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- ▶ Setting  $\hbar = 0$ , we recover classical physics.

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- ▶ There should be a special value ( $\hbar = 0 \Leftrightarrow q = 1$ ) such that  $A_1$  is commutative.
- ▶ We should study  $A_q$  (quantum) by exporting knowledge of  $A_{q=1}$  (classical), and vice versa.

# A determinant formula for quantum $GL(N)$

Masahiro Namiki  
*MIT PRIMES*

May 21, 2011

# DETERMINANTS

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Invertible matrices are characterized by non-zero determinant.

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- ▶ e.g.)

$\mathbb{C}$  itself

$Mat_2(\mathbb{C})$  (=  $2 \times 2$  matrices)

$\mathbb{C}[x, y]$  (= polynomials in two variables)

$= \mathbb{C}\langle x, y \rangle / (xy = yx)$

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The R-Matrix: 
$$R_{kl}^{ij} = q^{\delta_{ij}} \delta_{ik} \delta_{jl} + (q - q^{-1}) \theta(i - j) \delta_{il} \delta_{jk}$$

which

$$\theta(s) = \begin{pmatrix} 1 & \text{if } s > 0 \\ 0 & \text{otherwise} \end{pmatrix} \quad \delta_{mn} = \begin{pmatrix} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{pmatrix}$$

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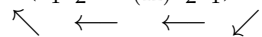
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$$\text{So, } t_{(12)} = q^2, \quad f((12)) = 2$$

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Using the information, we made a program which will change arbitrary order of elements in the right order.

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Thus, we got the exponents for each of the permutations.



# LIST

A part of data for  $N = 4$

Cycle notation	Permutation notation	Coefficient
$(1, 2)$	$[2, 1, 3, 4]$	$q^2$
$(2, 3)$	$[1, 3, 2, 4]$	$q^2$
$(3, 4)$	$[1, 2, 4, 3]$	$q^2$
$(1, 3, 2)$	$[3, 1, 2, 4]$	$q^3$
$(1, 3)$	$[3, 2, 1, 4]$	$q^4$
$(1, 2, 3)$	$[2, 3, 1, 4]$	$q^4$
$(1, 4, 3, 2)$	$[4, 1, 2, 3]$	$q^4$
$(1, 4, 3)$	$[4, 2, 1, 3]$	$q^5$
$(1, 3, 4, 2)$	$[3, 1, 4, 2]$	$q^5$
$(1, 2, 3, 4)$	$[2, 3, 4, 1]$	$q^6$
$(1, 2, 4)$	$[2, 4, 3, 1]$	$q^6$
$(1, 3, 4)$	$[3, 2, 4, 1]$	$q^6$
$(1, 3)(2, 4)$	$[3, 4, 1, 2]$	$q^6$
$(1, 4, 2, 3)$	$[4, 3, 1, 2]$	$q^7$

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$l(s)$  = "Length of the permutation"

which is the number of pairs out of order after  $s$ .

(  $i > j, s(i) < s(j)$  )

$e(s)$  = excedance, the number of  $i$  such that  $s(i) > i$ .

## FUTURE PLANS

We confirmed our conjecture formula through  $N = 11$ .

We are presently working on the general proof.

# ACKNOWLEDGMENTS

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Thank you all for listening to my presentation.