TOTAL CHIP NUMBERS AND MOTORS IN THE PARALLEL CHIP-FIRING GAME

ZIV SCULLY, YAN ZHANG, AND DAMIEN JIANG

Abstract. The *parallel chip-firing game* is an automaton on graphs in which vertices "fire" chips to their neighbors when they have enough chips to do so. In this work we first characterize positions that repeat every 2 turns. Using this we determine the eventual periods of games on trees given only the total number of chips. We introduce the concepts of motorized parallel chip-firing games and motor vertices, study the effects of motors connected to a tree, and show that some motorized games can be simulated by ordinary games. We end by proving the equivalence of two conjectures: one restricts the firing pattern of a single vertex over a period; the other restricts the set of vertices that fire at a single time step. These conjectures, if shown to be true, would greatly simplify the study of the parallel chip-firing game.

1. Introduction

Background. The *parallel chip-firing game*, also known as the *candy-passing game*, is a solitaire game played on a graph in which vertices that have at least as many chips as incident edges "fire" chips to their neighbors. It is a specific case of the *abelian sandpile model*, which is a generalization of a sandpile model introduced by Bak, Tang, and Weisenfeld [2, 3]. Bak, Tang, and Weisenfeld's model displays *self-organized criticality*: it is attracted towards a critical state in which adding one grain of sand can cause large avalanche-like reactions. The frequency of reactions of a given magnitude is approximately inversely proportional to that magnitude (called a *power law*), which gives the model *scale-free* properties. Self-organized criticality is a property that appears in a wide variety of natural systems, from earthquakes to traffic jams to the brain [1, 16].

The brain provides a particularly interesting example in that the mechanism by which neurons fire bears some resemblance to the firing rule in the abelian sandpile model. Neurons communicate by firing temporally-discrete electrical impulses ("action potentials") that are propagated to their neighbors. The electrical firing of a neuron's neighbors gradually increases the electrical potential difference across the neuron's membrane, until a threshold membrane potential is exceeded, causing the neuron to fire [18]. Analogously, during each time step of the abelian sandpile model, a vertex that has more particles than its critical threshold will "topple", passing one particle to each of its neighbors. Both models appear to display self-organized criticality [10, 16].

The graph of neuron connectivity displays the *small world* property: there are many densely connected clusters of neurons, each cluster being sparsely connected to other clusters [17]. In this paper we introduce a tool for analyzing the parallel chip-firing game that, by simulating external signals, may allow us to examine a game on a graph one cluster at a time, especially when the connections between clusters are sparse.

The parallel chip-firing game has also been the object of study in computer science; it is able to simulate any two-register machine and is thus universal [7].

The Game. The parallel chip-firing game is played on a graph as follows:

- At first, a nonnegative integer number of chips is placed on each vertex of the graph.
- The game then proceeds in discrete turns. Each turn, a vertex checks to see if it has at least as many chips as incident edges.
  - If so, that vertex *fires*.
  - Otherwise, that vertex *waits*.
- To fire, a vertex passes one chip along each of its edges. All vertices that fire in a particular turn do so in parallel.
- Immediately after firing or waiting, every vertex receives any chips that were fired to it.

Here we will only consider games on finite, undirected, connected graphs, though the definition of the game can be easily generalized for arbitrary multidigraphs. An example game is illustrated in Figure 1. We represent a parallel chip-firing game as σ, where the chip configuration at a particular time $t \in \mathbb{N}$ is denoted σ_t.

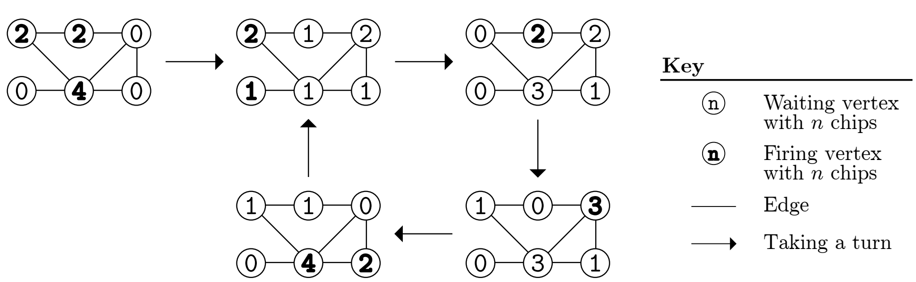

FIGURE 1. A parallel chip-firing game. From an initial position in the upper left the game eventually enters a period of length 4.

