and in the case of $\mathfrak{gl}_n$, are deformations of universal enveloping algebras of the Lie algebras $\mathfrak{sl}_{n+1}$. Despite these connections, infinitesimal Cherednik algebras are not widely-studied, and basic questions of intrinsic algebraic and representation theoretical nature remain open. In the first half of this paper, we construct the complete center of $H_\zeta(\mathfrak{gl}_n)$ for the case of $n = 2$ and give one particular generator of the center, the Casimir operator, for general n. We find the action of this Casimir operator on the highest weight modules to prove the formula for the Shapovalov determinant, providing a criterion for the irreducibility of Verma modules. We classify all irreducible finite dimensional representations and compute their characters. In the second half, we investigate Poisson-analogues of the infinitesimal Cherednik algebras and use them to gain insight on the center of $H_\zeta(\mathfrak{gl}_n)$. Finally, we investigate $H_\zeta(\mathfrak{sp}_{2n})$ and extend various results from the theory of $H_\zeta(\mathfrak{gl}_n)$, such as a generalization of Kostant's theorem.

Introduction

The main goal of this paper is to study the representation theory of the infinitesimal Cherednik algebra $H_\zeta(\mathfrak{gl}_n)$, a deformation of the representation theory of $\mathfrak{sl}_{n+1}$, with infinitely many deformation parameters $\zeta = (\zeta_0, \zeta_1, \zeta_2, ..., \zeta_m, ...)$. Namely, $\mathfrak{sl}_{n+1}$ can be represented as $\mathfrak{gl}_n \oplus V \oplus V^*$, where V, V^* are the natural representations of $\mathfrak{gl}_n$ on vectors and covectors. In this representation of $\mathfrak{sl}_{n+1}$, the elements of V commute with each other, as do the elements of V^*. The commutation relations of $\mathfrak{gl}_n$ with V, V^* are given by the usual action of matrices on vectors and covectors, while commutators of V with V^* produce elements of $\mathfrak{gl}_n$. To pass to the deformation $H_\zeta(\mathfrak{gl}_n)$, one needs to change only the last relation: commutators of V and V^* will now be not just elements of $\mathfrak{gl}_n$ but rather some polynomial $\zeta_0 r_0 + \zeta_1 r_1 + \cdots$ of them, where ζ_i are the deformation parameters mentioned above and r_i are basis polynomials introduced in [EGG]. This deformation turns out to be very interesting, since it unifies the representation theory of $\mathfrak{sl}_{n+1}$ with that of degenerate affine Hecke algebras (introduced by Drinfeld and Lusztig in [D],[L]) and of symplectic reflection algebras ([EG]).

The main results of this paper are the following. In Section 2, we generalize a classical result from the representation theory of Kac-Moody algebras by computing the determinant of the contravariant (or Shapovalov) form, thus determining when the Verma module over $H_\zeta(\mathfrak{gl}_n)$ is irreducible. This proof requires knowledge of the quadratic central element and its action on the Verma module. In Section 3, we find explicit formulas for all central elements of $H_\zeta(\mathfrak{gl}_2)$, and in Section 4, we find the quadratic central element for all $H_\zeta(\mathfrak{gl}_n)$. This extends the work of Tikaradze [T], who proved using methods of homological algebra that the center of $H_\zeta(\mathfrak{gl}_n)$ is a polynomial algebra in n generators, but did not get any explicit formulas for these generators.

In Section 5, we provide a complete classification and character formulas for finite dimensional representations of $H_\zeta(\mathfrak{gl}_n)$, generalizing Chmutova's unpublished work. In Sections 6 to 8, we introduce the Poisson analogue of the infinitesimal Cherednik algebras, compute their Poisson center, and use it to give a second proof of the formula for the quadratic central element of $H_\zeta(\mathfrak{gl}_n)$. We also present some results on the central elements of the Poisson analogue of $H_\zeta(\mathfrak{sp}_{2n})$; hopefully, these results could be extended to the noncommutative algebra $H_\zeta(\mathfrak{sp}_{2n})$. Finally, in Sections 9 and 10, we investigate the Harish-Chandra mapping and an analogue of Kostant's theorem.

It would be interesting to find explicit formulas for all central elements, and we expect that this can be done using the Duflo isomorphism. Other interesting problems include the study of infinite dimensional irreducible representations in category $\mathcal{O}$ (and possibly an analogue of the Kazhdan-Lusztig conjecture) and of the quantum analogues $H_\zeta^q(\mathfrak{gl}_n)$ and $H_\zeta^q(\mathfrak{sp}_{2n})$

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1 Basic Definitions

Let us formally define the infinitesimal Cherednik algebras of $\mathfrak{gl}_n$, which we denote by $H_\zeta(\mathfrak{gl}_n)$. Let $V = \mathrm{span}(y_1, \dots, y_n)$ be the basic n-dimensional representation of $\mathfrak{gl}_n$ and $V^* = \mathrm{span}(x_1, \dots, x_n)$ be the dual representation. For any $\mathfrak{gl}_n$ invariant pairing $\zeta : V \times V^* \to U(\mathfrak{gl}_n)$, define an algebra $H_\zeta(\mathfrak{gl}_n)$ as the quotient of the semi-direct product algebra $U(\mathfrak{gl}_n) \ltimes T(V \oplus V^*)$ by the relations $[y, x] = \zeta(y, x)$ and $[x, x'] = [y, y'] = 0$ for all $x, x' \in V^*$ and $y, y' \in V$.

Let us introduce an algebra filtration on $H_\zeta(\mathfrak{gl}_n)$ by setting $\deg(x) = \deg(y) = 1$ for $x \in V^*$, $y \in V$, and $\deg(g) = 0$ for $g \in U(\mathfrak{gl}_n)$. We say that $H_\zeta(\mathfrak{gl}_n)$ satisfies the PBW property if the natural surjective map $U(\mathfrak{gl}_n) \ltimes S(V \oplus V^*) \twoheadrightarrow \mathrm{gr} H_\zeta(\mathfrak{gl}_n)$ is an isomorphism, where S denotes the symmetric algebra; we call these $H_\zeta(\mathfrak{gl}_n)$ the infinitesimal Cherednik algebras of $\mathfrak{gl}_n$. In [EGG], it was shown that the pairings ζ satisfying the PBW property are given by $\zeta = \sum_{j=0}^k \zeta_j r_j$ where $\zeta_j \in \mathbb{C}$ and r_j is the symmetrization of the coefficient of τ^j in the expansion of $(x, (1 - \tau A)^{-1} y) \det(1 - \tau A)^{-1}$.

Note that for $\zeta = \zeta_0 r_0 + \zeta_1 r_1$ with $\zeta_1 \neq 0$, there is an isomorphism $\phi : H_\zeta(\mathfrak{gl}_n) \to U(\mathfrak{sl}_{n+1})$ given by $\phi(\alpha) = \alpha$ for $\alpha \in \mathfrak{sl}_n$, $\phi(y_i) = \sqrt{\zeta_1} e_{i,n+1}$, $\phi(x_i) = \sqrt{\zeta_1}\, e_{n+1,i}$, and

$$\phi(\mathrm{Id}) = \frac{1}{n+1} \left(e_{11} + \cdots + e_{nn} - n\, e_{n+1,n+1} - n \frac{\zeta_0}{\zeta_1} \right).$$

This isomorphism allows us to view $H_\zeta(\mathfrak{gl}_n)$ for general ζ as an interesting deformation of $U(\mathfrak{sl}_{n+1})$, even though any formal deformation of $U(\mathfrak{sl}_{n+1})$ is trivial.

Example 1.1. The infinitesimal Cherednik algebras of $\mathfrak{gl}_1$ are generated by elements e, f, and h, satisfying the relations $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = \phi(h)$ for some polynomial ϕ. In literature, these algebras are known as generalized Weyl algebras ([S]).

Similarly as in the representation theory of $\mathfrak{sl}_{n+1}$, we define the Verma module of $H_\zeta(\mathfrak{gl}_n)$ as

$$M(\lambda) = H_\zeta(\mathfrak{gl}_n) / \{ H_\zeta(\mathfrak{gl}_n) \cdot \mathfrak{n}^+ + H_\zeta(\mathfrak{gl}_n)(h - \lambda(h)) \}_{h \in \mathfrak{h}}$$

where the set of positive root elements $\mathfrak{n}^+$ is spanned by the positive root elements of $\mathfrak{gl}_n$ (i.e., matrix units e_{ij} with $i < j$) and elements of V; the set of negative root elements $\mathfrak{n}^-$ is spanned by the negative root elements of $\mathfrak{gl}_n$ (i.e., matrix units e_{ij} with $i > j$) and elements of V^*; and the Cartan subalgebra $\mathfrak{h}$ is spanned by diagonal matrices. The highest weight, λ, is an element of $\mathfrak{h}^*$.

Let us denote the set of positive roots by Δ^+, so $\Delta^+ = \{e_{ii}^* - e_{jj}^*\} \cup \{e_{ii}^*\}$ for $1 \leq i < j \leq n$. To denote the positive roots of $\mathfrak{gl}_n$, we use $\Delta^+(\mathfrak{gl}_n)$, and to denote the weights of y_i, we use $\Delta^+(V)$. We define $\rho = \frac{1}{2}\sum_{\lambda \in \Delta^+} \lambda = \left(\frac{n}{2}, \frac{n-2}{2}, \ldots, -\frac{n-2}{2}\right)$, a *quasiroot* to be an integral multiple of an element in Δ^+, and Q^+ to be the set of linear combinations of positive roots with non-negative integer coefficients. Finally, we denote the $-\nu$ weight-space of $U(\mathfrak{n}^-)$, where $\nu \in Q^+$, by $U(\mathfrak{n}^-)_\nu$.

2 Shapovalov Form

As in the classical representation theory of Lie algebras, the Shapovalov form can be used to investigate the basic structure of Verma modules. Similarly to the classical case, $M(\lambda)$ possesses a maximal proper submodule $\overline{M}(\lambda)$ and has a unique irreducible quotient $L(\lambda) = M(\lambda)/\overline{M}(\lambda)$. Define the Harish-Chandra projection $\mathrm{HC} : H_\zeta(\mathfrak{gl}_n) \to S(\mathfrak{h})$ with respect to the decomposition $H_\zeta(\mathfrak{gl}_n) = (H_\zeta(\mathfrak{gl}_n)\mathfrak{n}^+ + \mathfrak{n}^- H_\zeta(\mathfrak{gl}_n)) \oplus U(\mathfrak{h})$, and let $\sigma : H_\zeta(\mathfrak{gl}_n) \to H_\zeta(\mathfrak{gl}_n)$ be the anti-involution that takes y_i to x_i and e_{ij} to e_{ji}.

Definition 2.1. The *Shapovalov form* $S : H_\zeta(\mathfrak{gl}_n) \times H_\zeta(\mathfrak{gl}_n) \to U(\mathfrak{h}) \cong S(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*]$ is a bilinear form given by $S(a, b) = \mathrm{HC}(\sigma(a)b)$. The bilinear form $S(\lambda)$ on the Verma module $M(\lambda)$ is defined by $S(\lambda)(u_1 v_\lambda, u_2 v_\lambda) = S(u_1, u_2)(\lambda)$, for $u_1, u_2 \in U(\mathfrak{n}^-)$.

This definition is motivated by the following two properties (compare with [KK]):

Proposition 2.1. *1.* $S(U(\mathfrak{n}^-)_\mu, U(\mathfrak{n}^-)_\nu) = 0$ *for* $\mu \neq \nu$,
2. $\overline{M}(\lambda) = \ker S(\lambda)$.

Statement 1 of Proposition 2.1 reduces S to its restriction to $U(\mathfrak{n}^-)_\nu \times U(\mathfrak{n}^-)_\nu$, which we will denote as S_ν. Statement 2 of Proposition 2.1 gives a necessary and sufficient condition for the Verma module $M(\lambda)$ to be irreducible, namely that for any $\nu \in Q^+$, the bilinear form $S_\nu(\lambda)$ is non-degenerate, or equivalently, that $\det S_\nu(\lambda) \neq 0$, where the determinant is computed in any basis; note that this condition is independent of basis. For convenience, we choose the basis $\{f^{\mathbf{m}}\}$, where $\mathbf{m}$ runs over all partitions of ν into a sum of positive roots and $f^{\mathbf{m}} = \prod f_\alpha^{\mathbf{m}_\alpha}$ with $f_\alpha \in \mathfrak{n}^-$ of weight $-\alpha$. We will use the notation $a \vdash b$ to mean that $(a_1, \ldots, a_n)$ is a partition of b into a sum of n nonnegative integers when $b \in \mathbb{N}$, and $\mathbf{m} \vdash \nu$ to mean that $\mathbf{m}$ is a partition of ν into a sum of elements of Δ^+ when $\nu \in Q^+$. Then, the basis we will work with can be written concisely as $\{f^{\mathbf{m}}\}_{\mathbf{m} \vdash \nu}$.

Now, we will give a formula for the determinant of the Shapovalov form for $H_\zeta(\mathfrak{gl}_n)$ that generalizes the classical result presented in [KK]. This formula uses the following result proven in Section 3.3 for $H_\zeta(\mathfrak{gl}_2)$ and in Section 4.2 for general $H_\zeta(\mathfrak{gl}_n)$: if the deformation is given by $\zeta = \zeta_0 r_0 + \zeta_1 r_1 + \cdots + \zeta_m r_m$, the action of the Casimir element t_1' (introduced in Sections 3 and 4) on the Verma module $M(\lambda)$ can be written as $P(\lambda) = \sum_{j=0}^m w_j H_{j+1}(\lambda + \rho)$, where $H_j(\lambda) = \sum_{p \vdash j} \prod_{1 \leq i \leq n} \lambda_i^{p_i}$ are the complete symmetric functions. Note that each w_j is a linear combination of $\{\zeta_0, \ldots, \zeta_j\}$ and $(w_0, w_1, \ldots, w_j)$ can equal any point in $\mathbb{C}^{j+1}$; this observation is key to the proof of the Shapovalov determinant formula.

Define the Kostant partition function τ as $\tau(\nu) = \dim U(\mathfrak{n}^-)_\nu$. Then:

Theorem 2.1. *Up to a nonzero constant factor, the Shapovalov determinant computed in basis* $\{f^{\mathbf{m}}\}_{\mathbf{m}\vdash\nu}$ *is given by*

$$\det S_\nu(\lambda) = \left(\prod_{\alpha\in\Delta^+(V)}\prod_{k=1}^{\infty}(P(\lambda)-P(\lambda-k\alpha))^{\tau(\nu-k\alpha)}\right)\left(\prod_{\alpha\in\Delta^+(\mathfrak{gl}_n)}\prod_{k=1}^{\infty}((\lambda+\rho,\alpha)-k)^{\tau(\nu-k\alpha)}\right).$$

Remark 2.1. For the case of $\zeta = \zeta_0 r_0 + \zeta_1 r_1$ with $\zeta_1 \neq 0$, we get the classical formula from $[KK]$.

Proof. The proof of this theorem is quite similar to the classical case with a few technical details and differences that will be explained below. We begin with the following lemma, which shows that irreducible factors of $\det S_\nu(\lambda)$ must divide $P(\lambda)-P(\lambda-\mu)$ for some $\mu\in Q^+$.

Lemma 2.1. *Suppose* $\det S_\nu(\lambda) = 0$*. Then, there exists* $\mu\in Q^+\backslash\{0\}$ *such that* $P(\lambda)-P(\lambda-\mu) = 0$*.*

Proof. Note that $\det S_\nu(\lambda) = 0$ implies that the Verma module $M(\lambda)$ has a highest weight vector of weight $\lambda-\mu$ for some $\mu\in Q^+$ satisfying $0<\mu<\nu$. Thus, $M(\lambda-\mu)$ is embedded in $M(\lambda)$. Since t_1' acts by constants on both $M(\lambda)$ and $M(\lambda-\mu)$, which can be considered as a submodule of $M(\lambda)$, $P(\lambda)-P(\lambda-\mu) = 0$ as desired. □

The top term of the Shapovalov determinant $\det S_\nu(\lambda)$ in the basis $\{f^{\mathbf{m}}\}_{\mathbf{m}\vdash\nu}$ comes from the product of diagonal elements; that is, the top term is given by $\prod_{\mathbf{m}\vdash\nu}\prod[\sigma(f_\alpha),f_\alpha]^{\mathbf{m}_\alpha}(\lambda)$. We already know that the top term of $[e_{ij},e_{ji}](\lambda)$ for $i<j$ is $\lambda_i-\lambda_j = (\lambda,\alpha)$ where α is the weight of e_{ij}. The following lemma gives the top term of $[y_j,x_j](\lambda)$:

Lemma 2.2. *The highest term of* $[y_j,x_j](\lambda)$ *for* $\zeta = \zeta_0 r_0 + \ldots + \zeta_m r_m$ *is* $\zeta_m\sum_{\mathbf{p}}(\mathbf{p}_j+1)\prod\lambda_i^{\mathbf{p}_i}$*, where the sum is over all partitions* $\mathbf{p}$ *of* m *into* n *summands.*

Proof. From [EGG] Theorem 4.2, we know that the top term of $[y_j,x_j]$ for $\zeta = \zeta_0 r_0+\zeta_1 r_1+\cdots+r_m$ is given by the coefficient of τ^m in $\det(1-\tau A)^{-1}(x_j,(1-\tau A)^{-1}y_j)$. Because the set of diagonalizable matrices is dense in $\mathfrak{gl}_n$, we can assume A is a diagonal matrix $A = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ so that
$\det(1-\tau A)^{-1} = \prod\frac{1}{1-\tau\lambda_i} = \sum_k\sum_{p\vdash k}\prod_i\lambda_i^{\mathbf{p}_i}\tau^k$ and $x_j(1-\tau A)^{-1}y_j = \frac{1}{1-\tau\lambda_j} = 1+\lambda_j\tau+\cdots$. Multiplying these series gives the statement in the lemma. □

Thus, we see that the top term of the determinant computed in the basis $\{f^{\mathbf{m}}\}_{\mathbf{m}\vdash\nu}$, up to a scalar multiple, is of the form

$$\left(\prod_{\alpha\in\Delta^+(\mathfrak{gl}_n)}(\lambda,\alpha)^{\sum_{\mathbf{m}}\mathbf{m}_\alpha}\right)\left(\prod_{\alpha=y_j\in\Delta^+(V)}\left(\sum_{\mathbf{p}}(\mathbf{p}_j+1)\prod\lambda_i^{\mathbf{p}_i}\right)^{\sum_{\mathbf{m}}\mathbf{m}_\alpha}\right).$$

Since $\tau(\mu)$ is the number of partitions of a weight μ, the sum $\sum_{\mathbf{m}}\mathbf{m}_\alpha$ over all partitions $\mathbf{m}$ of ν with α fixed must equal $\sum_{k=1}^{\infty}\tau(\nu-k\alpha)$, so the expression above simplifies to

$$\left(\prod_{\alpha\in\Delta^+(\mathfrak{gl}_n)}\prod_{k=1}^{\infty}(\lambda,\alpha)^{\tau(\nu-k\alpha)}\right)\left(\prod_{\alpha=y_j\in\Delta^+(V)}\prod_{k=1}^{\infty}\left(\sum_{\mathbf{p}\vdash m}(\mathbf{p}_j+1)\prod\lambda_i^{\mathbf{p}_i}\right)^{\tau(\nu-k\alpha)}\right).$$

This highest term comes from the product of the highest terms of factors of $P(\lambda)-P(\lambda-\mu)$ for various $\mu\in Q^+$. Let us now prove that $P(\lambda)-P(\lambda-\mu)$ is irreducible as a polynomial in λ for all $\mu\neq k\alpha$, $\alpha\in\Delta^+(\mathfrak{gl}_n)$. If this claim is true, then all μ contributing to the above product should be quasiroots; if $\mu = k\alpha$ for $\alpha\in\Delta^+(\mathfrak{gl}_n)$, the linear factor of $P(\lambda)-P(\lambda-\mu)$,

$(\lambda+\rho,\alpha)-k$, has highest term (λ,α), which appears in the highest term of the Shapovalov determinant, while if $\mu \neq k\alpha$ for some $\alpha \in \Delta^+(\mathfrak{gl}_n)$, the highest term of the irreducible polynomial $P(\lambda)-P(\lambda-\mu)$, $\sum_{p\vdash m}\sum_j \mu_j(\mathbf{p}_j+1)\prod \lambda_i^{\mathbf{p}_i}$, does not match any factor in the highest term of the Shapovalov determinant unless μ is a V-quasiroot. Finally, since $\frac{P(\lambda)-P(\lambda-k\alpha)}{(\lambda+\rho,\alpha)-k}$ is irreducible for $\alpha \in \Delta^+(\mathfrak{gl}_n)$, only the linear factor $(\lambda+\rho,\alpha)-k$ of $P(\lambda)-P(\lambda-k\alpha)$ appears in the Shapovalov determinant.

