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- We should study A_q (quantum) by exporting knowledge of $A_{q=1}$ (classical), and vice versa.

A determinant formula for quantum GL(N)

Masahiro Namiki
MIT PRIMES

May 21, 2011

Determinants

The determinant for n×n matrix is

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Invertible matrices are characterized by non-zero determinant.

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  $Mat_2(\mathbb{C}) \qquad (= 2 \times 2$ matrices)
  $\mathbb{C}[x, y] \qquad$ (= polynomials in two variables)
  $= \mathbb{C}\langle x, y\rangle/(xy = yx)$

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So, $t_{(12)} = q^2, \; f((12)) = 2$

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Thus, we got the exponents for each of the pemutations.

LIST

A part of data for $N = 4$

Cycle notation	Permutation notation	Coefficient
$(1,2)$	$[2,1,3,4]$	q^2
$(2,3)$	$[1,3,2,4]$	q^2
$(3,4)$	$[1,2,4,3]$	q^2
$(1,3,2)$	$[3,1,2,4]$	q^3
$(1,3)$	$[3,2,1,4]$	q^4
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$(1,2,4)$	$[2,4,3,1]$	q^6
$(1,3,4)$	$[3,2,4,1]$	q^6
$(1,3)(2,4)$	$[3,4,1,2]$	q^6
$(1,4,2,3)$	$[4,3,1,2]$	q^7

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$l(s)$="Length of the permutation"
which is the number of pairs out of order after s.
$(i > j, s(i) < s(j))$

$e(s)$=excedance, the number of i such that $s(i) > i$.

FUTURE PLANS

We confirmed our conjecture formula through $N = 11$.

We are presently working on the general proof.

Acknowledgments

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Thank you all for listening to my presentation.