The total number of chips on all vertices of the graph is constant throughout a game, so there is a finite number of possible positions in every game.

Therefore, every game eventually reaches a position σ_t that is identical to a later position σ_{t+p} for some $t, p \in \mathbb{N}$. (We write $\sigma_t = \sigma_{t+p}$.) The game is deterministic, so $\sigma_{t+n} = \sigma_{t+n+p}$ for all $n \in \mathbb{N}$. Thus, every parallel chip-firing game is eventually periodic.

This gives rise to two questions. Firstly, what characteristics of a game and its underlying graph determine the length of a period? It is known exactly what periods are possible on certain classes of graphs, such as trees [5], simple cycles [9], the complete graph [15], and the complete bipartite graph [12]. Kiwi et al. [13] constructed graphs on which the period of games can grow exponentially with polynomial increase in the number of vertices. There are also results regarding the total number of chips in a game. Kominers and Kominers [14] showed that games with a very large density of chips must have period 1. Dall'Asta [9] and Levine [15], in their respective characterizations of periods on cycles and complete graphs, related the total number of chips to a game's *activity*, the fraction of turns during which a vertex fires. The activity, in turn, is closely tied to the period, which must be divisible by the denominator of the activity.

Secondly, we notice that some, but not all, positions σ_t are *periodic*, with $\sigma_t = \sigma_{t+p}$ for some p. What characterizes periodic positions? This problem has not been as extensively studied. Dall'Asta [9] characterized the periodic positions of games on cycles.

Our results advance the understanding of both of these questions. In Section 2 we precisely define the parallel chip-firing game, and in Section 3 we collect previous results that are used later on. In Section 4 we characterize 2-periodic positions, i.e. when $\sigma_t = \sigma_{t+2}$. Surprisingly, this allows us to show that the total number of chips alone determines the eventual period of games on trees.

The remainder of the paper develops a new tool for studying the chip-firing game: motors, vertices that fire with a regular pattern independently of normal chip-firing rules. Games with motors are called *motorized games*. Motors allow us to study the behavior of subgraphs in ordinary parallel chip-firing games. In Section 5 we show that in motorized games on trees, vertices are always "following" a motor. We also prove that motorized games are a subset of ordinary games, provided that the firing pattern of each motor occurs in an ordinary game. In Section 6 we define "clumpy" firing patterns that we conjecture cannot occur in ordinary games and prove that this is the case for a subset of these patterns. Finally, in Section 7 we use motors to show that games in which both the set of firing vertices and the set of waiting vertices have an interior at some turn (not necessarily simultaneously) exist iff a clumpy firing pattern can occur in some game.

2. Preliminaries

Definitions. A *parallel chip-firing game* σ on a graph $G = (V(G), E(G))$ is a sequence $(\sigma_t)_{t\in\mathbb{N}}$ of ordered tuples with natural number elements indexed by $V(G)$. Each tuple represents the chip configuration at a particular turn and each element of the tuple is the number of chips on the vertex. We define the following for all $v \in V(G)$:

$$\begin{aligned} N(v) &= \{w \in V(G) \mid \{v, w\} \in E(G)\} \\ \sigma_t(v) &= \text{number of chips on } v \text{ in position } \sigma_t \\ F_t^\sigma(v) &= \begin{cases} 0 & \text{if } \sigma_t(v) \le \deg(v) - 1 \\ 1 & \text{if } \sigma_t(v) \ge \deg(v) \end{cases} \\ \Phi_t^\sigma(v) &= \sum_{w \in N(v)} F_t^\sigma(w). \end{aligned}$$

In a parallel chip-firing game, σ_t induces σ_{t+1}. For all $v \in V(G)$,

$$\sigma_{t+1}(v) = \sigma_t(v) + \Phi_t^\sigma(v) - F_t^\sigma(v) \deg(v),$$

so it suffices to define a game on a given graph by its initial position σ_0. When $F_t^\sigma(v) = 0$, we say v *waits* at t, and when $F_t^\sigma(v) = 1$, we say v *fires* at t.