Now, let us prove the claim that $P(\lambda)-P(\lambda-\mu)$ is irreducible for $\mu \neq k\alpha$ ($\alpha \in \Delta^+(\mathfrak{gl}_n)$). Consider the parameters w_i as formal variables. Then, we have $P(\lambda)-P(\lambda-\mu) = \sum_{i\geq 0} w_i(H_{i+1}(\lambda+\rho)-H_{i+1}(\lambda+\rho-\mu))$. We can absorb the ρ vector into the λ vector. For this polynomial to be reducible in w_i and λ_j, the coefficient of w_0 should be zero: $H_1(\lambda)-H_1(\lambda-\mu) = H_1(\mu) = 0$. Also, since the coefficient of w_1 is linear in λ_j, it must divide the coefficients of every other w_i. In particular, the highest term of $H_2(\lambda)-H_2(\lambda-\mu)$ must divide that of $H_3(\lambda)-H_3(\lambda-\mu)$. The highest term of $H_2(\lambda)-H_2(\lambda-\mu)$ is $\sum_i \lambda_i(\mu_i+\sum_j \mu_j) = (\lambda,\mu)$ and the highest term of $H_3(\lambda)-H_3(\lambda-\mu)$ is given by $H_3'(\lambda)(\mu)$, the evaluation of the gradient $H_3'(\lambda)$ at μ. Since this term is quadratic and is divisible by (λ,μ), we can write $H_3'(\lambda)(\mu) = (\lambda,\mu)(\lambda,\xi)$ for some $\xi \in \mathfrak{h}^*$. Now, let us match coefficients of $\lambda_i\lambda_j$ for $i \neq j$ and of λ_i^2 on both sides of the equation. By doing so (and using the fact that $\sum \mu_i = 0$), we obtain $\mu_i\xi_j + \mu_j\xi_i = \mu_i + \mu_j$ and $\mu_i\xi_i = 2\mu_i$. Since $\mu_1+\cdots+\mu_n = 0$ and $\mu \neq 0$, at least two of μ_i are nonzero, say μ_{i_1} and μ_{i_2}. From the two equations, we obtain $\mu_{i_1}+\mu_{i_2} = 0$. If $\mu_{i_3} \neq 0$, then by similar arguments, $\mu_{i_1}+\mu_{i_3} = \mu_{i_2}+\mu_{i_3} = \mu_{i_1}+\mu_{i_2} = 0$, which is impossible since $\mu_{i_1},\mu_{i_2},\mu_{i_3} \neq 0$. Thus, $P(\lambda)-P(\lambda-\mu)$ is reducible only if exactly two of the μ_i are nonzero and opposite to each other—that is, $\mu = k\alpha$ for $\alpha \in \Delta^+(\mathfrak{gl}_n)$. For such μ, $(\lambda+\rho,\mu)-k$, is the linear factor of $P(\lambda)-P(\lambda-k\alpha)$. Similar arguments show that $\frac{P(\lambda)-P(\lambda-k\alpha)}{(\lambda+\rho,\alpha)-k}$ is irreducible for any $\alpha \in \Delta^+(\mathfrak{gl}_n)$, $k \in \mathbb{N}$.

To prove the power of each factor in the determinant formula of Theorem 2.1 is correct, we use an argument involving the Jantzen filtration, which we define as in [KK] page 101 (for our purposes, we switch $U(\mathfrak{g})$ to $H_\zeta(\mathfrak{gl}_n)$). The Jantzen filtration is a technique to track the order of zero of a bilinear form's determinant. Instead of considering the complex numbers, we consider the localized polynomials $\mathbb{C}\langle t\rangle$, defined as $p(t)/q(t)$ with $p(t),q(t) \in \mathbb{C}[t]$ and $q(0) \neq 0$. A word-to-word generalization of Lemma 3.3 in [KK] to our setting proves that the power of $P(\lambda)-P(\lambda-k\alpha)$ for $\alpha \in \Delta^+(V)$ and $(\lambda+\rho,\alpha)-k$ for $\alpha \in \Delta^+(\mathfrak{gl}_n)$ is given by $\tau(\nu-k\alpha)$, completing the proof of Theorem 2.1. □

3 The Center of $H_\zeta(\mathfrak{gl}_2)$

We first describe a basis for the center of $U(\mathfrak{gl}_n)$. Let $\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3,\ldots,\mathcal{Q}_n \in S(\mathfrak{gl}_n^*)$ (which can be identified as members of $S(\mathfrak{gl}_n)$ under the trace-map) be defined by the power series $\det(t\mathrm{Id}-X) = \sum_{j=0}^n (-1)^j t^{n-j}\mathcal{Q}_j(X)$, and let β_i be the image of $\mathcal{Q}_i$ under the symmetrization map from $S(\mathfrak{gl}_n)$ to $U(\mathfrak{gl}_n)$. To reduce the number of subscripts, we will use β to refer to β_1. The center of $U(\mathfrak{gl}_n)$ is a polynomial algebra generated by these β_i. We write equations in this section in terms of β_i because their commutativity simplifies computations.

Let $t_i = \sum_j x_j[\beta_i, y_j]$. In [T], it was shown that the center of $H_0(\mathfrak{gl}_n)$ is a polynomial algebra in t_i, $1 \leq i \leq n$, and that there exist unique (up to a constant) $c_i \in \mathfrak{z}(U(\mathfrak{gl}_n))$ such that the center of $H_\zeta(\mathfrak{gl}_n)$ is a polynomial algebra in $t_i' = t_i + c_i$.

Definition 3.1. The *Casimir element* of $H_\zeta(\mathfrak{gl}_n)$ is defined (up to a constant) as t_1'.

In this section, we will make use of the anti-involution σ, defined in the beginning of Section 2. Instead of working with t_i in this section, we shall work with $\tilde{t}_i = \sum_j [\beta_i, y_j]x_j$ instead for convenience. It is straightforward to see that $\mathfrak{z}(H_0(\mathfrak{gl}_n)) = \mathbb{C}[\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n]$ and that $\mathfrak{z}(H_\zeta(\mathfrak{gl}_n)) = \mathbb{C}[\tilde{t}_1 + C_1, \tilde{t}_2 + C_2, \ldots, \tilde{t}_n + C_n]$ for some $C_1, C_2, \ldots, C_n \in \mathfrak{z}(U(\mathfrak{gl}_n))$. For $\mathfrak{gl}_2$, $\tilde{t}_1 = y_1x_1 + y_2x_2$ and $\tilde{t}_2 = y_1e_{22}x_1 + y_2e_{11}x_2 - y_2e_{12}x_1 - y_1e_{21}x_2 - \frac{1}{2}(y_1x_1 + y_2x_2)$. Note (as in [T]) that $\tilde{t}_i$ and elements of $\mathfrak{z}(U(\mathfrak{gl}_2))$ are fixed by σ. Proofs involving technical computations are relegated to the Appendix.

3.1 Basis for PBW Deformations

We start by constructing an alternative basis s_m for PBW deformations such that the Casimir element's action on the Verma module of $H_{s_m}(\mathfrak{gl}_2)$ is given by

$$H_{m+1}(\lambda + \rho) = \textstyle\sum_{i=0}^{m+1} (\lambda_1 + 1)^i \lambda_2^{m+1-i}.$$

Let us define $\gamma = 1 + \beta^2 - 4\beta_2 \in \mathfrak{z}(U(\mathfrak{gl}_2))$ and $u_k = \frac{(1+\sqrt{\gamma})^k - (1-\sqrt{\gamma})^k}{2\sqrt{\gamma}} \in \mathbb{C}[\gamma]$.

Definition 3.2. Define $s_m = A_m(y, x) + B_m y \otimes x$ with

$$A_m = \frac{1}{2^{m+1}} \sum_{j=1}^{\lfloor \frac{m+2}{2} \rfloor} \sum_{k=0}^{j-1} \frac{4j - m - 1}{2j + 1} \binom{m+2}{2k+1} \binom{m+1-2k}{2j-2k-1} \beta^{m+2-2j} \gamma^k,$$

$$B_m = \frac{1}{2^m} \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{j-1} \binom{m+2}{2j+1} \binom{2j}{2k+1} \beta^{m+1-2j} \gamma^k.$$

Theorem 3.1. *The pairings s_j constitute a basis for PBW deformations.*

To prove Theorem 3.1 and other results of this section, we use the following lemma, which reduces many statements in this section to technical computations.

Lemma 3.1.

$$[\beta^m \gamma^n, x_i] = [\beta^m \gamma^n + (\beta + 1)^m (\beta u_{2n} - u_{2n+1})] x_i - 2(\beta + 1)^m u_{2n} (e_{1i}x_1 + e_{2i}x_2).$$

This lemma can be proved by finding a recursion linking $[\beta^m \gamma^n, x_i]$ to $[\beta^m \gamma^{n-1}, x_i]$ and solving it using the standard theory of linear recursions.

Proof of Theorem 3.1. In [EGG], it was shown that the PBW property is equivalent to the Jacobi identity: for all $v_1, v_2, v_3 \in V \oplus V^*$, $[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$. Since $[x_1, x_2] = 0$ for $x_1, x_2 \in V^*$, it suffices to show that $[x_1, [x_2, y_2]] + [x_2, [y_2, x_1]] + [y_2, [x_1, x_2]] = -[x_1, A_m + B_m e_{22}] + [x_2, B_m e_{21}] = [A_m, x_1] + [B_m e_{22}, x_1] + [x_2, B_m e_{21}] = 0$; the other cases would follow because of the anti-involution σ.

Since $[x_2, B_m e_{21}] = -[B_m, x_2]e_{21} + B_m x_1$ and $[B_m e_{22}, x_1] = [B_m, x_1]e_{22}$, the Jacobi identity becomes $[A_m, x_1] + [B_m, x_1]e_{22} - [B_m, x_2]e_{21} + B_m x_1 = 0$.

Let us try to expand $[B_m, x_1]e_{22} - [B_m, x_2]e_{21}$ first. Note that by Lemma 3.1, $[B_m, x_1] = K_1 x_1 + K_2(e_{11}x_1 + e_{21}x_2)$ and $[B_m, x_2] = K_1 x_2 + K_2(e_{12}x_1 + e_{22}x_2)$ where K_1 and K_2 are given by:

$$K_1 = \frac{1}{2^m} \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{j-1} \binom{m+2}{2j+1} \binom{2j}{2k+1} \left(\beta^{m+1-2j} \gamma^k + (\beta + 1)^{m+1-2j} (\beta u_{2k} - u_{2k+1}) \right),$$

$$K_2 = -\frac{1}{2^{m-1}} \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{j-1} \binom{m+2}{2j+1} \binom{2j}{2k+1} (\beta + 1)^{m+1-2j} u_{2k}.$$

By rearranging terms, we can then write

$$[K_1x_1+K_2(e_{11}x_1+e_{21}x_2)]\;\; e_{22}-[K_1x_2+K_2(e_{12}x_1+e_{22}x_2)]\,e_{21}= \\ -K_1(e_{11}x_1+e_{21}x_2)+\left[K_1(\beta-1)+K_2\left(\frac{\beta^2+1-\gamma-2\beta}{4}\right)\right]x_1.$$

We can evaluate $[A_m, x_1]$ using Lemma 3.1. Substituting this and the above expression into $[A_m,x_1]+[B_m,x_1]e_{22}-[B_m,x_2]e_{21}+B_mx_1$ gives $C(\beta,\gamma)x_1+D(\beta,\gamma)(e_{11}x_1+e_{21}x_2)$, where

$$\begin{aligned} C = \frac{1}{2^{m+1}}\sum_{j=1}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j-1}&\frac{(m+2)!}{(2j+1)(2k+1)!(2j-2k-1)!(m+2-2j)!}\\ &\times\Bigg((m+3)\Big(\beta^{m+2-2j}\gamma^k+(\beta+1)^{m+2-2j}\beta u_{2k}-(\beta+1)^{m+2-2j}u_{2k+1}\Big)\\ &-\;4(m+2-2j)\Big(\beta(\beta+1)^{m+1-2j}u_{2k}-(\beta+1)^{m+1-2j}u_{2k+1}\Big)\\ &-\;(m+2-2j)(\beta+1)^{m+1-2j}((\beta-1)^2-\gamma)u_{2k}\Bigg), \end{aligned}$$

$$\begin{aligned} D = -\frac{1}{2^{m}}\sum_{j=1}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j-1}&\frac{(m+2)!}{(2j+1)(2k+1)!(2j-2k-1)!(m+2-2j)!}\\ &\times\Bigg((m+2-2j)\beta^{m+1-2j}\gamma^k\\ &+\;(2j+1)(\beta+1)^{m+2-2j}u_{2k}-(m+2-2j)(\beta+1)^{m+1-2j}(u_{2k}+u_{2k+1})\Bigg). \end{aligned}$$

Lemma 3.2. *C and D are identically* 0.

The proof of this Lemma is given in the Appendix.

Thus, the Jacobi identity holds for $\zeta = s_m$ for all m. Since $\deg s_m = \deg r_m$ with respect to the grading where $\deg e_{ij}=1$, s_m constitute a basis for PBW pairings. □

If ζ is a PBW deformation, we will use ζ_i to refer exclusively to the components of ζ in the r_i basis: $\zeta=\zeta_0r_0+\zeta_1r_1+\cdots$. We will denote the components of ζ in the s_i basis by w_i instead: $\zeta=w_0s_0+w_1s_1+\cdots$.

3.2 Center of $H_\zeta(\mathfrak{gl}_2)$

We now give the following construction for the center of $H_\zeta(\mathfrak{gl}_2)$. First, define

$$\begin{aligned} C_1(m) &= \frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(\binom{m+2}{2j+1}\binom{2j+1}{2k+1}\beta^{m+1-2j}\gamma^k-\binom{m+2}{2j}\binom{2j}{2k+1}\beta^{m+2-2j}\gamma^k\right)\\ C_2(m) &= \frac{1}{2^{m+2}}\sum_{j=0}^{\lfloor\frac{m+1}{2}\rfloor}\sum_{k=0}^{j}\Bigg(\frac{m+1-2j-2k}{m+2-2j}\binom{m+2}{2k+1}\binom{m+1-2k}{2j-2k}\beta^{m+1-2j}\gamma^k\\ &\qquad -\frac{2j+2k-m}{2j-2k+1}\binom{m+2}{2k+1}\binom{m+1-2k}{2j-2k}\beta^{m+2-2j}\gamma^k\Bigg). \end{aligned}$$

Theorem 3.2. *For a deformation $\zeta = \sum_i w_i s_i$, the center is given by the polynomial algebra $\mathfrak{z}(H_\zeta(\mathfrak{gl}_2)) = \mathbb{C}\left[\tilde{t}_1 + \sum w_i C_1(i),\ \tilde{t}_2 + \sum w_i C_2(i)\right]$.*

Proof. It suffices to consider the case $\zeta = s_m$.

Lemma 3.3. $[x_1, \tilde{t}_1 + C_1(m)] = [x_1, \tilde{t}_2 + C_2(m)] = 0.$

The proof of this Lemma is given in the Appendix.

Thus, by the transitivity of GL_2 action on V^*, $[x, \tilde{t}_1 + C_1(m)]$ and $[x, \tilde{t}_2 + C_2(m)]$ for all $x \in V^*$. It was shown in [T] that $\tilde{t}_1$ and $\tilde{t}_2$ commute with $U(\mathfrak{gl}_2)$ and since $C_1(m), C_2(m) \in \mathfrak{z}(U(\mathfrak{gl}_2))$, $\tilde{t}_1 + C_1(m)$ and $\tilde{t}_2 + C_2(m)$ commute with $U(\mathfrak{gl}_2)$ also. Furthermore, by using the anti-involution σ, we see that if $[x, \tilde{t}_1 + C_1(m)] = 0$ and $[x, \tilde{t}_2 + C_2(m)] = 0$ for all $x \in V^*$, then $[y, \tilde{t}_1 + C_1(m)] = 0$ and $[y, \tilde{t}_2 + C_2(m)] = 0$ for all $y \in V$. This shows that $\tilde{t}_1' = \tilde{t}_1 + C_1(i)$ and $\tilde{t}_2' = \tilde{t}_2 + C_2(i)$. □

3.3 Action of Center on Verma Module

In [T], it was proven that the leading term of the Casimir element's action on the Verma module $M(\lambda)$ is given by $P(\lambda) = H_{m+1}(\lambda)$ for a deformation $\zeta = r_m$. In the new basis $\{s_i\}$ the action of the center can be written explicitly. Namely:

Theorem 3.3. *For a deformation $\zeta = \sum_i w_i s_i$, the actions of $\tilde{t}_1'$ and $\tilde{t}_2'$ on $M(\lambda)$ are given by $P(\lambda) = \sum_i w_i H_{i+1}(\lambda + \rho)$ and $\sum_i w_i \left(\frac{1}{2} H_{i+1}(\lambda+\rho) + H_{i+2}(\lambda+\rho) - (\lambda_1+1)^{i+2} - \lambda_2^{i+2}\right)$ respectively.*

The proof of this theorem is given in the Appendix.

4 The Casimir Element of $H_\zeta(\mathfrak{gl}_n)$

In this section, we construct the Casimir element of $H_\zeta(\mathfrak{gl}_n)$ and prove that its action on the Verma module $M(\lambda)$ is given by $P(\lambda) = \sum_{j=0}^m w_j H_{j+1}(\lambda + \rho)$.