A position σ_t is called *periodic* iff there exists $p \in \mathbb{N}$ such that $\sigma_t = \sigma_{t+p}$. The minimum such p for which this occurs is the *period* of σ and is denoted $p(\sigma)$. Abusing notation slightly, "a period" of a game σ can also refer to a set of times $\{t, t+1, \dots, t+p-1\}$ where σ_t is periodic. A periodic position σ_t is also called *n-periodic*, where $n = p(\sigma)$. Because the game is deterministic and there are a finite number of possible positions with a given number of chips, for any game σ there exists $t_0 \in \mathbb{N}$ such that σ_t is periodic for all $t \ge t_0$. If the initial position σ_0 of game σ is periodic, we may also say that σ is periodic.

Notation. We use the following notation throughout the paper. Precise definitions for invented notation are given in the section indicated in the last column.

Parallel Chip-Firing		*Defined in*
$\sigma_t(v)$	The number of chips on vertex v in position σ_t.	Section 2
$F_t^\sigma(v)$	Indicates whether or not vertex v fires in σ_t.	Section 2
$\Phi_t^\sigma(v)$	The number of chips vertex v will receive in position σ_t.	Section 2
$\delta_t^\sigma(v)$	The net gain of chips on vertex v from σ_0 to σ_t. In an ordinary game, this is equal to $\sigma_t(v) - \sigma_0(v)$.	Section 5

$T_f^\sigma(v)$	The set of times for which a vertex v waits in game σ (if $f=0$) or the set of times for which it fires (if $f=1$).	Section 3
$p(\sigma)$	The period of σ.	Section 2
$c(\sigma)$	The total number of chips in σ.	Section 4
$M(\sigma)$	The set of vertices that are motors in game σ. (Motors are defined later.)	Section 5
Graphs		
$V(G)$, $E(G)$	Vertex set and edge set of graph G, respectively.	
$N(v)$	The neighbors of vertex v.	
$\deg(v)$, $\deg_G(v)$	The degree of a vertex in graph G, which is not indicated in the subscript if there is no ambiguity.	
Other		
$[a,b]$	The integer interval $\{a, a+1, \ldots, b\}$.	

3. Known Properties

Let σ be a game on graph G. We define $T_f^\sigma(v) = \{t \in \mathbb{N} \mid F_t^\sigma(v) = f\}$ for all $v \in V(G)$ and $t \in \mathbb{N}$. For $f=0$ this is the set of times at which v waits, and for $f=1$ this is the set of times at which v fires.

Lemma 3.1 ([12, Proposition 2.5])**.** *During a period in a game on a connected graph, every vertex fires the same number of times.*

Lemma 3.2 ([5, Lemma 1])**.** *Let σ be a game on G. For all $v \in V(G)$ and $f = 0$ or 1, if $[a,b] \in T_f^\sigma(v)$, then there exists $w \in N(v)$ such that $[a-1, b-1] \in T_f^\sigma(w)$.*

Lemma 3.3 ([12, Lemma 2.3])**.** *Let σ and $\overline{\sigma}$ be games on G. If $\overline{\sigma}_0(v) = 2\deg(v) - 1 - \sigma_0(v)$ for all $v \in V(G)$, then $\overline{\sigma}_t(v) = 2\deg(v) - 1 - \sigma_t(v)$ for all $v \in V(G)$ and $t \in \mathbb{N}$.*

4. 2-Periodic Positions

Lemma 4.1. *A position σ_0 on graph G is 2-periodic iff for all $v \in V(G)$,*

$$\deg(v) \leq \sigma_0(v) + \Phi_0^\sigma(v) \leq 2\deg(v) - 1.$$

Proof. Suppose the inequality is true for all $v \in V(G)$. Choose any $v \in V(G)$. If v waits on turn 0, then

$$\sigma_1(v) = \sigma_0(v) + \Phi_0^\sigma(v) \geq \deg(v),$$

so v fires on turn 1. If v instead fires on turn 0, then

$$\sigma_1(v) = \sigma_0(v) + \Phi_0^\sigma(v) - \deg(v) \leq \deg(v) - 1,$$

so v waits on turn 1. Thus, each vertex fires exactly once during the first two turns, so $\sigma_0 = \sigma_2$.

Suppose σ_0 is 2-periodic and there exists $v \in V(G)$ with $\sigma_0(v) + \Phi_0^\sigma(v) \leq \deg(v) - 1$. We have $\sigma_0(v) \leq \deg(v) - 1$ because $\Phi_0^\sigma(v) \geq 0$, so

$$\sigma_1(v) = \sigma_0(v) + \Phi_0^\sigma(v) \leq \deg(v) - 1.$$

Therefore, v waits on turns 0 and 1. By Lemma 3.1, all vertices wait on turns 0 and 1 as well, but then σ_0 is 1-periodic, a contradiction.

Suppose σ_0 is 2-perioidc and there exists $v \in V(G)$ with $\sigma_0(v) + \Phi_0^\sigma(v) \geq 2\deg(v)$. We have $\sigma_0(v) \geq \deg(v)$ because $\Phi_0^\sigma(v) \leq \deg(v)$, so

$$\sigma_1(v) = \sigma_0(v) + \Phi_0^\sigma(v) - \deg(v) \geq \deg(v).$$

Therefore, v fires on turns 0 and 1 and σ_0 is 1-periodic, a contradiction. □

Noting that a 1-periodic position can only occur if every vertex is waiting or every vertex is firing and that the only periods possible in games on trees are 1 and 2 [5], Lemma 4.1 completes the characterization of periodic positions on trees.

Let $c(\sigma) = \sum_{v \in V(G)} \sigma_t(v)$ for some $t \in \mathbb{N}$ be the total number of chips in a game σ on G. Note that the choice of t is irrelevant because no chips are ever gained or lost.

Lemma 4.2. *Let σ be a game on G. If $p(\sigma) = 1$, then*

$$c(\sigma) \leq 2|E(G)| - |V(G)| \quad \text{or} \quad c(\sigma) \geq 2|E(G)|.$$

Proof. The period of σ is 1 iff there is some t for which σ_t is 1-periodic, which is the case iff all or none of the vertices are firing at t. In the first case, $\sigma_t(v) \leq \deg(v)$ for all $v \in V(G)$. Summing these inequalities over all vertices, we get

$$\sum_{v \in V(G)} \sigma_t(v) \leq \sum_{v \in V(G)} (\deg(v) - 1),$$

or $c(\sigma) \leq 2|E(G)| - |V(G)|$. In the second case, $\sigma_t(v) \geq \deg(v)$, and summing this over $v \in V(G)$ gives us $c(\sigma) \geq 2|E(G)|$. □

Lemma 4.3. *Let σ be a game on G. If $p(\sigma) = 2$, then*

$$|E(G)| \leq c(\sigma) \leq 3|E(G)| - |V(G)|.$$

Proof. The approach is similar to that of Lemma 4.2. The period of σ is 2 iff there is some t for which σ_t is 2-periodic. By Lemma 4.1, a position σ_t is 2-periodic only if $\deg(v) \leq \sigma_t(v) + \Phi_t^\sigma(v) \leq 2\deg(v) - 1$ for all $v \in V(G)$. Summing over all vertices gives us

$$2|E(G)| \leq c(\sigma) + \sum_{v \in V(G)} \Phi_t^\sigma(v) \leq 4|E(G)| - |V(G)|.$$

Note that σ_{t+1} is also 2-periodic, which means

$$2|E(G)| \leq c(\sigma) + \sum_{v \in V(G)} \Phi_{t+1}^\sigma(v) \leq 4|E(G)| - |V(G)|.$$

Every vertex fires at exactly one of steps t and $t+1$, so $\Phi_t^\sigma(v) + \Phi_{t+1}^\sigma(v) = \deg(v)$ for all $v \in V(G)$. Adding the above inequalities yields

$$4|E(G)| \leq 2c(\sigma) + 2|E(G)| \leq 8|E(G)| - 2|V(G)|,$$

or $|E(G)| \leq c(\sigma) \leq 3|E(G)| - |V(G)|$. □

Theorem 4.4. *Let σ be a game on a tree T. Then $p(\sigma) = 2$ iff*

$$|E(T)| \leq c(\sigma) \leq 2|E(T)| - 1.$$

Proof. We apply Lemmas 4.2 and 4.3 to T. Note that $|V(T)| = |E(T)| + 1$, so the necessary conditions for having period 1 exactly complement the necessary conditions for having period 2. Games on trees can only have those period lengths [5], so the conditions are also sufficient.[1] □