4.1 Center

Let us switch to the approach elaborated in [EGG] Section 4, where all deformations satisfying the PBW property were determined. Define $\delta^{(m)} = (i\partial)^m \delta$ with δ being a standard delta-function at 0, i.e., $\int \delta(\theta)\phi(\theta)d\theta = \phi(0)$. Let $f(z)$ be a polynomial satisfying $f(z) - f(z-1) = \partial^n(z^n \zeta(z))$, where $\zeta(z)$ is the generating series of the deformation parameters: $\zeta(z) = \zeta_0 + \zeta_1 z + \zeta_2 z^2 + \cdots$. Recall from [EGG], Section 4.2, that for $\hat{f}(\theta) = \sum_{m\geq 0} f_m \delta^{(m)}(\theta)$,

$$[y, x] = \frac{1}{2\pi^n} \int_{v\in\mathbb{C}^n:|v|=1} (x, (v\otimes\bar{v})y) \int_{-\pi}^{\pi} \left(1 - e^{-i\theta}\right) \hat{f}(\theta) e^{i\theta(v\otimes\bar{v})}\, d\theta\, dv.$$

Theorem 4.1. *Let $g(z) = \sum g_m z^m = \sum \frac{f_m}{(m+1)(m+2)\cdots(m+n-1)} z^m$. The Casimir element of $H_\zeta(\mathfrak{gl}_n)$ is given by $t_1' = \sum x_i y_i + \mathrm{Res}_{z=0}\, g(z^{-1}) \det(1 - zA)^{-1} dz/z$.*

Proof. Define $C' = \text{Res}_{z=0} g(z^{-1}) \det(1 - zA)^{-1} dz/z$. Let us compute $[y, t_1 + C']$. Then, $[y, t_1 + C'] = \sum_i [y, x_i] y_i + [y, C']$. Let us rewrite the first sum:

$$\begin{aligned}\sum_i [y, x_i] y_i &= \frac{1}{2\pi^n} \sum_i \int_{v \in \mathbb{C}^n : |v|=1} \int_{-\pi}^{\pi} (1 - e^{-i\theta}) \hat{f}(\theta) e^{i\theta(v \otimes \bar{v})} (x_i, (v \otimes \bar{v}) y) y_i \, d\theta \, dv \\ &= \frac{1}{2\pi^n} \int_{|v|=1} \int_{-\pi}^{\pi} (1 - e^{-i\theta}) \hat{f}(\theta) e^{i\theta(v \otimes \bar{v})} \otimes (v \otimes \bar{v}) y \, d\theta \, dv.\end{aligned}$$

We define $F_m(A) = \int_{|v|=1} \langle Av, v \rangle^{m+1} dv = \int_{|v|=1} (v \otimes \bar{v})^{m+1} dv$ as in [EGG] Section 4.2; it was shown there that $\sum_m f_m F_{m-1}(A) = 2\pi^n \text{Res}_{z=0} g(z^{-1}) \det(1 - zA)^{-1} z^{-1} dz = 2\pi^n C'$. Thus, we can write

$$C' = \frac{1}{2\pi^n} \sum_m f_m \int_{|v|=1} (v \otimes \bar{v})^m dv = \frac{1}{2\pi^n} \int_{|v|=1} \int_{-\pi}^{\pi} \hat{f}(\theta) e^{i\theta(v \otimes \bar{v})} \, d\theta \, dv,$$

so that $[y, C'] = \frac{1}{2\pi^n} \int_{|v|=1} \int_{-\pi}^{\pi} \hat{f}(\theta) [y, e^{i\theta(v \otimes \bar{v})}] \, d\theta \, dv$. Since

$$e^{-i\theta(v \otimes \bar{v})} [y, e^{i\theta(v \otimes \bar{v})}] = e^{-i\theta(v \otimes \bar{v})} y e^{i\theta(v \otimes \bar{v})} - y = e^{-i\theta \text{ad}(v \otimes \bar{v})} y - y = (e^{-i\theta} - 1)(v \otimes \bar{v}) y,$$

we get $[y, C'] = \frac{1}{2\pi^n} \int_{|v|=1} \int_{-\pi}^{\pi} \hat{f}(\theta) e^{i\theta(v \otimes \bar{v})} (e^{-i\theta} - 1)(v \otimes \bar{v}) y \, d\theta \, dv$, so $\sum_i [y, x_i] y_i + [y, C'] = 0$ as desired. By using the anti-involution σ, this implies $[x, t_1 + C'] = 0$ for any $x \in V^*$, while $[e_{ij}, t_1 + C'] = 0$ by [T], and hence, $t_1' = t_1 + C'$. □

4.2 Action of Casimir Element on Verma Module

In this section, we justify our claim that the action of the Casimir element t_1' is given by $P(\lambda) = \sum_{j=0}^m w_j H_{j+1}(\lambda + \rho)$. Obviously, t_1' acts by a scalar on $M(\lambda - \rho)$, which we will denote by $t_1'(\lambda)$. Since $t_1' = \sum x_i y_i + C'$, $C' \in \mathfrak{z}(U(\mathfrak{g})) \cong S(\mathfrak{g})^G$, we see that $t_1'(\lambda) = C'(\lambda)$ where $C'(\lambda)$ denotes the constant by which C' acts on $M(\lambda - \rho)$.

Theorem 4.2. *Let* $w(z) = \sum w_p z^p = z^{1-n} \left(\frac{1}{2 \sinh(\partial/2)} \right)^{n-1} f(z)$. *Then,* $t_1'(\lambda) = \sum w_p H_p(\lambda)$.

Proof. Instead of considering the Verma module $M(\lambda - \rho)$ of $H_\zeta(\mathfrak{gl}_n)$, we can use a finite-dimensional representation of $U(\mathfrak{gl}_n)$ in the proof since $C'(\lambda)$ is a polynomial in λ. For a dominant weight $\lambda - \rho$ (so that the highest weight $\mathfrak{gl}_n$-module $V_{\lambda-\rho}$ is finite dimensional) we define the normalized trace $T(\lambda, \theta) = \text{tr}_{V_{\lambda-\rho}} (e^{i\theta(v \otimes \bar{v})}) / \dim V_{\lambda-\rho}$ for any v satisfying $|v| = 1$ (note that $T(\lambda, \theta)$ does not depend on v). To compute $T(\lambda, \theta)$, we will use the Weyl Character formula (see [Ful]): $\chi_{\lambda-\rho} = \frac{\sum_{w \in W} (-1)^w e^{w\lambda}}{\sum_{w \in W} (-1)^w e^{w\rho}}$ where W denotes the Weyl group (which is S_n for $\mathfrak{gl}_n$). However, direct substitution of $e^{i\theta(v \otimes \bar{v})}$ into this formula gives zero in the denominator, and thus, we compute the limit $\lim_{\epsilon \to 0} \chi_{\lambda-\rho}(e^{i\theta(v \otimes \bar{v}) + \epsilon\mu})$ for a general diagonal matrix μ.

Without loss of generality, we may suppose $v = y_1$, so

$$v \otimes \bar{v} = w_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then,

$$\lim_{\epsilon\to 0}\chi_{\lambda-\rho}(e^{i\theta(v\otimes\bar{v})+\epsilon\mu}) = \lim_{\epsilon\to 0}\frac{\sum_{w\in S_n}(-1)^w e^{\langle w\lambda, i\theta w_1+\epsilon\mu\rangle}}{\sum_{w\in S_n}(-1)^w e^{\langle w\rho, i\theta w_1+\epsilon\mu\rangle}}$$
$$= \lim_{\epsilon\to 0}\frac{\sum_{w\in S_n}(-1)^w e^{\langle w\lambda, i\theta w_1+\epsilon\mu\rangle}}{\prod_{\alpha\in\Delta^+(\mathfrak{gl}_n)}(e^{\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle})}$$

(recall that $\Delta^+(\mathfrak{gl}_n)$ is defined as the set of positive roots of $\mathfrak{gl}_n$, given by $\Delta^+(\mathfrak{gl}_n) = \{e_{ii}^* - e_{jj}^* : 1\le i<j\le n\}$).

We first compute the denominator. Partition $\Delta^+(\mathfrak{gl}_n)$ into $\Delta_1\sqcup\Delta_2 = \Delta^+(\mathfrak{gl}_n)$, where $\Delta_1 = \{e_{11}^* - e_{jj}^* : 1<j\le n\}$ and $\Delta_2 = \Delta^+(\mathfrak{gl}_n)\backslash\Delta_1$. For $\alpha\in\Delta_1$,

$$\lim_{\epsilon\to 0}\left(e^{\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle}\right) = e^{i\theta/2} - e^{-i\theta/2} = 2i\sin\left(\frac{\theta}{2}\right),$$

so $\lim_{\epsilon\to 0}\prod_{\alpha\in\Delta_1}(e^{\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle})^{-1} = \left(2i\sin\left(\frac{\theta}{2}\right)\right)^{1-n}$.

Next, we compute the numerator. We can divide $S_n = \bigsqcup_{1\le j\le n} B_j$, where $B_j = \{w\in S_n | w(j) = 1\}$. Note that $B_j = \sigma_j\cdot S_{n-1}$, where $\sigma_j = (1,2,\dots,j)$ and S_{n-1} denotes the subgroup of S_n corresponding to permutations of $\{1,2,\dots,j-1,j+1,\dots,n\}$. We can then write

$$\sum_{w\in B_j}(-1)^w e^{\langle w\lambda, i\theta w_1+\epsilon\mu\rangle} = \sum_{\sigma\in S_{n-1}}(-1)^{\sigma_j}(-1)^{\sigma} e^{i\theta\lambda_j} e^{\epsilon\langle\sigma_j\circ\sigma(\lambda),\mu\rangle}$$
$$= (-1)^{j-1} e^{i\theta\lambda_j} e^{\epsilon\lambda_j\mu_1}\sum_{\sigma\in S_{n-1}}(-1)^{\sigma} e^{\epsilon\langle\sigma(\tilde{\lambda}_j),\tilde{\mu}\rangle}$$

where $\widetilde{\lambda_j} = (\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1},\dots,\lambda_n)$ and $\tilde{\mu} = (\mu_2,\dots,\mu_n)$.

Combining the results of the last two paragraphs,

$$\lim_{\epsilon\to 0}\frac{\sum_{w\in S_n}(-1)^w e^{\langle w\lambda, i\theta w_1+\epsilon\mu\rangle}}{\prod_{\alpha\in\Delta^+(\mathfrak{gl}_n)}(e^{\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle})}$$
$$= \lim_{\epsilon\to 0}\sum_{1\le j\le n}(-1)^{j-1}\frac{e^{i\theta\lambda_j+\epsilon\lambda_j\mu_1}}{(2i\sin\frac{\theta}{2})^{n-1}}\frac{\sum_{\sigma\in S_{n-1}}(-1)^{\sigma} e^{\epsilon\langle\sigma(\tilde{\lambda}_j),\tilde{\mu}\rangle}}{\prod_{\alpha\in\Delta_2}(e^{\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle})}.$$

Using the Weyl character formula again, we see that

$$\frac{\sum_{\sigma\in S_{n-1}}(-1)^{\sigma} e^{\epsilon\langle\sigma(\tilde{\lambda}_j),\tilde{\mu}\rangle}}{\prod_{\alpha\in\Delta_2}(e^{\langle\alpha/2,\epsilon\mu\rangle} - e^{-\langle\alpha/2,\epsilon\mu\rangle})} = \mathrm{tr}_{V_{\tilde{\lambda}_j-\tilde{\rho}}}(e^{\epsilon\tilde{\mu}})$$

where $\tilde{\rho}$ is half the sum of all positive roots of $\mathfrak{gl}_{n-1}$. Thus,

$$\lim_{\epsilon\to 0}\frac{\sum_{\sigma\in S_{n-1}}(-1)^{\sigma} e^{\epsilon\langle\sigma(\tilde{\lambda}_j),\tilde{\mu}\rangle}}{\prod_{\alpha\in\Delta_2}(e^{\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta w_1+\epsilon\mu\rangle})} = \mathrm{tr}_{V_{\tilde{\lambda}_j-\tilde{\rho}}}(1) = \dim V_{\tilde{\lambda}_j-\tilde{\rho}}.$$

We substitute to obtain

$$\mathrm{tr}_{V_{\lambda-\rho}}(e^{i\theta(v\otimes\bar{v})}) = \sum_{1\le j\le n}(-1)^{j-1}\frac{e^{i\theta\lambda_j}\dim V_{\tilde{\lambda}_j-\tilde{\rho}}}{(2i\sin\frac{\theta}{2})^{n-1}}.$$

Our original goal was to calculate $T(\lambda,\theta) = \mathrm{tr}_{V_{\lambda-\rho}}(e^{i\theta(v\otimes\bar{v})})/\dim V_{\lambda-\rho}$. We obtain

$$T(\lambda,\theta) = \sum_{1\le j\le n} (-1)^{j-1} \frac{e^{i\theta\lambda_j}\dim V_{\tilde{\lambda}_j-\tilde{\rho}}}{(2i\sin\frac{\theta}{2})^{n-1}\dim V_{\lambda-\rho}}.$$

We can use the following dimension formula from [Ful]:

$$\dim V_{\lambda-\rho} = \prod_{1\le i<j\le n} \frac{\lambda_i-\lambda_j}{j-i}$$

where $V_{\lambda-\rho}$ is a $\mathfrak{gl}_n$-module. Hence, $T(\lambda,\theta) = (2i\sin(\theta/2))^{1-n}(n-1)!\sum_{j=1}^n \frac{e^{i\lambda_j\theta}}{\prod_{k\ne j}(\lambda_j-\lambda_k)}$. Since one can prove by induction that $\sum_{j=1}^n \frac{x_j^m}{\prod_{k\ne j}(x_j-x_k)} = H_{m-n+1}(x_1,...,x_n)$, we get

$$T(\lambda,\theta) = (2i\sin(\theta/2))^{1-n}(n-1)!\textstyle\sum_{p\ge0}\frac{H_p(\lambda)(i\theta)^{p+n-1}}{(p+n-1)!}.$$

Thus we get,

$$\begin{aligned} t_1'(\lambda) &= C'(\lambda) = \left(\frac{1}{2\pi^n}\int_{|v|=1}\int_{-\pi}^{\pi}\hat{f}(\theta)e^{i\theta(v\otimes\bar{v})}d\theta dv\right)(\lambda) = \frac{1}{(n-1)!}\int_{-\pi}^{\pi}\hat{f}(\theta)T(\lambda,\theta)d\theta \\ &= \int_{-\pi}^{\pi}\hat{f}(\theta)(2i\sin(\theta/2))^{1-n}\sum_{p\ge0}\frac{H_p(\lambda)(i\theta)^{p+n-1}}{(p+n-1)!}d\theta = \sum_{p\ge0} w_p' H_p(\lambda), \end{aligned}$$

where $w_p' = \int_{-\pi}^{\pi}\hat{f}(\theta)(2i\sin(\theta/2))^{1-n}\frac{(i\theta)^{p+n-1}}{(p+n-1)!}d\theta$. Let $w'(z) = \sum w_p' z^p$. Note that

$$\begin{aligned} \left(e^{\partial/2}-e^{-\partial/2}\right)^{n-1} z^{n-1}w'(z) &= \int_{-\pi}^{\pi}\hat{f}(\theta)\sum_{p\ge0}(2i\sin(\theta/2))^{1-n}\left(e^{\partial/2}-e^{-\partial/2}\right)^{n-1}\frac{(iz\theta)^{p+n-1}}{(p+n-1)!}\,d\theta \\ &= \int_{-\pi}^{\pi}\hat{f}(\theta)(2i\sin(\theta/2))^{1-n}\left(e^{\partial/2}-e^{-\partial/2}\right)^{n-1}e^{iz\theta}\,d\theta \\ &= \int_{-\pi}^{\pi}\hat{f}(\theta)(2i\sin(\theta/2))^{1-n}\left(e^{i\theta/2}-e^{-i\theta/2}\right)^{n-1}e^{iz\theta}\,d\theta \\ &= \int_{-\pi}^{\pi}\hat{f}(\theta)e^{iz\theta}\,d\theta = f(z). \end{aligned}$$

Hence, $w'(z) = z^{1-n}\left(\frac{1}{2\sinh(\partial/2)}\right)^{n-1}f(z) = w(z)$ and $w_p' = w_p$ as claimed. □

5 Finite Dimensional Representations

In this section, we investigate when the irreducible $H_\zeta(\mathfrak{gl}_n)$ representation $L(\lambda)$ is finite dimensional. As in the case for classical Lie algebras, this representation is a quotient of a Verma module $M(\lambda)$. In Section 5.1, we show that the finite dimensional $L(\lambda)$ must be *rectangular* and have characters (with respect to $\mathfrak{h}$) of the form

$$\sum_{0\le\lambda-\lambda'<\nu}\frac{\sum_{w\in S_n}(-1)^w e^{w(\lambda'+\rho')}}{\sum_{w\in S_n}(-1)^w e^{w\rho'}}$$

where the summation is over all dominant $\mathfrak{gl}_n$ weights λ', and $\nu\in\mathbb{N}^n$ is a parameter depending on ζ and λ. In Section 5.2, we show the existence of deformations ζ such that the representation $L(\lambda)$ of $H_\zeta(\mathfrak{gl}_n)$ has the above character.

5.1 Rectangular Nature of Irreducible Representations

Theorem 5.1. *All finite-dimensional irreducible representations of* $H_\zeta(\mathfrak{gl}_n)$ *must be of the form* $L(\lambda) = M(\lambda)/\left\{\sum_{i=1}^{n-1} e_{i+1,i}^{\lambda_i-\lambda_{i+1}+1}M(\lambda) + \sum_{i=1}^{n} x_i^{\nu_i}M(\lambda)\right\}$*, where* $\nu = (\nu_1,\ldots,\nu_n) \in \mathbb{N}^n$*, and* $\lambda - \nu'$ *are dominant* $\mathfrak{gl}_n$*-weights for all* $0 \le \nu' \le \nu$*.*

Proof. Let $L'(\lambda) = M(\lambda)/\left\{\sum_{i=1}^{n-1} e_{i+1,i}^{\lambda_i-\lambda_{i+1}+1}M(\lambda) + \sum_{i=1}^{n} x_i^{b'_i}M(\lambda)\right\}$ be a quotient (not necessarily irreducible) of $M(\lambda)$ for some integers b'_i. Since any finite dimensional representation is a quotient of some $L'(\lambda)$ (for some sufficiently large b'_i), it suffices to show that the maximal proper submodule of $L'(\lambda)$ is of form $\sum x_i^{l_i} L'(\lambda)$, in which case the irreducible representation $L'(\lambda)/\overline{L'(\lambda)}$ is of the form described in Theorem 5.1.

Regard $L'(\lambda)$ as a representation of $\mathfrak{gl}_n$. Since the decomposition of any positive weight of $\mathfrak{gl}_n$ into the sum of positive roots does not include weights of V, there exists a finite-dimensional irreducible $\mathfrak{gl}_n$ submodule $V_\lambda \subset L'(\lambda)$ that contains only the vectors generated by the action of $\mathfrak{gl}_n$ on the highest weight vector of $L'(\lambda)$. If we let $S_k = \mathrm{Sym}^k(x_1, x_2, ..., x_n)$, then $L'(\lambda)$ decomposes as $L'(\lambda) = V_\lambda \oplus (V_\lambda \otimes S_1) \oplus (V_\lambda \otimes S_2) \oplus \cdots$. We can further decompose each $V_\lambda \otimes S_i$ into irreducible modules of $\mathfrak{gl}_n$; once we do so, we find that $L'(\lambda)$ has a simple $\mathfrak{gl}_n$ spectrum. Note that $V_\mu \otimes S_1$ can be decomposed as $V_{\mu-e^*_{11}} \oplus V_{\mu-e^*_{22}} \oplus \cdots \oplus V_{\mu-e^*_{nn}}$ for $\alpha_i = e^*_{ii}$ (taking $V_{\mu-e^*_{ii}} = \{0\}$ if $\mu - e^*_{ii}$ is not dominant). We can thus associate each V_μ for $\mu = \lambda - a_1 e^*_{11} - \ldots - a_n e^*_{nn}$ in the decomposition of $L'(\lambda)$ with a lattice point $P_\mu = (-a_1, -a_2, \cdots, -a_n) \in \mathbb{Z}^n$. We draw a directed edge from P_μ to $P_{\mu'}$ if $V_{\mu'}$ is in the decomposition of $V_\mu \otimes S_1$, and we call the $P_{\mu'}$ smaller than the P_μ. A key property of this graph is that any $H_\zeta(\mathfrak{gl}_n)$-submodule of $L'(\lambda)$ intersecting the module V_μ must necessarily contain V_μ and all $V_{\mu'}$ such that $P_{\mu'}$ is reachable from P_μ by a walk along directed edges.