5. Motors

Let G be a graph. Suppose we wish to study the periodic behavior of games on G, focusing on a particular subgraph $H \subseteq G$. Consider

$$X = \{v \in V(G) \setminus V(H) \mid N(v) \cap V(H) \neq \emptyset\},$$

the set of vertices "just outside" of H. Knowing the initial chip configuration on $V(H) \cup X$ is in general not enough to determine all subsequent configurations because vertices in X may have interactions with vertices outside of $V(H) \cup X$. However, we do know that every vertex assumes a pattern of firing and waiting that repeats periodically as soon as a game reaches a periodic position. Therefore, we can simulate the presence of the rest of G by having each vertex in X fire with a regular pattern regardless of the number of chips it receives.

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[1] We are aware of an alternative proof that makes use of [6, Lemma 2.3], which states that a chip-firing game on G terminates (i.e. no vertex can fire) if it has less than $E(G)$ chips. By applying Lemma 3.3 we find that the period of a game is 1 with every vertex firing if it has more than $3E(G) - V(G)$ chips. It was shown in [14] that if $\sigma_{t_0}(v) \geq 2\deg(v)$ for some $t_0 \in \mathbb{N}$, then $\sigma_t(v) \geq 2\deg(v)$ for all $t \in [0, t_0 - 1]$, so this is also the case if using Lemma 3.3 is impossible. From these facts Theorem 4.4 follows easily. However, we think the argument given may be useful in other contexts.

Let σ be a game on G. We define $\delta_t^\sigma(v) = \sum_{i=0}^{t-1}(\Phi_i^\sigma(v) - F_i^\sigma(v)\deg(v))$ for all $v \in V(G)$ and $t \in \mathbb{N}$ to be the net flux of chips into v over the time interval $[0, t-1]$.

A *periodic firing pattern* of $v \in V(G)$ is a sequence $(F_t^\sigma(v))_{t\in[t_0,\infty)}$ such that $F_t^\sigma(v) = F_{t+p(\sigma)}^\sigma(v)$ for all $t \in [t_0, \infty)$. (For brevity, the "periodic" is often omitted.)

A *motorized game* on G is a game σ with a non-empty set $M(\sigma) \subseteq V(G)$. Call each $m \in M(\sigma)$ a *motor*. We associate a periodic firing pattern starting from $t_0 = 0$ with each motor, which the motor follows without regard to normal chip-firing rules. Motors have no chip count; they destroy all chips they receive and create new chips whenever they fire. See Figure 2 for an example. Motorized games σ on G have the requirement that $\{\delta_t^\sigma(v) \mid t \in \mathbb{N}\}$ be bounded for all $v \in V(G)$. The values $c(\sigma)$ and $\sigma_t(m)$ are undefined for motorized games σ for all $m \in M(\sigma)$ and $t \in \mathbb{N}$. The term "ordinary game" refers to a game with no motors when there is ambiguity.

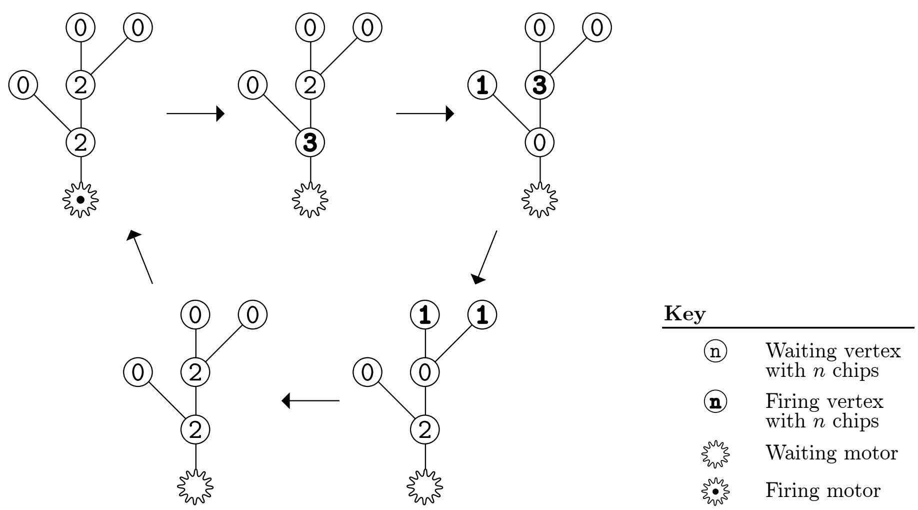

FIGURE 2. A motorized parallel chip-firing game. The motor has firing pattern $(1, 0, 0, 0, 0, 1, 0, 0, 0, 0, \dots)$.

Call an interval $[a, b]$ with $a < b$ a *clump* of $v \in V(G)$ iff for $f = 0$ or 1, $[a, b] \in T_f^\sigma(v)$ and $F_{a-1}^\sigma(v) = F_{b+1}^\sigma(v) = 1 - f$. Given $v \in V(G)$, we can express $\mathbb{N}$ as the union of clumps of v and times during which v alternates between firing and waiting.

In the following two theorems we consider periodic motorized games on trees. In particular, this means that we may go arbitrarily far back in time because given a periodic position there is a unique periodic position that can precede it.

Theorem 5.1. *Let σ be a periodic motorized game on tree T. For all $v \in V(T)$, if $[a,b] \in T_f^\sigma(v)$ and $a < b$, then $[a-d, b-d] \in T_f^\sigma(m)$ for some $m \in M(\sigma)$, where d is the distance from m to v.*

Proof. Let $v_0 = v$ and let $[a_0, b_0] \supseteq [a,b]$ be a clump of v_0. By Lemma 3.2, given a vertex $v_{i-1} \notin M(\sigma)$ with $[a_{i-1}, b_{i-1}] \in T_f^\sigma(v_{i-1})$, we can pick a vertex $v_i \in N(v_{i-1})$ and integers a_i and b_i such that $[a_i, b_i]$ is a clump of v_i and $[a_{i-1}-1, b_{i-1}-1] \subseteq [a_i, b_i] \subseteq T_f^\sigma(v_i)$. If there is a maximum i for which such a v_i exists, that v_i must be a motor, which would mean $[a-d, b-d] \subseteq [a_d, b_d] \subseteq T_f^\sigma(m)$, where d is the maximum i and $m = v_d \in M(\sigma)$. Thus, it suffices to show that there are finitely many v_i.

Suppose that $v_i = v_{i-2}$. Then $[a_i, b_i] \cup [a_{i-2}, b_{i-2}] \subseteq T_f^\sigma(v_i)$. But $[a_{i-2} - 2, b_{i-2} - 2] \subseteq [a_i, b_i]$, so $[a_{i-2} - 2, b_{i-2}] \subseteq T_f^\sigma(v_i)$. Therefore, $[a_{i-2}, b_{i-2}]$ is not a clump, a contradiction, so $v_i \neq v_{i-2}$ for all i. Because T has no cycles, the v_i are distinct, so there are finitely many v_i. □

Call a firing pattern *clumpy* if it contains two consecutive 0s and two consecutive 1s; otherwise, call it *nonclumpy.*

Corollary 5.2. *Let σ be a periodic motorized game on tree t with $M(\sigma) = \{m\}$. If m has a nonclumpy firing pattern but has at least one clump, then $F_{t+d}^\sigma(v) = F_t^\sigma(m)$ for all $v \in V(T)$ and $t \in \mathbb{N}$, where d is the distance from v to m.*

Proof. Let $v \in V(T)$. By Theorem 5.1, v has a nonclumpy firing pattern because m does. By Lemma 3.1, v must have at least one clump, again because m does. For every clump $[a,b] \subseteq T_f^\sigma(v)$, $[a-d, b-d] \subseteq T_f^\sigma(m)$, where $f = 0$ or 1. The non-clump intervals of v's firing pattern are alternations between 0 and 1, starting and ending with $1-f$. The same must be true of m because it is nonclumpy and fires the same number of times each period as v. □

We call a firing pattern $(f_t)_{t \in [t_0, \infty)}$ *possible* if there exists an ordinary periodic game σ on some graph G such that $F_t^\sigma(v) = f_{t_0+t}$ for all $t \in \mathbb{N}$. Our next theorem states that we can simulate motorized games with ordinary games as long as every motor's firing pattern is possible. Figure 3 shows an example.