Now suppose that $L'(\lambda)$ has a proper maximal submodule L''_μ (over $H_\zeta(\mathfrak{gl}_n)$) with highest weight vector v_μ of weight μ associated with $P_\mu = (-a_1, \ldots, -a_n)$ (we are not assuming, however, that this submodule is generated by v_μ). Then, because the quotient module $L'(\lambda)/L''_\mu$ is finite dimensional and irreducible, it must have a lowest weight $\tilde\lambda$ from which all other points in the subgraph associated with $L'(\lambda)/L''_\mu$ can be reached by walking along reverse edges. Without loss of generality, suppose a_1 and a_2 are nonzero. Then, consider the two points $P_{\mu+e^*_{11}}$ and $P_{\mu+e^*_{22}}$. Both points are larger than μ and less than or equal to the origin P_λ, and so both points lie in the subgraph associated with $L'(\lambda)/L''_\mu$. However, since we can walk along reverse edges from the point corresponding to the lowest weight $\tilde\lambda$ to $\mu + e^*_{11}$ and $\mu + e^*_{22}$, we can also walk along reverse edges to μ, implying that $\tilde\lambda \in L''_\mu$, a contradiction. We conclude that any maximum submodule of $L'(\lambda)$ must be of the form given in Theorem 5.1. □

From the proof, we get a decomposition of $L(\lambda)$ into the sum of $\mathfrak{gl}_n$ modules $V_{\lambda'}$ for all dominant $\mathfrak{gl}_n$ weights λ' satisfying $0 \le \lambda - \lambda' < \nu$, where $\nu \in \mathbb{N}^n$ is some parameter depending on ζ and λ (using notations from the proof, $\nu = \lambda - \tilde\lambda + (1, 1, \ldots, 1)$). By the results of Section 2, we find that $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, where each ν_i is the smallest positive integer such that

$$P(\lambda) - P(\lambda - (0, \ldots, \underbrace{\nu_i}_{\text{i-th}}, 0, \ldots, 0)) = 0;$$

if the $H_\zeta(\mathfrak{gl}_n)$-module $L(\lambda)$ is finite dimensional, ν necessarily exists.

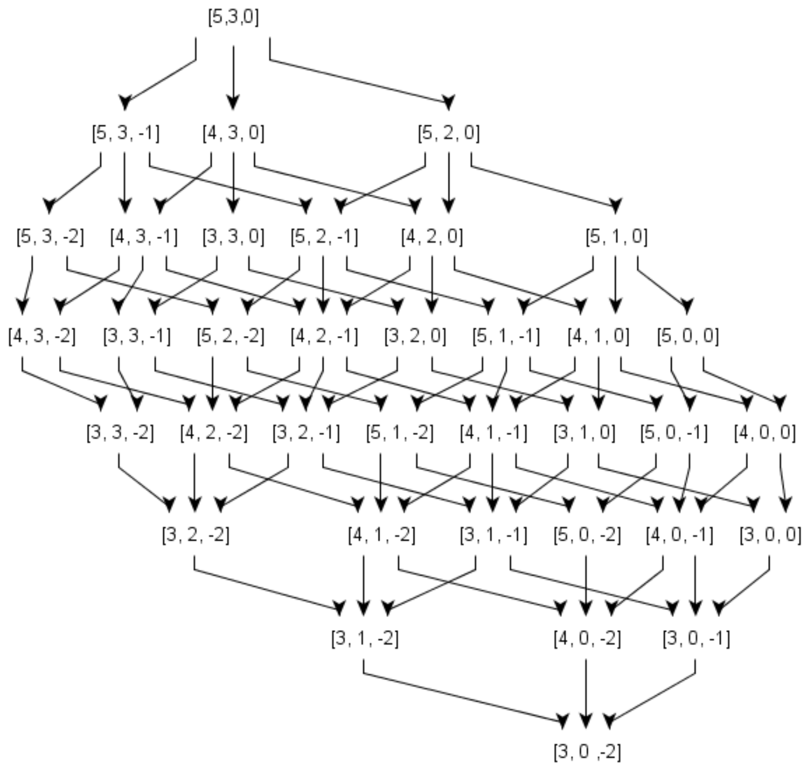


Figure 1: In this diagram, we use a graph to represent the three dimensional rectangular prism corresponding to a finite dimensional representation $L((5,3,0))$ of $H_\zeta(\mathfrak{gl}_3)$, with the highest weight of each $\mathfrak{gl}_3$ module indicated. Notice that an arrow points from a module of weight μ to one of weight $\mu - e_{ii}^*$ for some i. The rectangular nature of the representation is clear.

The decomposition of $L(\lambda)$ as a $\mathfrak{gl}_n$ module provides the character formula for $L(\lambda)$ as the sum of the characters of the $\mathfrak{gl}_n$ modules:

$$\chi_{\lambda;\zeta} = \sum_{0 \le \lambda - \lambda' < \nu} \frac{\sum_{w \in S_n} (-1)^w e^{w(\lambda' + \rho')}}{\sum_{w \in S_n} (-1)^w e^{w\rho'}} \qquad (*)$$

where ρ' is half the sum of positive roots of $\mathfrak{gl}_n$. As in the classical theory, the character allows us to calculate the decomposition of finite dimensional representations into irreducible ones.

Example 5.1. For $H_\zeta(\mathfrak{gl}_1)$, the irreducible finite dimensional representation $L(\lambda)$, for $\lambda \in \mathbb{C}$, has character $\chi_{\lambda,\zeta} = \sum_{\nu'=0}^{\nu-1} e^{\lambda-\nu'}$, where ν is some positive integer. If we describe $H_\zeta(\mathfrak{gl}_1)$ as in Example 1.1, we can easily calculate the Casimir element to be $fe + g(h)$, where g satisfies the equation $g(x) - g(x-1) = \phi(x)$. Then, ν is the smallest positive integer such that $g(\lambda) - g(\lambda - \nu) = 0$,

For $H_\zeta(\mathfrak{gl}_2)$, the irreducible finite dimensional representations are of the form $L(\lambda)$, with $\lambda = (\lambda_2 + m, \lambda_2) \in \mathbb{C}^2$ for some nonnegative integer m. The character of $L(\lambda)$ is of the form

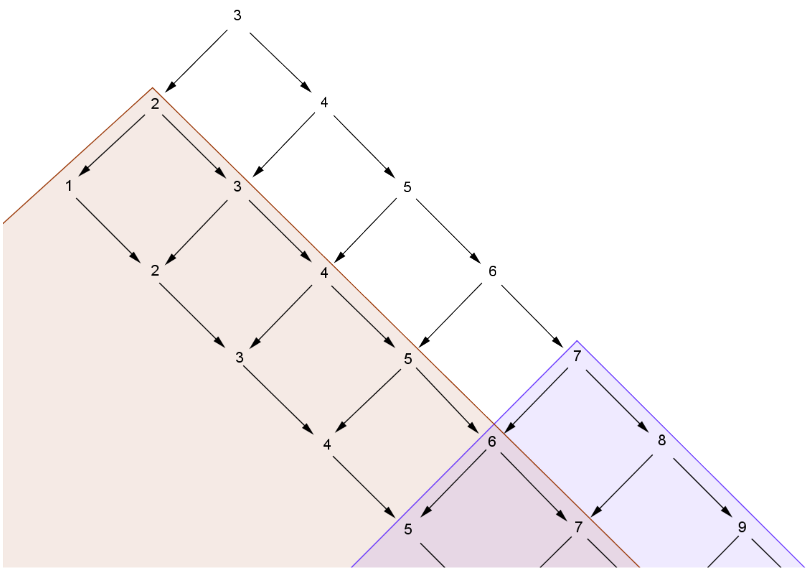
$$\chi_{\lambda;\zeta} = \sum_{\substack{(0,0) \le (\nu_1', \nu_2') < \nu \\ \nu_1' - \nu_2' \le m}} \frac{e^{(\lambda_2 + m - \nu_1', \lambda_2 - \nu_2')} - e^{(\lambda_2 - \nu_2' - 1, \lambda_2 + m - \nu_1' + 1)}}{1 - e^{(-1,1)}}.$$

for the minimal element $\nu \in \mathbb{N}^2$ such that $f_1(\lambda,\nu_1) = P(\lambda_2+m+1,\lambda_2) - P(\lambda_2+m+1-\nu_1,\lambda_2) = 0$ and $f_2(\lambda,\nu_2) = P(\lambda_2+m+1,\lambda_2) - P(\lambda_2+m+1,\lambda_2-\nu_2) = 0$. For instance, if $\zeta = \zeta_0 r_0$ with $\zeta_0 \neq 0$, then f_1 and f_2 are linear in ν_1 and ν_2, and so the only solution to the equations $f_1(\lambda,\nu_1) = 0$ and $f_2(\lambda,\nu_2) = 0$ is $\nu_1 = \nu_2 = 0$. Thus, $H_{\zeta_0 r_0}(\mathfrak{gl}_2)$ has no finite dimensional irreducible representations. If $\zeta = \zeta_0 r_0 + \zeta_1 r_1$ with $\zeta_1 \neq 0$, $P(\lambda) = \zeta_0(\lambda_1+1+\lambda_2) + \zeta_1((\lambda_1+1)^2 + (\lambda_1+1)\lambda_2 + \lambda_2^2)$, so $f_1(\lambda,\nu_1) = \zeta_1\nu_1\left(\frac{\zeta_0}{\zeta_1} + \lambda_2 + 2\lambda_1 + 2 - \nu_1\right)$ and $f_2(\lambda,\nu_2) = \zeta_1\nu_2\left(\frac{\zeta_0}{\zeta_1} + \lambda_1 + 1 + 2\lambda_2 - \nu_2\right)$. Thus, $L(\lambda)$ is finite dimensional if and only if $\frac{\zeta_0}{\zeta_1} + \lambda_2 + 2\lambda_1 + 2$ and $\frac{\zeta_0}{\zeta_1} + \lambda_1 + 2\lambda_2 + 1$ are positive integers; moreover, since $\frac{\zeta_0}{\zeta_1} + \lambda_2 + 2\lambda_1 + 2 = \left(\frac{\zeta_0}{\zeta_1} + \lambda_1 + 2\lambda_2 + 1\right) + \lambda_1 + 1 - \lambda_2$ and $\lambda_1 + 1 - \lambda_2$ is a positive integer, it suffices to show that $\frac{\zeta_0}{\zeta_1} + \lambda_1 + 2\lambda_2 + 1$ is a positive integer.

Now, we will illustrate the decomposition of $L(\lambda)$ in the proof of Theorem 5.1; for clarity, we will work with $\mathfrak{sl}_2$ representations instead of $\mathfrak{gl}_2$ representations. Using the notations of the proof, $S_k = S^k(x_1,x_2) \cong V_k$, where we used the fact that $V^* \cong V$ as $\mathfrak{sl}_2$ representations. We then have, by the Clebsch-Gordon formula,

$$V_m \otimes V_k \cong V_{m+k} \oplus V_{m+k-2} \oplus \cdots \oplus V_{m+k-2\min(k,m)}.$$

We can use the above formula to draw the graph, shown below, representing the decomposition of $L((3,0))$, with $\nu = (1,4)$, into $\mathfrak{sl}_2$ modules. This representation is the quotient of $M(3,0)/e_{21}^3 M(3,0)$ by the submodules represented by the red and blue areas of the diagram, and we can write $L((3,0)) \cong V_3 \oplus V_4 \oplus V_5 \oplus V_6$ as $\mathfrak{sl}_2$ modules.



5.2 Existence of $L(\lambda)$ with given shape

Theorem 5.2. *For all $\mathfrak{gl}_n$-weights λ and $\nu \in \mathbb{N}^n$ such that $\lambda - \nu'$ is dominant for all $0 \le \nu' \le \nu$, there exists a deformation ζ, such that the irreducible representation $L(\lambda)$ of $H_\zeta(\mathfrak{gl}_n)$ is finite dimensional and its character is given by (*).*

Proof. Let $\lambda' = \lambda + \rho$. We can write $\lambda'_i = \lambda'_n + k_i$ for $k_1 > k_2 > k_3 > \ldots > k_{n-1} > k_n = 0$ (we have strict inequalities because of the shift by ρ). Recall that $P(\lambda) = \sum w_m H_{m+1}(\lambda')$ for w_i defined as in Theorem 4.2. Let $\mu_i = (0,\ldots,\nu_i,0,\ldots,0)$. We will find w_i such that $P(\lambda') - P(\lambda' - \mu_i) = 0$, while for all $0 < \mu_i' < \mu_i$, $P(\lambda') - P(\lambda' - \mu_i') \neq 0$. This implies that $M(\lambda')/\sum_i M(\lambda' - \mu_i)$ is irreducible.

We first define $P_{mj} = P(\lambda') - P(\lambda' - \mu)$ for $\mu = (0, \dots, m, 0, \dots, 0)$ with the m at the j-th location. We must prove that there exist w such that $P_{\nu_1 1} = \dots = P_{\nu_n n} = 0$ and $P_{\nu_1' 1}, \dots, P_{\nu_n' n} \neq 0$ for all $0 < \nu_i' < \nu_i$. We can write $P_{mj} = \sum_{i \geq 0} w_i R^i_{mj}$, where

$$R^N_{mj} = \sum_{i_1 + \dots + i_n = N+1} (\lambda_n' + k_1)^{i_1} \cdots (\lambda_n' + k_{j-1})^{i_{j-1}} ((\lambda_n' + k_j)^{i_j} - (\lambda_n' + k_j - m)^{i_j})(\lambda_n' + k_{j+1})^{i_{j+1}} \cdots (\lambda_n' + k_n)^{i_n}.$$

Note that the condition $P_{kj} = 0$ determines a hyperplane Π_{kj} in the space $(w_0, w_1, \dots)$ (Π_{kj} might in fact be the entire space, but the following argument would be unaffacted). Hence, the intersection $\bigcap \Pi_{\nu_j j}$ belongs to the union $\bigcup_{j, 0 < \nu_j' < \nu_j} \Pi_{\nu_j', j}$ if and only if it belongs to some $\Pi_{\nu_j', j}$. Thus, if we show that $\{P_{\nu_1 1}, \dots, P_{\nu_n n}, P_{\nu_l' l}\}$ are linearly independent as functions of w_i, then there exist deformations w that belong to all the hyperplanes $\Pi_{\nu_j j}$ but not to $\Pi_{\nu_l' l}$ for all $1 \leq l \leq n$ and $0 < \nu_l' < \nu_l$. This condition of linear independence is satisfied if

$$\det \begin{pmatrix} R^0_{\nu_1 1} & R^1_{\nu_1 1} & \cdots & R^n_{\nu_1 1} \\ R^0_{\nu_2 2} & R^1_{\nu_2 2} & \cdots & R^n_{\nu_2 2} \\ \vdots & \vdots & \ddots & \vdots \\ R^0_{\nu_n n} & R^1_{\nu_n n} & \cdots & R^n_{\nu_n n} \\ R^0_{\nu_l' l} & R^1_{\nu_l' l} & \cdots & R^n_{\nu_l' l} \end{pmatrix} \neq 0.$$

Now we shall prove that using column transformations, we can reduce the above matrix to its evaluation at $\lambda_n' = 0$. We proceed by induction on the column number. The elements of the first column, R^0_{mj}, are of degree zero with respect to λ_n', so $R^0_{mj} = R^0_{mj}(0)$. Suppose that using column transformations, all columns before column p are reduced to their constant terms. Now, we note that

$$\begin{aligned} \frac{\partial R^p_{mj}}{\partial \lambda_n'} &= \frac{\partial}{\partial \lambda_n'} \left(\sum_{i_1 + \dots + i_n = p+1} (\lambda_n' + k_1)^{i_1} \cdots ((\lambda_n' + k_j)^{i_j} - (\lambda_n' + k_j - m)^{i_j}) \cdots \lambda_n'^{i_n} \right) \\ &= \sum_{i_1 + \dots + i_n = p} (i_1 + i_2 + \dots + i_n + n)(\lambda_n' + k_1)^{i_1} \cdots ((\lambda_n' + k_j)^{i_j} - (\lambda_n' + k_j - m)^{i_j}) \cdots \lambda_n'^{i_n} \\ &= (p + n) R^{p-1}_{mj}. \end{aligned}$$

Thus, we see that $R^p_{mj} - R^p_{mj}(0)$ ia a linear combination of $R^{p-i}_{mj}(0)$, the entries of the other columns:

$$R^p_{mj} = \sum_i \frac{1}{i!} \lambda_n'^i \left. \frac{\partial^i R^p_{mj}}{\partial \lambda_n'^i} \right|_{\lambda_n' = 0} = \sum_i \frac{(p+n) \cdots (p+n-i+1)}{i!} \lambda_n'^i R^{p-i}_{mj}(0) = \sum_i \binom{p+n}{i} R^{p-i}_{mj}(0) \lambda_n'^i.$$

By selecting pivots of $\binom{p+n}{i} \lambda_n'^i$, we can eliminate every term except $R^p_{mj}(0)$. By repeating this step, we reduce the matrix to its evaluation at $\lambda_n' = 0$:

$$\det \begin{pmatrix} R^0_{\nu_1 1}(\lambda') & R^1_{\nu_1 1}(\lambda') & \cdots & R^n_{\nu_1 1}(\lambda') \\ R^0_{\nu_2 2}(\lambda') & R^1_{\nu_2 2}(\lambda') & \cdots & R^n_{\nu_2 2}(\lambda') \\ \vdots & \vdots & \ddots & \vdots \\ R^0_{\nu_n n}(\lambda') & R^1_{\nu_n n}(\lambda') & \cdots & R^n_{\nu_n n}(\lambda') \\ R^0_{\nu_l' l}(\lambda') & R^1_{\nu_l' l}(\lambda') & \cdots & R^n_{\nu_l' l}(\lambda') \end{pmatrix} = \det \begin{pmatrix} R^0_{\nu_1 1}(0) & R^1_{\nu_1 1}(0) & \cdots & R^n_{\nu_1 1}(0) \\ R^0_{\nu_2 2}(0) & R^1_{\nu_2 2}(0) & \cdots & R^n_{\nu_2 2}(0) \\ \vdots & \vdots & \ddots & \vdots \\ R^0_{\nu_n n}(0) & R^1_{\nu_n n}(0) & \cdots & R^n_{\nu_n n}(0) \\ R^0_{\nu_l' l}(0) & R^1_{\nu_l' l}(0) & \cdots & R^n_{\nu_l' l}(0) \end{pmatrix}.$$

Let us now rewrite $R^N_{mj}(0)$:

$$R^N_{mj}(0) = \sum_{i_1+\ldots+i_n=N+1} k_1^{i_1}\cdots k_{j-1}^{i_{j-1}}(k_j^{i_j} - (k_j-m)^{i_j})k_{j+1}^{i_{j+1}}\cdots k_n^{i_n} = \sum_{i=0}^{N} H'_{N-i}\left(k_j^{i+1} - (k_j-m)^{i+1}\right)$$
$$= \sum_{i=0}^{N} H_{N-i}\left(k_j^{i+1} - (k_j-m)^{i+1} - k_j(k_j^i - (k_j-m)^i)\right) = \sum_{i=0}^{N} H_{N-i}\left(m(k_j-m)^i\right)$$

where $H_{N-i} = \sum_{i_1+\ldots+i_n=N-i} k_1^{i_1}\cdots k_n^{i_n}$ and $H'_{N-i} = \sum_{i_1+\ldots+\widehat{i_j}+\ldots+i_n=N-i} k_1^{i_1}\cdots \widehat{k_j^{i_j}}\cdots k_n^{i_n}$. The third equality is because $H'_{N-i} = H_{N-i} - k_j H_{N-i-1}$. It is now easy to see that the determinant can be reduced further to

$$\det\begin{pmatrix} \nu_1 & \nu_1(k_1-\nu_1) & \cdots & \nu_1(k_1-\nu_1)^n \\ \nu_2 & \nu_2(k_2-\nu_2) & \cdots & \nu_2(k_2-\nu_2)^n \\ \vdots & \vdots & \ddots & \cdots \\ \nu_n & \nu_n(k_n-\nu_n) & \cdots & \nu_n(k_n-\nu_n)^n \\ \nu'_l & \nu'_l(k_l-\nu'_l) & \cdots & \nu'_l(k_l-\nu'_l)^n \end{pmatrix} = \nu_1\nu_2\cdots\nu_n\nu'_l \det\begin{pmatrix} 1 & k_1-\nu_1 & \cdots & (k_1-\nu_1)^n \\ 1 & k_2-\nu_2 & \cdots & (k_2-\nu_2)^n \\ \vdots & \vdots & \ddots & \cdots \\ 1 & k_n-\nu_n & \cdots & (k_n-\nu_n)^n \\ 1 & k_l-\nu'_l & \cdots & (k_l-\nu'_l)^n \end{pmatrix},$$

which equals $\nu_1\ldots\nu_n\nu'_l\prod_{i=1}^n(k_l - k_i + \nu_i - \nu'_l)\prod_{1\le i<j\le n}(k_j - k_i + \nu_i - \nu_j)$ by the Vandermonde determinant formula. Now, recall the condition that $\lambda-\nu$ is dominant integral, or $\lambda'_n + k_i - \nu_i > \lambda'_n + k_j - \nu_j$ for $i < j$ (we have strict inequality because λ' is the weight shifted by ρ). Thus, $\prod_{1\le i<j\le n}(k_j - k_i + \nu_i - \nu_j)$ is nonzero. Also, since $\lambda-\mu$ is dominant for all $0 < \mu < \nu$, if we let $\mu = (\nu_1, \nu_2, \ldots, \nu'_l, \ldots, \nu_n)$, then we see that $\lambda'_n + k_i - \nu_i \neq \lambda'_n + k_l - \nu'_l$. Thus, the dominance of $\lambda - \nu'$ for $0 \le \nu' < \nu$ ensures the determinant is nonzero, and so $\{P_{\nu_1,1}, \ldots, P_{\nu_n,n}, P_{\nu'_l,l}\}$ are linearly independent as desired. □

6 Poisson Infinitesimal Cherednik Algebras

Now we will introduce a new way to study infinitesimal Cherednik algebras by using Poisson analogues. The Poisson infinitesimal Cherednik algebras are as natural as $H_\zeta(\mathfrak{gl}_n)$, and their theory goes along the same lines with some simplifications. Although these algebras have not been defined before in the literature, the authors of [EGG] were aware of them, and technical calculations with these algebras are similar to those made in [T]. Using the Poisson infinitesimal Cherednik algebra, we will get a new proof of Theorem 4.1, which provides the formula for the Casimir element.

Let ζ be a deformation parameter, $\zeta : V \times V^* \to S(\mathfrak{gl}_n)$. The Poisson infinitesimal Cherednik algebra $H'_\zeta(\mathfrak{gl}_n)$ is defined to be the algebra $S\mathfrak{gl}_n \ltimes S(V \oplus V^*)$ with a bracket defined on the generators by:

$$\{a,b\} = [a,b] \text{ for } a,b \in \mathfrak{gl}_n$$
$$\{g,v\} = g(v) \text{ for } g \in \mathfrak{gl}_n, v \in V \oplus V^*$$
$$\{y,y'\} = \{x,x'\} = 0 \text{ for } y,y' \in V, x,x' \in V^*$$
$$\{y,x\} = \zeta(y,x) \text{ for } y \in V, x \in V^*.$$

This bracket extends to a Poisson bracket on $H'_\zeta(\mathfrak{gl}_n)$ if and only if the Jacobi identity $\{\{x,y\},z\} + \{\{y,z\},x\} + \{\{z,x\},y\} = 0$ holds for any $x,y,z \in \mathfrak{gl}_n \ltimes (V \oplus V^*)$. As can be verified by doing analogous computations as in [EGG], the pairings ζ satisfying the PBW property are given by $\zeta = \sum_{j=0}^k \zeta_j r_j$ where $\zeta_j \in \mathbb{C}$ and r_j is the coefficient of τ^j in the expansion of $(x,(1-\tau A)^{-1}y)\det(1-\tau A)^{-1}$. Actually, we can consider the specialized infinitesimal Cherednik algebra as a quantization of $H'_\zeta(\mathfrak{gl}_n)$.

Remark 6.1 (Due to Pavel Etingof). Note that

$$\{y_i, x_j\} = \sum \zeta_l r_l(y_i, x_j) = \sum \zeta_l \frac{\partial \operatorname{tr}(S^{l+1}A)}{\partial e_{ij}};$$

this follows from

$$\frac{\partial}{\partial B}(\det(1-\tau A)^{-1}) = \frac{\operatorname{tr}(\tau B(1-\tau A)^{-1})}{\det(1-\tau A)}$$

when $B = y_i \otimes x_j$. In fact, if $\{y_i, x_j\} = F_{ij}(A)$, the Jacobi identity implies that $F_{ij}(A) = \frac{\partial F}{\partial A}$ for some $GL(n)$ invariant function F, and that $\Lambda^2 D_A(F) = 0$, where D_A is the matrix with $(D_A)_{ij} = \frac{\partial}{\partial e_{ij}}$. One can then show that the only $GL(n)$ invariant functions F satisfying this partial differential equation are linear combinations of $\operatorname{tr}(S^l A)$.

Our main goal is to compute the Poisson center of the algebra $H'_\zeta(\mathfrak{gl}_n)$. As before, we set β_k as the coefficient of $(-t)^k$ in the expansion of $\det(1-tA)$ and $\tau_k = \sum_{i=1}^n x_i\{\beta_k, y_i\}$. It follows from [T] that $\tau_k \in \mathfrak{z}_{\mathrm{Pois}}(H'_0(\mathfrak{gl}_n))$.

Theorem 6.1. *The Poisson center* $\mathfrak{z}_{\mathrm{Pois}}(H'_\zeta(\mathfrak{gl}_n)) = \mathbb{C}[\tau_1+c_1, \tau_2+c_2, \dots, \tau_n+c_n]$*, where* $(-1)^i c_i$ *is the coefficient of* t^i *in the series*

$$c(t) = \operatorname{Res}_{z=0} \zeta(z^{-1}) \frac{\det(1-tA)}{\det(1-zA)} \frac{1}{1-t^{-1}z} \frac{dz}{z}.$$

Proof. Because t_k lies in the center of $H_0(\mathfrak{gl}_n)$, $\tau_k \in \mathfrak{z}_{\mathrm{Pois}}(H'_0(\mathfrak{gl}_n))$. All Poisson-central elements of $H'_0(\mathfrak{gl}_n)$ belong to $\mathbb{C}[\tau_1, \dots, \tau_n]$ because of the structure of the coadjoint action of the Lie group corresponding to the Lie algebra $\mathfrak{gl}_n \ltimes (V \oplus V^*)$ (for a detailed explanation, refer to the proof of Theorem 2 in [T]).

We wish to prove that the Poisson center of $H'_0(\mathfrak{gl}_n)$ can be lifted to the Poisson center of $H'_\zeta(\mathfrak{gl}_n)$, with τ_k being lifted to $\tau_k + c_k$. In [T], this was done using methods of homological algebra for noncommutative infinitesimal Cherednik algebras, but the proof did not yield formulas for c_k. We will take a more direct approach by deriving a formula for c_k. Since $\tau_k \in \mathfrak{z}_{\mathrm{Pois}}(H_0(\mathfrak{gl}_n))$, $\tau_k + c_k$ Poisson-commutes with elements of $S(\mathfrak{gl}_n)$ for any $c_k \in \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{gl}_\mathfrak{n}))$. We can define an anti-involution on $H'_\zeta(\mathfrak{gl}_n)$ that acts on basis elements by taking e_{ij} to e_{ji} and y_i to x_i. By using the arguments explained in the proof of Theorem 2 in [T], we can show that τ_k is fixed by this anti-involution, while c_k is also fixed since it lies in $\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{gl}_\mathfrak{n}))$. Applying this anti-involution, we see that if $\tau_k + c_k$ commutes with elements of V, then $\tau_k + c_k$ also commutes with elements of V^*. Thus, it suffices to find c_k such that $\{\tau_k + c_k, y\} = 0$ for all $y \in V$.

First, notice that if $g \in S(\mathfrak{gl}_n)$, then $\{g, y\} = \sum_{i,j=1}^n \frac{\partial g}{\partial e_{ij}}\{e_{ij}, y\}$. Second, notice that $\{\{\beta_k, y_i\}, y\} = 0$ (see the proof of Lemma 2.1 in [T]), so that

$$\{\tau_k, y\} = \left\{\sum_{i=1} x_i\{\beta_k, y_i\}, y\right\} = \sum_{i=1}\{x_i, y\}\{\beta_k, y_i\} = -\sum_{i=1}\left(\operatorname{Res}_{z=0} \zeta(z^{-1}) \frac{\operatorname{tr}(x_i(1-zA)^{-1}y)}{z\det(1-zA)} dz\right)\{\beta_k, y_i\}.$$

Thus, we have

$$\{\tau_k + c_k, y\} = \sum_{i,j=1}^n \frac{\partial c_k}{\partial e_{ij}}\{e_{ij}, y\} - \sum_{i=1}\left(\operatorname{Res}_{z=0} \zeta(z^{-1}) \frac{\operatorname{tr}(x_i(1-zA)^{-1}y)}{z\det(1-zA)} dz\right)\{\beta_k, y_i\}.$$

We get a system of partial differential equations for c_k:

$$\sum_{i,j=1}^n \frac{\partial c_k}{\partial e_{ij}}\{e_{ij}, y\} = \sum_{i=1}\left(\operatorname{Res}_{z=0} \zeta(z^{-1}) \frac{\operatorname{tr}(x_i(1-zA)^{-1}y)}{z\det(1-zA)} dz\right)\{\beta_k, y_i\}.$$

Multiplying both sides by $(-t)^k$ and summing over $k = 1, \dots, n$, we obtain an equivalent single equation

$$\sum_{i,j=1}^{n} \frac{\partial c(t)}{\partial e_{ij}} \{e_{ij}, y\} = \sum_{i=1} \left(\mathrm{Res}_{z=0}\, \zeta(z^{-1}) \frac{\mathrm{tr}(x_i(1-zA)^{-1}y)}{z \det(1-zA)} dz \right) \{\det(1-tA), y_i\},$$

where $c(t) = \sum_{k=1}^{n} c_k(-t)^k$ is the generating function for the c_k.

Since all terms above are $GL(n)$ invariant and diagonalizable matrices are dense in $\mathfrak{gl}_n$, we can set A as a diagonal matrix, $A = \mathrm{diag}(a_1, \dots, a_n)$. Furthermore, we substitute $y = y_l$. Using this simplification, we obtain

$$\begin{aligned}
\frac{\partial c(t)}{\partial a_l} y_l &= \left(\mathrm{Res}_{z=0} \frac{\zeta(z^{-1})}{z(1-za_l)\det(1-zA)} dz \right) \{\det(1-tA), y_l\} \\
&= \left(\mathrm{Res}_{z=0} \frac{\zeta(z^{-1})}{z(1-za_l)\det(1-zA)} dz \right) \frac{\partial \det(1-tA)}{\partial a_l} y_l \\
&= -\left(\mathrm{Res}_{z=0} \frac{\zeta(z^{-1})}{z(1-za_l)\det(1-zA)} dz \right) \frac{t\det(1-tA)}{1-ta_l} y_l \\
&= \frac{\partial}{\partial a_l} \left(\mathrm{Res}_{z=0}\, \zeta(z^{-1}) \frac{\det(1-tA)}{\det(1-zA)} \frac{1}{1-t^{-1}z} \frac{dz}{z} \right) y_l,
\end{aligned}$$

providing the formula for c_i. □

Example 6.1. By taking the coefficient of t in the above formula, we get

$$c_1 = \sum_{i=0}^{k} \zeta_i \,\mathrm{tr}\, S^{i+1} A,$$

where $\zeta(z) = \zeta_0 + \ldots + \zeta_k z^k$.

Remark 6.2. Another way of writing the formula for c_k is

$$c_k = \mathrm{Res}_{z=0}\, \zeta(z^{-1}) F_k(z) \frac{dz}{z^2},$$

where $F_k(z) = \sum z^m y_{m,k}(A)$ and $y_{m,k}(A) = \chi\underbrace{(m, 1, \dots, 1)}_{k}$, the character of an irreducible $\mathfrak{gl}_n$ module corresponding to a hook Young diagram. This rewriting of the formula gives better insight for the quantization construction.

7 Passing from Commutative to Noncommutative Algebras

Note that $\{g, y\} \in S(\mathfrak{gl}_n) \otimes V$ for $g \in S(\mathfrak{gl}_n)$ and $y \in V$; we can thus identify $\{g, y\} = \sum_{i=1}^{n} h_i \otimes y_i \in H'_\zeta(\mathfrak{gl}_n)$ with the element $\sum_{i=1}^{n} \mathrm{Sym}(h_i) y_i \in H_\zeta(\mathfrak{gl}_n)$.

Lemma 7.1.

$$[\mathrm{tr}\, S^{k+1} A, y] = \left\{ \sum_{j=0}^{k} \frac{(-1)^j}{k+n+1} \binom{k+n+1}{j+1} \mathrm{tr}\, S^{k+1-j} A, y \right\}.$$

Proof. It is enough to consider the case $y = y_1$. Recall that $\operatorname{tr} S^{k+1}(A)$ can be written as a sum of degree $k+1$ monomials of form $e_{1,i_1}\cdots e_{1,i_{s_1}} e_{2,i_{s_1+1}}\cdots e_{2,i_{s_1+s_2}}\cdots e_{n,i_{s_1+\cdots+s_n}}$ where $s_1+\cdots+s_n = k+1$ and the sequence i_k is a permutation of the sequence of s_1 ones, s_2 twos, and so forth; for conciseness, we will denote the above monomial by $e_{1,i_1}\cdots e_{n,i_{k+1}}$. The only terms of $\operatorname{tr} S^{k+1}A$ that contribute to $[\operatorname{tr} S^{k+1}A, y_1]$ and to $\{\operatorname{tr} S^{k+1}A, y_1\}$ have $s_1 \geq 1$. Since to compute $[\operatorname{tr} S^{k+1}A, y_1]$ we first symmetrize $\operatorname{tr} S^{k+1}A$, we will compute $[\operatorname{Sym}(e_{1,i_1}\cdots e_{n,i_{k+1}}), y_1] - \{\operatorname{Sym}(e_{1,i_1}\cdots e_{n,i_{k+1}}), y_1\}$. For both the Lie bracket and the Poisson bracket, we use Leibniz's rule to compute the bracket, but whereas in the Poisson case we can transfer the resulting elements of V to the right since the Poisson algebra is commutative, in the Lie case when we do so extra terms appear.

Consider a typical term that may appear after we use Leibniz's rule to compute $[\operatorname{tr} S^{k+1}A, y_1]$:

$$\cdots y_{j_0}\cdots e_{j_1 j_0}\cdots e_{j_2 j_1}\cdots e_{j_N j_{N-1}}\cdots$$

When we move y_{j_0} to the right, we get, besides $\cdots e_{j_1 j_0}\cdots e_{j_2 j_1}\cdots e_{j_N j_{N-1}}\cdots y_{j_0}$, additional residual terms like $-\cdots e_{j_2 j_1}\cdots e_{j_N j_{N-1}}\cdots y_{j_1}$ and $\cdots e_{j_3 j_2}\cdots e_{j_N j_{N-1}}\cdots y_{j_2}$, up to $(-1)^N\cdots y_{j_N}$. Without loss of generality, we can consider only the last expression, since the others will appear in the smaller chains

$$\cdots y_{j_0}\cdots e_{j_1 j_0}\cdots \widehat{e_{j_2 j_1}}\cdots \widehat{e_{j_3 j_2}}\cdots \widehat{e_{j_N j_{N-1}}}$$

and

$$\cdots y_{j_0}\cdots e_{j_1 j_0}\cdots e_{j_2 j_1}\cdots \widehat{e_{j_3 j_2}}\cdots \widehat{e_{j_N j_{N-1}}}$$

and so forth with the same coefficients. For notation, we let z_1 denote the coefficient of y_{j_N} in the residual term, i.e., the term represented by the ellipsis: $(-1)^N \underbrace{\cdots}_{z_1} y_{j_N}$. Then, $z_1 y_{j_N}$ is a term in the expression $(-1)^N\{z_1 e_{j_N 1}, y_1\}$, which appears in $(-1)^N\{\operatorname{tr} S^{k+1-N}A, y_1\}$. Thus, we can write

$$[\operatorname{tr} S^{k+1}A, y_1] = \left\{\sum_{N=0}^{k+1}(-1)^N C_N \operatorname{tr} S^{k+1-N}A, y_1\right\}$$

for some coefficients C_N.

Next, we compute C_N. We first count how many times $z_1 y_{j_N}$ appears in $\{\operatorname{tr} S^{k+1-N}A, y_1\}$. Notice that since z_1 is the product of $k-N$ e_{jl}'s, we can insert $e_{j_N 1}$ in $k-N+1$ places to obtain z_2 such that $\{z_2, y_1\}$ contains $z_1 y_{j_N}$.

Now we compute the coefficient of z_2 in $\operatorname{tr} S^{k+1-N}A$. As noted before, $\operatorname{tr} S^{k+1-N}(A)$ can be written as a sum of degree $k+1-N$ monomials of form $e_{1,i_1}\cdots e_{1,i_{s_1}} e_{2,i_{s_1+1}}\cdots e_{2,i_{s_1+s_2}}\cdots e_{n,i_{k+1-N}}$. Any term that is a permutation of those $k+1-N$ unit matrices will appear in the symmetrization of $\operatorname{tr} S^{k+1-N}A$. We count the number of sequences $i_1,\ldots,i_{k+1-N}$ such that z_2 is the product of the elements $e_{1,i_1},\ldots,e_{n,i_{k+1-N}}$ (in some order); this tells us the multiplicity of z_2 in the symmetrization of $\operatorname{tr} S^{k+1-N}A$. Suppose $z_2 = e_{1,i_1}\cdots e_{n,i_{k+1-N}}$ for a certain sequence $i_1,\ldots,i_{k+1-N}$. Then, $z_2 = e_{1,i'_1}\cdots e_{n,i'_{k+1-N}}$ if and only if $i'_{s_1+\cdots+s_{j-1}+1},\ldots,i'_{s_1+\cdots+s_j}$ is a permutation of $i_{s_1+\cdots+s_{j-1}+1},\ldots,i_{s_1+\cdots+s_j}$ for all j. Thus, z_2 appears $s_1!s_2!\cdots s_n!$ times in $\operatorname{tr} S^{k+1-N}A$. Since each term has coefficient $\frac{1}{(k-N+1)!}$ in the symmetrization, z_2 appears with coefficient

$$\frac{s_1!s_2!\cdots s_n!}{(k-N+1)!}$$

in the symmetrization of $\operatorname{tr} S^{k+1-N}A$. In conjunction with the previous paragraph, we see that $z_1 y_{j_N}$ appears

$$\frac{s_1!s_2!\cdots s_n!}{(k-N+1)!}\times(k-N+1) = \frac{s_1!s_2!\cdots s_n!}{(k-N)!}$$

times in $\{\operatorname{tr} S^{k+1-N}A, y_1\}$.

It remains to calculate how many times $z_1 y_{j_N}$ appears in $[\operatorname{tr} S^{k+1}A, y_1]$. Recall that z_1 is obtained from a term like:

$$\cdots e_{j_0 1} \cdots e_{j_1 j_0} \cdots e_{j_2 j_1} \cdots e_{j_N j_{N-1}} \cdots$$

where the ordered union of the ellipsis equals z_1. Thus, z_1 comes from terms of the following form: we choose arbitrary numbers $j_0, \ldots, j_{N-1}$, and insert $e_{j_0 1}, e_{j_1 j_0}, \ldots e_{j_N j_{N-1}}$ into z_1. There are

$$\frac{(k+1)(k)\cdots(k+1-N)}{(N+1)!}$$

ways for this choice for any fixed $j_0, \ldots, j_{N-1}$. Any such term z_3 appears in $\operatorname{tr} S^{k+1}A$ with coefficient

$$\frac{s_1'!\cdots s_n'!}{(k+1)!}$$

where s_l' is the total number e_{li}'s (for some i) in z_3, i.e., $s_l +$ number of j_i's with $j_i = l$, $0 \le i < N$.

Combining the results of the last two paragraphs, we see that $\{\operatorname{tr} S^{k+1-N}A, y_1\}$ must appear with coefficient

$$\left(\frac{(k+1)(k)\cdots(k+1-N)}{(N+1)!} \sum \frac{s_1'!\cdots s_n'!}{(k+1)!}\right) \Big/ \frac{s_1!s_2!\cdots s_n!}{(k-N)!} = \frac{1}{(N+1)!} \sum \frac{s_1'!\cdots s_n'!}{s_1!s_2!\cdots s_n!},$$

where the summation is over all length-N sequences $\{j_l\}$ of integers from 1 to n. We claim that

$$\frac{\sum s_1'!\cdots s_n'!}{s_1!\cdots s_n!} = (k+n)\cdots(k+n-N+1).$$

To see this, notice that $\frac{\sum s_1'!\cdots s_n'!}{s_1!\cdots s_n!}$ is the coefficient of t^N in the expression

$$N! \prod_{i=1}^{n} \left(1 + (s_i+1)t + \frac{(s_i+1)(s_i+2)}{2!}t^2 + \cdots\right).$$

The above generating function equals $N! \prod_{i=1}^{n}(1-t)^{-(s_i+1)} = N!(1-t)^{-(k+1-N+n)}$, and the coefficient of t^N in this expression is $(k+n)\cdots(k+n-N+1)$.

Finally, we arrive at the simplified coefficient of $\{\operatorname{tr} S^{k+1-N}A, y_1\}$:

$$C_N = \frac{1}{(N+1)!} \sum \frac{s_1'!\cdots s_n'!}{s_1!s_2!\cdots s_n!} = \frac{(k+n)\cdots(k+n-N+1)}{(N+1)!},$$

as desired. □

Now we will give an alternative proof of Theorem 4.1.

Proof. Let $f(z)$ be the polynomial satisfying $f(z) - f(z-1) = \partial^n(z^n\zeta(z))$ and $g(z) = z^{1-n}\frac{1}{\partial^{n-1}}f(z)$ (in the expression for $g(z)$, we discard any negative powers of z). Note that if $g(z) = g_{k+1}z^{k+1} + \ldots + g_1 z$,

$$\zeta(z) = \sum_{j=1}^{k+1}\sum_{i=0}^{j-1} \frac{1}{j+n}\binom{j+n}{i+1}(-1)^i g_j z^{j-1-i},$$

so

$$\zeta_{j-1} = \sum_{i=0}^{k-j+1} \frac{1}{j+i+n}\binom{j+n+i}{i+1}(-1)^i g_{j+i}.$$

Lemma 7.1 allows us to write

$$\left[\sum_{j=1}^{k+1} g_j \operatorname{tr} S^j A, y\right] = \left\{\sum_{j=1}^{k+1}\sum_{i=0}^{j-1} \frac{1}{j+n}\binom{j+n}{i+1}(-1)^i g_j \operatorname{tr} S^{j-i} A, y\right\}$$
$$= \left\{\sum_{j=1}^{k+1}\sum_{i=0}^{k-j+1} \frac{1}{j+i+n}\binom{j+i+n}{i+1}(-1)^i g_{j+i} \operatorname{tr} S^{j} A, y\right\}$$
$$= \left\{\sum_{j=1}^{k+1} \zeta_{j-1} \operatorname{tr} S^j A, y\right\}.$$

Hence,

$$[t_1, y] = \sum_{i=1}^{n}[x_i, y]y_i = \sum_{i=1}^{n}\{x_i, y\}y_i = -\left\{\sum_{j=1}^{k+1} \zeta_{j-1} \operatorname{tr} S^j A, y\right\} = -\left[\sum_{j=1}^{k+1} g_j \operatorname{tr} S^j A, y\right],$$

where the third equality follows from the fact that $\tau_1 + \sum_{j=1}^{k+1} \zeta_{j-1} \operatorname{tr} S^j A$ is Poisson-central in $H'_\zeta(\mathfrak{gl}_n)$ (see Example 6.1). Thus, we get $t'_1 = t_1 + C'$, where

$$C' = \sum_{j=1}^{k+1} g_j \operatorname{tr} S^j A = \operatorname{Res}_{z=0} g(z^{-1}) \det(1 - zA)^{-1} z^{-1} dz.$$

□

We expect a similar approach to work for determining the other central elements of $H_\zeta(\mathfrak{gl}_n)$.

8 Cherednik Algebras of $\mathfrak{sp}_{2n}$

Let V be the standard $2n$-dimensional representation of $\mathfrak{sp}_{2n}$ with symplectic form ω, and let $\zeta : V \times V \to U(\mathfrak{sp}_{2n})$ be an $\mathfrak{sp}_{2n}$-invariant bilinear form. The infinitesimal Cherednik algebra $H_\zeta(\mathfrak{sp}_{2n})$ is defined as the quotient of $U(\mathfrak{sp}_{2n}) \ltimes T(V)$ by the relation $[x, y] = \zeta(x, y)$ for all $x, y \in V$, such that $H_\zeta(\mathfrak{sp}_{2n})$ satisfies the PBW property. In [EGG], it was shown that $H_\zeta(\mathfrak{sp}_{2n})$ satisfies the PBW property if and only if $\zeta = \sum_{j=0}^{k} \zeta_{2j} r_{2j}$ where r_j is the coefficient of z^j in the expansion of

$$\omega(x, (1 - z^2A^2)^{-1}y) \det(1 - zA)^{-1} = r_0(x, y) + r_2(x, y)z^2 + \cdots.$$

Note that since $A \in \mathfrak{sp}_{2n}$, the expansion $\det(1 - zA)^{-1}$ only contains even powers of z. There is an isomorphism between $H_{\zeta_0 r_0}(\mathfrak{sp}_{2n})$ for nonzero ζ_0 and $U(\mathfrak{sp}_{2n}) \ltimes A_n$, where A_n is the n-th Weyl algebra (see [EGG]); thus, we can regard $H_\zeta(\mathfrak{sp}_{2n})$ as a deformation of $U(\mathfrak{sp}_{2n}) \ltimes A_n$.

Remark 8.1. It would be desirable to develop the representation theory of $H_\zeta(\mathfrak{sp}_{2n})$ analogously to how we developed the theory for $H_\zeta(\mathfrak{gl}_n)$, but while $H_\zeta(\mathfrak{gl}_n)$ has a natural triangular decomposition (with V assigned positive weights and V^* assigned negative weights), there is no natural way to assign elements of V positive or negative weights for $H_\zeta(\mathfrak{sp}_{2n})$ when $n > 1$. The reason is that the set of positive elements A^+ of $H_\zeta(\mathfrak{sp}_{2n})$ form a subalgebra, and for linearly independent $v_1, v_2 \in A^+ \cap V$ (which could be found if $n > 1$), $[v_1, v_2]$ lies in $U(\mathfrak{sp}_{2n})$ but not $U(\mathfrak{sp}_{2n})^+$, contradicting the fact that A^+ is a subalgebra. Thus, a reasonable category $\mathcal{O}$ cannot be defined for $H_\zeta(\mathfrak{sp}_{2n})$, and so a different approach must be taken to study the representations of $H_\zeta(\mathfrak{sp}_{2n})$. Another way to see that $\mathcal{O}$ cannot be defined reasonably is the fact that $H_\zeta(\mathfrak{gl}_n)$

is a deformation of $U(\mathfrak{sl}_{n+1})$, which has a category $\mathcal{O}$, whereas $H_\zeta(\mathfrak{sp}_{2n})$ is a deformation of $U(\mathfrak{sp}_{2n}) \ltimes A_n$, which does not have a reasonable category $\mathcal{O}$. Note however that when $n = 1$ the above arguments are not valid, and a corresponding theory of the category $\mathcal{O}$ of $H'_\zeta(\mathfrak{sp}_{2n})$ has been elaborated in [KT].

Choose a basis v_j of V, so that

$$\omega(x, y) = x^T J y,$$

with

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

As before, we study the noncommutative infinitesimal Cherednik algebra $H_\zeta(\mathfrak{sp}_{2n})$ by considering its Poisson analogue $H'_\zeta(\mathfrak{sp}_{2n})$. We define $\sum_{i=1}^n \beta_i z^{2i} = \det(1 - zA)$ and

$$\tau_i = (-1)^{i-1} \sum_{j=1}^{2n} \{\beta_i, v_j\} v_j^*,$$

where v_j^* denotes the element of V that satisfies $\omega(v_i, v_j^*) = \delta_{ij}$. Note that

$$\tau_i = -\sum_{j=0}^{i-1} \beta_j \omega(A^{2i-1-2j} v, v),$$

(where $\beta_0 = 1$) so τ_i is $\mathfrak{sp}_{2n}$-invariant and independent of the choice of basis for V.

Proposition 8.1. *The Poisson center of $H'_0(\mathfrak{sp}_{2n})$ is $\mathbb{C}[\tau_1, \ldots, \tau_n]$.*

Proof. We will follow a similar approach as in the proof of Theorem 2.1 in [T]. Let L be the Lie algebra $\mathfrak{sp}_{2n} \ltimes V$ and S be the Lie group of L. We need to verify that $\mathbb{C}[\tau_1, \ldots, \tau_n] = \mathfrak{z}_{\mathrm{Pois}}(H'_0(\mathfrak{sp}_{2n}))$, with the latter being identified with $\mathbb{C}[L^*]^S$. Let M be the $2n$-dimensional subspace of L containing all elements of the form

$$y = \left\{ \begin{pmatrix} 0 & y_{12} & 0 & \cdots & 0 & 0 \\ y_{21} & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & 0 & y_{2n-3,2n-2} & 0 & 0 \\ 0 & 0 & y_{2n-2,2n-3} & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & y_{2n-1,2n} \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y_{2n} \end{pmatrix} \right\},$$

where all the y's belong to $\mathbb{C}$. In what follows, we identify L^* and L via the non-degenerate pairing, so that the coadjoint action of S is on L. We use the following two facts proved in [K]: first, that the orbit of M under the coadjoint action of S on L^* is dense in L^*; and second, that $\mathbb{C}[L^*]^S \cong \mathbb{C}[f_1, \ldots, f_n]$, where

$$f_i(y) = \sigma_{i-1}(y_{2,1}y_{1,2}, y_{3,2}y_{2,3}, \ldots, y_{2n-2,2n-3}y_{2n-3,2n-2}) y_{2n-1,2n} y_{2n}^2$$

and σ_j is the j-th elementary symmetric function. It is straightforward to see that $\tau_i|_M = f_i$, and so $\mathbb{C}[L^*]^S \cong \mathbb{C}[\tau_1, \ldots, \tau_n]$ as desired. □

Theorem 8.1. *The Poisson center* $\mathfrak{z}_{\mathrm{Pois}}(H'_\zeta(\mathfrak{sp}_{2n})) = \mathbb{C}[\tau_1 + c_1, \tau_2 + c_2, \dots, \tau_n + c_n]$, *where* $(-1)^{i-1}c_i$ *is the coefficient of* t^{2i} *in the series*

$$c(t) = 2\,\mathrm{Res}_{z=0}\,\zeta(z^{-1})\frac{\det(1-tA)}{\det(1-zA)}\frac{z^{-1}}{1-z^2t^{-2}}dz.$$

Proof. We first note that $c_i \in \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{sp}_{2n}))$. Since $\{\tau_i + c_i, g\} = 0$ for any $g \in S(\mathfrak{sp}_{2n})$, we just need to find c_i satisfying $\{\tau_i + c_i, v\} = 0$ for all $v \in V$. By the Jacobi rule,

$$\{\tau_i, v\} = (-1)^{i-1}\sum_j \{\beta_i, v_j\}\{v_j^*, v\} + (-1)^{i-1}\sum_j \{\{\beta_i, v_j\}, v\}v_j^*.$$

Thus,

$$0 = \{\tau_i + c_i, v\} = (-1)^{i-1}\sum_j \{\beta_i, v_j\}\{v_j^*, v\} + (-1)^{i-1}\sum_j \{\{\beta_i, v_j\}, v\}v_j^* + \{c_i, v\}. \tag{1}$$

In the case of $H'_\zeta(\mathfrak{gl}_n)$, $\sum_j \{\{\beta_i, y_j\}, y\}x_j = 0$ by straightforward application of properties of the determinant; however, for $H'_\zeta(\mathfrak{sp}_{2n})$, $\sum_j \{\{\beta_i, v_j\}, v\}v_j^* \neq 0$. To calculate this sum, let B be a basis for $\mathfrak{sp}_{2n}$ (the basis elements are given in Appendix A.4, but for the purposes of this section, the specific elements are not needed). Write

$$\sum_j \{\{\beta_i, v_j\}, v\}v_j^* = \sum_j \left\{\sum_{e\in B}\frac{\partial\beta_i}{\partial e}e(v_j), v\right\}v_j^* = \sum_j \left(\sum_{e\in B}\frac{\partial\beta_i}{\partial e}\{e(v_j), v\}v_j^* + \left\{\frac{\partial\beta_i}{\partial e}, v\right\}e(v_j)v_j^*\right).$$

Lemma 8.1.

$$\sum_j \sum_{e\in B}\left\{\frac{\partial\beta_i}{\partial e}, v\right\}e(v_j)v_j^* = 0.$$

The proof of this Lemma is given in the Appendix.

Using the fact that $\sum_j \{\{\beta_i, v_j\}, v\}v_j^* = \sum_j \sum_{e\in B}\frac{\partial\beta_i}{\partial e}\{e(v_j), v\}v_j^*$, we can restrict (1) to diagonal matrices, which are spanned by elements $e_i = \mathrm{diag}(0, \dots, 1, -1, 0, \dots, 0)$ with 1 at the $2i-1$-th coordinate. We get:

$$0 = (-1)^{i-1}\sum_j\sum_k \frac{\partial\beta_i}{\partial e_k}\{e_k, v_j\}\{v_j^*, v\} + (-1)^{i-1}\sum_k\left(\frac{\partial\beta_i}{\partial e_k}\{v_{2k-1}, v\}v_{2k} + \frac{\partial\beta_i}{\partial e_k}\{v_{2k}, v\}v_{2k-1}\right) + \sum_k \frac{\partial c_i}{\partial e_k}\{e_k, v\}$$

$$= 2(-1)^{i-1}\sum_k \frac{\partial\beta_i}{\partial e_k}(v_{2k-1}\{v_{2k}, v\} + v_{2k}\{v_{2k-1}, v\}) + \sum_k \frac{\partial c_i}{\partial e_k}\{e_k, v\}.$$

Multiplying by $(-1)^{i-1}t^{2i}$ and summing over i for $i = 1, \dots, n$, we get

$$0 = 2\sum_k \frac{\partial\det(1-tA)}{\partial e_k}(v_{2k-1}\{v_{2k}, v\} + v_{2k}\{v_{2k-1}, v\}) + \sum_k \frac{\partial c(t)}{\partial e_k}\{e_k, v\}.$$

Now, we can alternatively set $v = v_{2s-1}$ and $v = v_{2s}$ to get

$$0 = 2\sum_k \frac{\partial\det(1-tA)}{\partial e_k}(v_{2k-1}\{v_{2k}, v_{2s-1}\} + v_{2k}\{v_{2k-1}, v_{2s-1}\}) + \frac{\partial c(t)}{\partial e_s}v_{2s-1}$$

and

$$0 = 2\sum_k \frac{\partial\det(1-tA)}{\partial e_k}(v_{2k-1}\{v_{2k}, v_{2s}\} + v_{2k}\{v_{2k-1}, v_{2s}\}) - \frac{\partial c(t)}{\partial e_s}v_{2s}.$$

These last two formulas both reduce to

$$\frac{\partial c(t)}{\partial e_s} = -2\frac{\partial \det(1-tA)}{\partial e_s}\{v_{2s}, v_{2s-1}\}.$$

We now recall that

$$\begin{aligned}\{v_{2s}, v_{2s-1}\} &= \operatorname{Res}_{z=0}\zeta(z^{-1})\omega(v_{2s},(1-z^2A^2)^{-1}v_{2s-1})\det(1-zA)^{-1}z^{-1}dz\\ &= -\operatorname{Res}_{z=0}\zeta(z^{-1})\frac{1}{1-z^2\lambda_s^2}\det(1-zA)^{-1}z^{-1}dz,\end{aligned}$$

so integrating, we get

$$c(t) = 2\operatorname{Res}_{z=0}\zeta(z^{-1})\frac{\det(1-tA)}{\det(1-zA)}\frac{z^{-1}}{1-z^2t^{-2}}dz.$$

□

We now briefly consider the center of $H_\zeta(\mathfrak{sp}_{2n})$. Let

$$t_i = (-1)^{i-1}\sum_{j=1}^{2n}[\beta_i, v_j]v_j^*,$$

where $\beta_i \in U(\mathfrak{sp}_{2n})$ is the symmetrization of the coefficient of z^{2i} in the series $\sum_{i=1}^n \beta_i z^{2i} = \det(1-zA)$. Clearly, t_i is independent of the choice of basis $\{v_j\}$, and it is straightforward to see that it is $\mathfrak{sp}_{2n}$-invariant.

Conjecture 8.1. *For any deformation ζ there exist $c_i \in \mathfrak{z}(U(\mathfrak{sp}_{2n}))$ that are unique up to a constant, such that $\mathfrak{z}(H_\zeta(\mathfrak{sp}_{2n})) = \mathbb{C}[t_1 + c_1, \dots, t_n + c_n]$.*

9 Harish-Chandra Map

Recall that the Harish-Chandra map is defined as $HC : \mathbb{C}^n \to \mathbb{C}^n$, with $HC(\lambda) = (t_1'(\lambda), t_2'(\lambda), \dots, t_n'(\lambda))$. The HC-map is of degree $(m+1)(m+2)\cdots(m+n)$, where m is the largest index with $\zeta_m \neq 0$. In this section, we will examine the fibers of the HC-map. We will always shift λ by ρ to simplify the formulas.

9.1 $\mathfrak{gl}_2$

We first consider the case of $H_\zeta(\mathfrak{gl}_2)$ with $\zeta = w_0s_0+w_1s_1+\cdots+w_ms_m$. We know that $t_1'(\lambda) = P_1(\lambda)$ and $t_2'(\lambda) = P_2(\lambda)$ as defined in Section 3.3.

Note that $\sigma : \lambda = (\lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1)$ preserves both $t_1'(\lambda)$ and $t_2'(\lambda)$ by symmetry. Suppose $P_1(\lambda_1, \lambda_2') = P_1(\lambda_1, \lambda_2)$. Since $P_2(\lambda) = \frac{1}{2}P_1(\lambda) + \lambda_1 P_1(\lambda) - \sum_{0\le i\le m} a_i\lambda_1^{i+2}$, a transformation $(\lambda_1, \lambda_2) \to (\lambda_1, \lambda_2')$ that fixes P_1 will also fix P_2. The polynomial $P_1(\lambda_1, \lambda_2') - P_1(\lambda_1, \lambda_2) = 0$ is of degree $m+1$ in λ_2', and for generic λ, it has $m+1$ pairwise distinct roots. We let τ be the multivalued transformation taking (λ_1, λ_2) to any (λ_1, λ_2') with $P_1(\lambda_1, \lambda_2') = P_1(\lambda_1, \lambda_2)$.

Proposition 9.1. *The fiber of HC over $\mu = HC(\lambda)$ for a generic λ can be written as the disjoint union*

$$\{\tau\lambda\} \sqcup \{\tau\sigma\tau\lambda\}.$$

Proof. It is straightforward to check that both sets are disjoint and that the elements in both sets are pairwise distinct for generic λ. Moreover, since $|\{\tau\lambda\}| = m+1$ and $|\{\tau\sigma\tau\lambda\}| = (m+1)^2$, $|\{\tau\lambda\} \sqcup \{\tau\sigma\tau\lambda\}| = (m+1)(m+2) = \deg HC$, so in fact, we have found all points in the fiber over $\mu = HC(\lambda)$. □

9.2 $\mathfrak{gl}_n$

Although we do not yet know the formulas for t'_i, we know their highest terms from Section 6, and so we can consider the highest term $Q_i(\lambda)$ of their action on a Verma module $M(\lambda-\rho)$. Let the deformation ζ be of degree m with the leading coefficient equal to 1. Then, it follows from Theorem 6.1 that $Q_i(\lambda)$ is the coefficient of $t^i z^m$ in

$$(-1)^i \frac{(1-t\lambda_1)\cdots(1-t\lambda_n)}{(1-z\lambda_1)\cdots(1-z\lambda_n)} \frac{1}{1-t^{-1}z}.$$

Proposition 9.2. *For any $1 \le k \le n$ and deformation $\zeta = r_m + \zeta_{j-1}r_{j-1} + \cdots + \zeta_0 r_0$,*

$$\lambda_k^{n-1}Q_1 - \lambda_k^{n-2}Q_2 + \cdots + (-1)^{n-1}\lambda_k^0 Q_n = \lambda_k^{m+n}.$$

Proof. Let σ_j and H_j be given by the following two equations:

$$\prod_{j=1}^n (1-\lambda_j t) = \sum_{j=0}^n (-1)^j \sigma_j t^j$$

$$\prod_{j=1}^n (1-\lambda_j t)^{-1} = \sum_{j=0}^\infty H_j t^j.$$

We shall prove this proposition for λ_1 as the statement is totally symmetric in k.

We have $Q_i(\lambda) = \sum_{j=i}^n (-1)^{i+j}\sigma_j H_{m+i-j}$. This proposition is equivalent to the equation

$$\lambda_1^{-1}(\sigma_1 H_m - \sigma_2 H_{m-1} + \cdots) + \lambda_1^{-2}(-\sigma_2 H_m + \cdots) + \cdots = \lambda_1^m.$$

Now, notice that

$$(-1)^{l-1}\lambda_1^{-l}(\sigma_l H_m - \sigma_{l+1}H_{m-1} + \cdots) = \sum_{\substack{d_1+\cdots+d_n=m,\\ d_1\ge -l,\, d_2,\ldots,d_n\ge 0}} C_l(d_1,\ldots,d_n)\lambda_1^{d_1}\cdots\lambda_n^{d_n}$$

where

$$C_l(d_1,\ldots,d_n) = \sum_{i=0}^{n-l}(-1)^{l+i+1}|\{(j_1<j_2<\cdots<j_{l+i})|\ d_{j_\iota} > -l\delta_{1,j_\iota}\,\forall \iota\}|$$

Thus, the proposition reduces to showing that

$$C_1(d_1,\ldots,d_n) + C_2(d_1,\ldots,d_n) + \cdots + C_n(d_1,\ldots,d_n) = 0$$

for all $(d_1,\ldots,d_n) \neq (m,0\ldots,0)$ and $C_1(m,0,\ldots,0) + C_2(m,0,\ldots,0) + \cdots + C_n(m,0,\ldots,0) = 1$. This can be done using counting arguments. □

Define $R(t) \in \mathbb{C}(Q_1,\ldots,Q_n)[t]$ as

$$R(t) = t^{n+m} - t^{n-1}Q_1 + t^{n-2}Q_2 - \cdots + (-1)^n Q_n.$$

Proposition 9.2 states that λ_j is a root of $R(t)$ for all j. We naturally consider the field extension $K \subset L$, with $K = \mathbb{C}(Q_1,\ldots,Q_n)$ and $L = \mathbb{C}(\lambda_1,\ldots,\lambda_n)$, where we treat $Q_1,\ldots,Q_n$ as formal variables.

Theorem 9.1. *Let $\tilde{L}$ be the splitting field of $R(t) \in K[t]$, so that we have a tower of fields $K \subset L \subset \tilde{L}$. Then,* $\mathrm{Gal}(\tilde{L} : K) \cong S_{n+m}$ *for $n \geq 2$, and* $\mathrm{Gal}(\tilde{L} : K) \cong \mathbb{Z}/(m+1)\mathbb{Z}$ *for $n = 1$.*

Corollary 9.1. *If $n > 1$, the subgroup of* $\mathrm{Gal}(\tilde{L} : K)$ *that fixes L is S_m. If $n = 1$, the stabilizer of L is trivial, so $L = \tilde{L}$.*

Thus, a generic fiber will contain $\frac{(m+n)!}{m!}$ points, and we should be able to decompose this fiber similarly as in the case of $\mathfrak{gl}_2$.

Proof. If $n = 1$, we get $R(t) = t^{m+1} - Q_1 = 0$, and so $\mathrm{Gal}(\tilde{L} : K) \cong \mathbb{Z}/(m+1)\mathbb{Z}$. Now, suppose $n \geq 2$. Let us specialize $Q_1 = Q_2 = \cdots = Q_{n-2} = 0$ and set $a = (-1)^n Q_{n-1}$ and $b = (-1)^{n+1} Q_n$. Then, $R(t)$ specializes to $t^{n+m} - at - b$, whose splitting field over $\mathbb{C}(a, b)$ is well known to have Galois group S_{m+n}. The statement that the Galois group in the specialized case is S_{n+m} is equivalent to the assertion that

$$\{z_1^{\nu_1} z_2^{\nu_2} \cdots z_{n+m}^{\nu_{n+m}} \mid 0 \leq \nu_1 < n+m, 0 \leq \nu_2 < n+m-1, \ldots, 0 \leq \nu_{n+m} < 1\},$$

where $\{z_i\}$ are the roots of $R(t)$, are linearly independent over $\mathbb{C}(Q_{n-1}, Q_n)$ after specialization. Label the elements of the above set by γ. If

$$\alpha_1 \gamma_1 + \cdots + \alpha_l \gamma_l = 0, \alpha_i \in \mathbb{C}[Q_1, \ldots, Q_n]$$

in the unspecialized case, we can divide by the largest factor of Q_1 that divides all α_i and specialize $Q_1 = 0$, and then repeat this process for $Q_2, \ldots, Q_{n-2}$, to obtain, in the end, a linear combination with non-zero coefficients from $\mathbb{C}(Q_{n-1}, Q_n)$ that equals zero, contradicting the above result. Thus, the Galois group in the unspecialized case must also be S_{n+m}. □

Now, we will prove an analogous theorem for $H_\zeta(\mathfrak{sp}_{2n})$. Though we do not yet know the existence of c_i for $H_\zeta(\mathfrak{sp}_{2n})$, we can consider the highest term of c_i computed in Section 8 and label by Q_i the evaluation of c_i at the diagonal matrix $\mathrm{diag}(\lambda_1, -\lambda_1, \ldots, \lambda_n, -\lambda_n)$. We define $HC : \mathbb{C}^n \to \mathbb{C}^n$ by $HC(\lambda) = (Q_1(\lambda), Q_2(\lambda), \ldots, Q_n(\lambda))$.

Let $R(t) = t^{2n+2m} - \frac{Q_1}{2} t^{2n-2} + \frac{Q_2}{2} t^{2n-4} - \cdots + (-1)^n \frac{Q_n}{2}$.

Proposition 9.3. *$R(\pm\lambda_k) = 0$ for all $1 \leq k \leq n$.*

Because the formula for the top term of c_i in the $\mathfrak{sp}_{2n}$ case is very similar to the formula in the $\mathfrak{gl}_n$ case, the proof of this proposition follows exactly the same lines as the proof of Proposition 9.2.

As before, define $L = \mathbb{C}(\lambda_1, \ldots, \lambda_n)$ and $K = \mathbb{C}(Q_1, \ldots, Q_n)$.

Theorem 9.2. *Let $\tilde{L}$ be the splitting field of $R(t) \in K[t]$, so that we have a tower of fields $K \subset L \subset \tilde{L}$. Then,* $\mathrm{Gal}(\tilde{L} : K) \cong S_{n+m} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n+m}$ *for $n \geq 2$, and* $\mathrm{Gal}(\tilde{L} : K) \cong \mathbb{Z}/(2m+2)\mathbb{Z}$ *for $n = 1$.*

Corollary 9.2. *The subgroup of* $\mathrm{Gal}(\tilde{L} : K)$ *that fixes L is* $S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ *for $n > 1$ and trivial for $n = 1$.*

Proof. When $n = 1$, $R(t) = t^{2m+2} - \frac{Q_1}{2}$, so $\mathrm{Gal}(\tilde{L} : K) \cong \mathbb{Z}/(2m+2)\mathbb{Z}$.

Now assume $n > 1$. Label the roots of $R(t)$ by $\gamma_1, \gamma_2, \ldots, \gamma_{2m+2n}$. If γ is a root of $R(t)$, then $-\gamma$ is also a root, so without loss of generality, assume $\gamma_{2i} = -\gamma_{2i-1}$. Let $N \cong (\mathbb{Z}/2\mathbb{Z})^{n+m}$ be the subgroup of $\mathrm{Gal}(\tilde{L} : K)$ consisting of automorphisms that send each γ_i to $\pm\gamma_i$. Then,

$\tilde{L}^N = K(\gamma_2^2, \gamma_4^2 \dots, \gamma_{2m+2n}^2)$, which is the splitting field of $t^{n+m} - \frac{Q_1}{2} t^{n-1} + \frac{Q_2}{2} t^{n-2} - \dots + (-1)^n \frac{Q_n}{2}$. Thus, N is a normal subgroup of $\mathrm{Gal}(\tilde{L} : K)$. Using the same reasoning underlying the proof of Theorem 9.1, the Galois group of $\tilde{L}^N$ over K is isomorphic to S_{m+n}. Now, it is clear that $\mathrm{Gal}(\tilde{L} : K) \cong S_{n+m} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n+m}$.

□

Thus, for $\mu = HC(\lambda)$, the HC fiber $HC^{-1}(\mu)$ can be noncanonically identified with $(S_{n+m} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n+m})/(S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m)$ for $n > 1$ and with $\mathbb{Z}/(2m+2)\mathbb{Z}$ for $n = 1$.

10 Kostant's Theorem

Recall Kostant's theorem in the classical case ([BL]):

Theorem. *Let $\mathfrak{g}$ be a reductive Lie algebra with adjoint-type Lie group G, and let J be the ideal in $\mathbb{C}[\mathfrak{g}^*]$ generated by the homogeneous elements of $\mathbb{C}[\mathfrak{g}^*]^G$ of positive degree. Then,*

1. *$U(\mathfrak{g})$ is a free module over its center $\mathfrak{z}(U(\mathfrak{g}))$;*
2. *the subscheme of $\mathfrak{g}$ defined by J is a normal reduced irreducible subvariety that corresponds to the set of nilpotent elements in $\mathfrak{g}$.*

In [T2], Kostant's theorem was generalized to $H_\zeta(\mathfrak{gl}_n)$. In this section, we will prove Kostant's theorem for $H_\zeta(\mathfrak{sp}_{2n})$ assuming Conjecture 8.1: $\mathfrak{z}(H_\zeta(\mathfrak{sp}_{2n})) = \mathbb{C}[t_1 + c_1, \dots, t_n + c_n]$.

Introduce a filtration on $H_\zeta(\mathfrak{sp}_{2n})$ with $\deg g = 1$ for all $g \in \mathfrak{sp}_{2n}$ and $\deg v = m + \frac{1}{2}$ for all $v \in V$. Let

$$B_m = S(V \oplus \mathfrak{sp}_{2n}) / \left(\sum_j \{\beta_i, v_j\} v_j^* + c_i^{\text{top,m}} \right)_{1 \le i \le n}$$

where $c_i^{\text{top,m}}$ are the generators of the Poisson-center given in Theorem 8.1; if Conjecture 8.1 is true, $c_i^{\text{top,m}}$ is also the highest term of c_i.

Theorem 10.1. 1. *Assuming that Conjecture 8.1 is true, $H_\zeta(\mathfrak{sp}_{2n})$ is a free module over its center.*

2. *B_m is a normal complete-intersection integral domain.*

Proof. Introduce a filtration on B_m with $\deg g = 1$ for $g \in \mathfrak{sp}_{2n}$ and $\deg v = 0$ for $v \in V$. Define $B_m^{(1)}$ by $B_m^{(1)} = \mathrm{gr}\, B_m = S(V \oplus \mathfrak{sp}_{2n})/(c_i^{\text{top,m}})_{1 \le i \le n}$. To show that $S(V \oplus \mathfrak{sp}_{2n})$ is free over $\mathbb{C}[\sum_j \{\beta_1, v_j\} v_j^* + c_1^{\text{top,m}}, \dots, \sum_j \{\beta_n, v_j\} v_j^* + c_n^{\text{top,m}}]$, it suffices to show that $S(V \oplus \mathfrak{sp}_{2n})$ is free over $\mathbb{C}[c_1^{\text{top,m}}, \dots, c_n^{\text{top,m}}]$.

From the formulas for c_i, we see that $\mathbb{C}[\lambda_1, \dots, \lambda_n]$ is a free and finite module over $\mathbb{C}[\mathrm{gr}\, c_1, \dots, \mathrm{gr}\, c_n]$, so $\mathbb{C}[\mathfrak{h}]^W$ is finite and free over $\mathbb{C}[c_1, \dots, c_n]$. Since $S(\mathfrak{sp}_{2n})$ is free over $\mathbb{C}[\mathfrak{h}]^W$ by the classical Kostant's theorem, we conclude that $S(\mathfrak{sp}_{2n})$, and hence $S(\mathfrak{sp}_{2n}) \otimes SV$, is free over $\mathbb{C}[c_1, \dots, c_n]$.

To show that B_m is a normal integral domain, it suffices to show that the smooth locus of the zero set of $t_1', t_2', \dots, t_n'$ has codimension 2 and is irreducible. Let $Z = \mathrm{Spec}(B_m)$ be a closed subscheme of $V \oplus \mathfrak{sp}_{2n}$ defined by $\mathrm{gr}\, t_i' = 0$, and let

$$U := Z \backslash Z_{sm} = \{(v, A) \in V \oplus \mathfrak{sp}_{2n} | (v, A) \in Z \text{ and } \mathrm{rank}(\mathrm{Jac}) < n\},$$

where Jac is the Jacobi matrix of $t'_1, t'_2, \dots, t'_n$ at (v, A) with respect to some basis of V and $\mathfrak{sp}_{2n}$. We need to show that U is a codimension 2 subvariety of Z and that Z is irreducible.

Now, recall that

$$\sum\{\beta_i, v_j\}v_j^* = -(\omega(A^{2i-1}v, v) + \beta_1\omega(A^{2i-3}v, v) + \beta_2\omega(A^{2i-5}v, v) + \cdots).$$

Thus, the ideal $(\sum\{\beta_i, v_j\}v_j^* + (c_i^{\mathrm{top,m}})_{1\le i\le n})$ is equal to $(S_i)_{1\le i\le n}$ for

$$S_i = (\omega(A^{2i-1}v, v) - (c_i^{\mathrm{top,m}} - \beta_1 c_{i-1}^{\mathrm{top,m}} - \beta_2 c_{i-2}^{\mathrm{top,m}} - \cdots));$$

we can and will use the Jacobian of S_i instead of t'_i to describe U. We can calculate the differentials of $\omega(A^{2i-1}v, v)$ with respect to $y_j \in V$ and $\gamma \in \mathfrak{sp}_{2n}$:

$$\frac{\partial}{\partial y_j}(\omega(A^{2i-1}v, v)) = 2\omega(A^{2i-1}v, y_j)$$
$$\frac{\partial}{\partial \gamma}(\omega(A^{2i-1}v, v)) = \omega(A^{2i-2}\gamma v + A^{2i-3}\gamma A v + \cdots + \gamma A^{2i-2}v, v).$$

Thus, if

$$\mu_1 \mathrm{grad}(S_1) + \mu_2 \mathrm{grad}(S_2) + \cdots + \mu_n \mathrm{grad}(S_n) = 0$$

for some $\mu_1, \mu_2, \dots, \mu_n \in \mathbb{C}$, then

$$\omega(\mu_1 A v, y_j) + \omega(\mu_2 A^3 v, y_j) + \cdots + \omega(\mu_n A^{2n-1}v, y_j) = 0$$

for all $1 \le j \le 2n$. Equivalently, $(\mu_1 A + \mu_2 A^3 + \cdots + \mu_n A^{2n-1})v = 0$.

Now we will consider the situation in $B_m^{(1)} = \mathrm{gr}\, B_m$. We know that $\dim Z = \dim \tilde{Z}$, where $\tilde{Z} = V \times \mathcal{N}$ and $\mathcal{N}$ is the nilpotent cone of $\mathfrak{sp}_{2n}$. Since V and $\mathcal{N}$ are irreducible, $\tilde{Z}$, and hence Z, is irreducible. Recall that U was defined as the locus of points $(v, A) \in Z \subset V \oplus \mathfrak{sp}_{2n}$ such that $\mathrm{rank}(\mathrm{Jac}) < n$, or in other words, all $n \times n$ minors of the Jacobian matrix have determinant 0. Since each of those determinants is homogeneous with respect to our second filtration, it is natural to define $\tilde{U} \subset \tilde{Z}$ as a locus of points where $\mathrm{rank}(\mathrm{Jac}) < n$. Then, $\dim U \le \dim \tilde{U}$. Note that $\tilde{U} = \tilde{U}_1 \sqcup \tilde{U}_2$, where $\tilde{U}_1 = \tilde{U} \cap \{(v, A) | A \text{ is regular nilpotent}\}$ and $\tilde{U}_2 = \tilde{U} \cap \{(v, A) | A \text{ is non-regular nilpotent}\}$. The codimension of a regular nilpotent's orbit is 2, so $\mathrm{codim}_{\tilde{Z}}(\tilde{U}_2) \ge 2$. It suffices to show that $\mathrm{codim}_{\tilde{Z}}(\tilde{U}_1) \ge 2$ as well. We shall do this by showing that given a regular nilpotent A, $\dim(V_{A,\mathrm{sing}}) \le 2n - 2$ in V, where $V_{A,\mathrm{sing}} = \{v \in V | (v, A) \in \tilde{U}\}$.

Let us switch to a basis of $\mathfrak{sp}_{2n}$ where

$$J = \begin{pmatrix} 0 & \cdots & 0 & 0 & -1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & -1 & 0 & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If we define

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

then $AJ + JA^T = 0$ so $A \in \mathfrak{sp}_{2n}$. Now, suppose that $\sum_{1\le j\le n} \mu_j \mathrm{grad}(S_j) = 0$ at (A, v), for $v = (a_1, \dots, a_{2n})$. By examining the $\frac{\partial}{\partial y_j}$ components of $\mathrm{grad}(S_j)$, we get $a_{2n} = 0$; moreover, either $a_{2n-1} = 0$, or $\mu_1 = \cdots = \mu_{n-1} = 0$. The conditions $a_{2n} = a_{2n-1} = 0$ define a codimension two subspace as desired. We thus need to show that if $a_{2n} = 0$ and $\mu_1 = \cdots = \mu_{n-1} = 0$, then $\sum_{1\le j\le n} \mu_j \mathrm{grad}(S_j) = 0$ implies a non-trivial condition on v. To find such a condition, note that

$$\frac{\partial}{\partial\gamma}(\omega(A^{2n-1}v, v)) = \omega(A^{2n-2}\gamma v, v) + \omega(A^{2n-3}\gamma Av, v) + \cdots + \omega(\gamma A^{2n-2}v, v),$$

and that $\frac{\partial}{\partial\gamma}(c_n^{\text{top,m}} - \beta_1 c_{n-1}^{\text{top,m}} - \cdots)$ does not depend on v and is just a number for a fixed A. Now, let us take $\gamma = e_{2n,1}$; we can verify that $e_{2n,1}J + Je_{2n,1}^T = 0$, so $e_{2n,1} \in \mathfrak{sp}_{2n}$. We note that $e_{2n,1}A^{2n-2} = e_{2n,2n-1}$, $Ae_{2n,1}A^{2n-3} = e_{2n-1,2n-2}$, $A^2 e_{2n,1}A^{2n-4} = e_{2n-2,2n-3}$ and so forth. Thus, $\frac{\partial}{\partial\gamma}(\omega(A^{2n-1}v, v)) = \omega(A^T v, v)$. However, if $v = (a_1, \dots, a_{2n-1}, 0)$,

$$\omega(A^T v, v) = \omega((0, a_1, \dots, a_{2n-1}), (a_1, \dots, a_{2n-1}, 0)),$$

is a nontrivial degree two polynomial in $a_1, \dots, a_{2n-1}$ that should equal the number $\frac{\partial}{\partial\gamma}(c_n^{\text{top,m}} - \beta_1 c_{n-1}^{\text{top,m}} - \cdots)(A)$. This gives the other codimension 1 condition, and so $\tilde{U}_1$ is at least codimension 2 as desired. □

A Appendix

In the appendix, we give examples of the technical computations from the main body of the paper.

A.1 Proof of Lemma 3.2

We shall present the proof that $D = 0$; the proof that $C = 0$ goes along exactly analogous lines.

Let us write $D = -D_1 - D_2 + D_3$, where

$$D_1 = \frac{1}{2^m} \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{j-1} \frac{(m+2)!}{(2j+1)(2k+1)!(2j-2k-1)!(m+1-2j)!} \beta^{m+1-2j}\gamma^k,$$

$$D_2 = \frac{1}{2^m} \sum_{j=1}^{\lfloor \frac{m+2}{2} \rfloor} \sum_{k=0}^{j-1} \frac{(m+2)!}{(2k+1)!(2j-2k-1)!(m+2-2j)!} (\beta+1)^{m+2-2j} u_{2k},$$

and

$$D_3 = \frac{1}{2^m} \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{j-1} \frac{(m+2)!}{(2j+1)(2k+1)!(2j-2k-1)!(m+1-2j)!} (\beta+1)^{m+1-2j}(u_{2k} + u_{2k+1}),$$

Using the binomial theorem, we find that

$$\begin{aligned} D_1 &= F_1(\sqrt{\gamma}) + F_1(-\sqrt{\gamma}) \\ D_2 &= F_2(\sqrt{\gamma}) + F_2(-\sqrt{\gamma}) \\ D_3 &= F_3(\sqrt{\gamma}) + F_3(-\sqrt{\gamma}), \end{aligned}$$

where

$$F_1(\sqrt{\gamma}) = \frac{1}{4\sqrt{\gamma}(1+\sqrt{\gamma})}\left((\beta+1+\sqrt{\gamma})^{m+2} - (\beta-1-\sqrt{\gamma})^{m+2}\right)$$

$$F_2(\sqrt{\gamma}) = \frac{1}{8\sqrt{\gamma}(1+\sqrt{\gamma})}\Bigg((\beta+3+\sqrt{\gamma})^{m+2} - (\beta-1-\sqrt{\gamma})^{m+2} + \frac{2+\sqrt{\gamma}}{\sqrt{\gamma}}\left((\beta+2+\sqrt{\gamma})^{m+2} - (\beta+1-\sqrt{\gamma})^{m+2}\right)\Bigg)$$

$$F_3(\sqrt{\gamma}) = \frac{1}{8\sqrt{\gamma}(1+\sqrt{\gamma})}\left((\beta+3+\sqrt{\gamma})^{m+2} - (\beta+1-\sqrt{\gamma})^{m+2} + (\beta-1-\sqrt{\gamma})^{m+2} - (\beta+1+\sqrt{\gamma})^{m+2}\right).$$

Then, $D = F(\sqrt{\gamma}) + F(-\sqrt{\gamma})$, where

$$\begin{aligned} F(\sqrt{\gamma}) &= -F_1(\sqrt{\gamma}) - F_2(\sqrt{\gamma}) + F_3(\sqrt{\gamma}) \\ &= \frac{1}{4\gamma}\left((\beta+1-\sqrt{\gamma})^{m+2} - (\beta+1+\sqrt{\gamma})^{m+2}\right). \end{aligned}$$

Since F is an odd function in $\sqrt{\gamma}$, $D = 0$ as desired.

A.2 Proof of Lemma 3.3

We show that $[y_1x_1 + y_2x_2 + C_1(m), x_1] = 0$ in $H_{s_m}(\mathfrak{gl}_2)$. Using Theorem 3.1, we get

$$\begin{aligned} [y_1x_1 + y_2x_2, x_1] = &\left(\frac{1}{2^{m+1}}\sum_{j=1}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j-1}\frac{4j-m-1}{2j+1}\binom{m+2}{2k+1}\binom{m+1-2k}{2j-2k-1}\beta^{m+2-2j}\gamma^k\right)x_1 \\ &+\left(\frac{1}{2^m}\sum_{j=1}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j-1}\binom{m+2}{2j+1}\binom{2j}{2k+1}\beta^{m+1-2j}\gamma^k\right)(e_{11}x_1 + e_{21}x_1). \end{aligned}$$

Using Lemma 3.1, we get

$$\begin{aligned} [C_1(m), x_1] = &\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\Bigg(\binom{m+2}{2j+1}\binom{2j+1}{2k+1}(\beta^{m+1-2j}\gamma^k + (\beta+1)^{m+1-2j}(\beta u_{2k} - u_{2k+1})) \\ &- \binom{m+2}{2j}\binom{2j}{2k+1}(\beta^{m+2-2j}\gamma^k + (\beta+1)^{m+2-2j}(\beta u_{2k} - u_{2k+1}))\Bigg)x_1 \\ &+ \frac{1}{2^m}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(\left(\binom{m+2}{2j+1}\binom{2j+1}{2k+1} + \binom{m+2}{2j}\binom{2j}{2k+1}(\beta+1)u_{2k}\right)(\beta+1)^{m+1-2j}u_{2k}\times\right. \\ &\left.(e_{11}x_1 + e_{21}x_1).\right. \end{aligned}$$

We shall prove that the coefficient of x_1 in $[y_1x_1 + y_2x_2 + C_1(m), x_1]$ vanishes. This coefficient simplifies to

$$\begin{aligned} &\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\Bigg(-\binom{m+2}{2j+1}\binom{2j}{2k+1}\beta^{m+2-2j}\gamma^k + \binom{m+2}{2j+1}\binom{2j+1}{2k+1}\beta^{m+1-2j}\gamma^k \\ &+ \binom{m+2}{2j+1}\binom{2j+1}{2k+1}(\beta+1)^{m+1-2j}(\beta u_{2k} - u_{2k+1}) - \binom{m+2}{2j}\binom{2j}{2k+1}(\beta+1)^{m+2-2j}(\beta u_{2k} - u_{2k+1})\Bigg). \end{aligned}$$

As in A.1, we can find closed form expressions for the above quantity:

$$\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(-\binom{m+2}{2j+1}\binom{2j}{2k+1}\beta^{m+2-2j}\gamma^k\right)=F_1(\sqrt{\gamma})+F_1(-\sqrt{\gamma})$$

$$\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(\binom{m+2}{2j+1}\binom{2j+1}{2k+1}\beta^{m+1-2j}\gamma^k\right)=F_2(\sqrt{\gamma})+F_2(-\sqrt{\gamma})$$

$$\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(\binom{m+2}{2j+1}\binom{2j+1}{2k+1}(\beta+1)^{m+1-2j}\beta u_{2k}\right)=F_3(\sqrt{\gamma})+F_3(-\sqrt{\gamma})$$

$$\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(-\binom{m+2}{2j+1}\binom{2j+1}{2k+1}(\beta+1)^{m+1-2j}u_{2k+1}\right)=F_4(\sqrt{\gamma})+F_4(-\sqrt{\gamma})$$

$$\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(-\binom{m+2}{2j}\binom{2j}{2k+1}(\beta+1)^{m+2-2j}\beta u_{2k}\right)=F_5(\sqrt{\gamma})+F_5(-\sqrt{\gamma})$$

$$\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(\binom{m+2}{2j}\binom{2j}{2k+1}(\beta+1)^{m+2-2j}u_{2k+1}\right)=F_6(\sqrt{\gamma})+F_6(-\sqrt{\gamma}).$$

Then, some elementary algebra will show that $F_1(\gamma)+F_2(\gamma)+F_3(\gamma)+F_4(\gamma)+F_5(\gamma)+F_6(\gamma)=0$.

A.3 Proof of Theorem 3.3

The trick, once again, is to find the closed form expressions for $C_1(i)$ and $C_2(i)$. We shall demonstrate by computing the action of $\tilde{t}_1'$ on $M(\lambda)$ for the deformation $\zeta = s_m$.

Note that $\tilde{t}_1' = x_1y_1+x_2y_2+[y_1,x_1]+[y_2,x_2]+C_1(m)$, so $\tilde{t}_1'(\lambda) = ([y_1,x_1]+[y_2,x_2]+C_1(m))(\lambda)$. Using Theorem 3.1 and 3.2, we can write $\tilde{t}_1'(\lambda) = (2A_m+\beta B_m+C_1(m))(\lambda)$, so after some easy simplifications, we get

$$\tilde{t}_1'(\lambda)=\frac{1}{2^{m+1}}\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(\binom{m+2}{2j+1}\binom{2j+1}{2k+1}\beta^{m+1-2j}\gamma^k+\binom{m+2}{2j}\binom{2j}{2k+1}\beta^{m+2-2j}\gamma^k\right)(\lambda).$$

We can write

$$\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(\binom{m+2}{2j+1}\binom{2j+1}{2k+1}\beta^{m+1-2j}\gamma^k\right)$$
$$=\frac{1}{4\sqrt{\gamma}}\left((\beta+1+\sqrt{\gamma})^{m+2}-(\beta-1-\sqrt{\gamma})^{m+2}-(\beta+1-\sqrt{\gamma})^{m+2}+(\beta-1+\sqrt{\gamma})^{m+2}\right)$$

and

$$\sum_{j=0}^{\lfloor\frac{m+2}{2}\rfloor}\sum_{k=0}^{j}\left(\binom{m+2}{2j}\binom{2j}{2k+1}\beta^{m+2-2j}\gamma^k\right)$$
$$=\frac{1}{4\sqrt{\gamma}}\left((\beta+1+\sqrt{\gamma})^{m+2}+(\beta-1-\sqrt{\gamma})^{m+2}-(\beta+1-\sqrt{\gamma})^{m+2}-(\beta-1+\sqrt{\gamma})^{m+2}\right).$$

Thus,

$$\tilde{t}_1'(\lambda)=\frac{1}{2^{m+2}\sqrt{\gamma}}\left((\beta+1+\sqrt{\gamma})^{m+2}-(\beta+1-\sqrt{\gamma})^{m+2}\right)(\lambda)=\frac{(2\lambda_1+2)^{m+2}-(2\lambda_2)^{m+2}}{2^{m+2}(\lambda_1-\lambda_2+1)}=H_{m+1}(\lambda)$$

as desired.

A.4 Proof of Lemma 8.1

In this section, we will outline the proof of Lemma 8.1, which states:

$$\sum_{j=1}^{2n} \sum_{e \in B} \left\{ \frac{\partial \beta_i}{\partial e}, v \right\} e(v_j) v_j^* = 0. \tag{2}$$

We use the basis for V defined in Section 8, in which ω is represented by the matrix J.

Let us multiply (2) by t^{2i} and sum over i to get the equivalent assertion that

$$\sum_j \sum_{e \in B} \left\{ \frac{\partial \det(1 - tA)}{\partial e}, v \right\} e(v_j) v_j^* = 0.$$

Since the whole sum is $\mathfrak{sp}_{2n}$-invariant (even though each term considered separately is not), we can look at the restriction of the sum to $\mathfrak{h}$. Thus, this sum equals zero if and only if

$$\sum_j \sum_{e \in B} \left\{ \frac{\partial \det(1 - tA)}{\partial e}, v \right\} e(v_j) v_j^* \bigg|_{\mathfrak{h}} = 0.$$

We choose the following basis B for $\mathfrak{sp}_{2n}$: $e_{2j-1,2j}$, $e_{2j,2j-1}$, $e_{2j-1,2j-1} - e_{2j,2j}$, for all $1 \le j \le n$, and for all $1 \le k < l \le n$, the elements $e_{2l+1,2k} + e_{2k-1,2l+2}$, $e_{2l,2k} - e_{2k-1,2l-1}$, $e_{2l+1,2k-1} - e_{2k,2l+2}$, and $e_{2l,2k-1} + e_{2k,2l-1}$. We observe that for any $1 \le j, j' \le 2n$, there exists a unique basis vector in B that takes v_j to $\pm v_{j'}$; we shall denote this element by $v_{j',j} \in \mathfrak{sp}_{2n}$. These $v_{j',j}$ are not pairwise distinct since there are basis vectors with two nonzero entries.

Since Sp_{2n} acts transitively on V, we can assume $v = v_1$. Together with our choice of basis, we can then write

$$\sum_j \sum_{e \in B} \left\{ \frac{\partial \det(1 - tA)}{\partial e}, v_1 \right\} e(v_j) v_j^* = \sum_{j,j',k} \frac{\partial^2 \det(1 - tA)}{\partial v_{k,1} \partial v_{j',j}} v_{j'} v_k v_j^* (-1)^{\iota_{jj'}},$$

where

$$\iota_{jj'} = \begin{cases} 1 & \text{if } j \equiv j' \bmod 2 \text{ and } j' < j, \text{ or if } j' = j \text{ and } j \text{ is even}, \\ 0 & \text{otherwise.} \end{cases}$$

We now restrict to $\mathfrak{h}$ and only keep track of the non-zero terms. We have $\frac{\partial^2 \det(1-tA)}{\partial v_{k,1} \partial v_{j',j}}\Big|_{\mathfrak{h}} \neq 0$ only when the matrices for $v_{k,1}$ and $v_{j',j}$ have nonzero entries on the diagonal, or if $v_{k,1}$ and $v_{j',j}$ have nonzero entries at the i-th row j-th column and j-th row i-th column respectively. This can only happen when $v_{j'} v_k v_j^* = v_1 v_a v_a^*$ for some a. We can list all the ways this can happen for $a = 2b$ or $a = 2b - 1$ with $b \neq 1$ (keeping in mind that $v_{2b-1}^* = v_{2b}$ and $v_{2b}^* = -v_{2b-1}$):

1. $\frac{\partial^2 \det(1-tA)}{\partial v_{1,1} \partial v_{2b-1,2b-1}} v_1 v_{2b-1} v_{2b}$,

2. $\frac{\partial^2 \det(1-tA)}{\partial v_{1,1} \partial v_{2b,2b}} v_{2b} v_1 v_{2b-1}$,

3. $\frac{\partial^2 \det(1-tA)}{\partial v_{2b-1,1} \partial v_{1,2b-1}} (-v_1 v_{2b-1} v_{2b})$,

4. $\frac{\partial^2 \det(1-tA)}{\partial v_{2b,1} \partial v_{1,2b}} (-v_1 v_{2b} v_{2b-1})$,

5. $\frac{\partial^2 \det(1-tA)}{\partial v_{2b,1} \partial v_{2b-1,2}} (-v_{2b-1} v_{2b} v_1)$,

6. $\frac{\partial^2 \det(1-tA)}{\partial v_{2b-1,1}\partial v_{2b,2}}(-v_{2b-1}v_{2b}v_1)$.

To calculate the derivatives, let A_1 be the copy of $\mathfrak{sp}_4$ formed by the intersections of the first, second, $2k-1$-th, and $2k$-th rows and columns of $1-tA$, and let A_2 be what remains after we throw out those rows and columns. Then, note that all the above derivatives evaluate to the same polynomial in the Cartan of A_2 times the corresponding derivative in $\mathfrak{sp}_4$; for instance, $\frac{\partial^2 \det(1-tA)}{\partial v_{1,1}\partial v_{2b-1,2b-1}} = h\frac{\partial^2 \det A_1}{\partial v'_{1,1}\partial v'_{3,3}}$ with $v'_{1,1}, v'_{3,3} \in \mathfrak{sp}_4$ and $h \in S(\mathfrak{h}(A_2))$. Thus, we can reduce our problem to $\mathfrak{sp}_4$, and straightforward computation can show that (2) is true for $\mathfrak{sp}_4$. Similarly, when $b=1$ (that is, when the term is of form $v_1v_1v_2$), all computations will reduce to analogous ones in $\mathfrak{sp}_2$.

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