Theorem 5.3. *Let σ be a motorized game on G with σ_0 periodic. If every motor's firing pattern is possible, then there exists an ordinary game σ' on graph $H \supseteq G$ such that $\deg_H(u) = \deg_G(u)$ for all $u \in V(G) \setminus M(\sigma)$ and $F_t^{\sigma'}(v) = F_t^\sigma(v)$ for all $t \in \mathbb{N}$ and $v \in V(G)$. In addition, H contains no paths between distinct vertices of G that are not also in G.*

Proof. For each $m \in M(\sigma)$, let A_m be a graph such that there exists a game σ^m and some vertex $u_m \in V(A_m)$ such that for all t, $F_t^{\sigma^m}(u_m) = F_t^\sigma(m)$. Let a_m and b_m be the minimum and maximum respectively of $\{\delta_t^\sigma(m) \mid t \in \mathbb{N}\}$. Let $k_m = b_m - a_m + 1$. Let H be the union of G and k_m copies of each A_m,

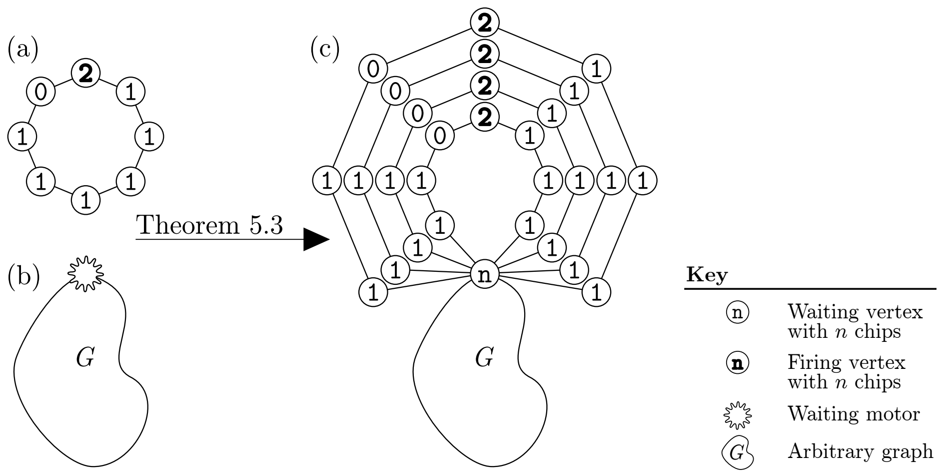

FIGURE 3. The motor in motorized game (b) has firing pattern $(0,0,0,0,1,0,0,0,\dots)$. This firing pattern is possible because it occurs in ordinary game (a). By using sufficiently many copies of (a) and carefully choosing n, we construct (c). The behavior of G in (c) is identical to the behavior of G in (b).

with G and the A_m copies disjoint except for $m = u_m$ for each $m \in M(\sigma)$. It is clear by construction that H contains no new paths between distinct vertices of G and that

- $\deg_H(m) = k_m \deg_{A_m}(m) + \deg_G(m)$ for all $m \in M(\sigma)$.
- $\deg_H(u) = \deg_{A_m}(u)$ for all $u \in V(A_m) \setminus \{m\}$.
- $\deg_H(v) = \deg_G(v)$ for all $v \in V(G) \setminus M(\sigma)$.

Suppose that for some $t \in \mathbb{N}$, σ'_t satisfies the following:

(1) $\sigma'_t(m) = k_m \sigma^m_t(m) + \deg_G(m) + \delta^\sigma_t(m) - a_m$ for all $m \in M(\sigma)$.
(2) $\sigma'_t(u) = \sigma^m_t(u)$ for all $u \in V(A_m) \setminus \{m\}$.
(3) $\sigma'_t(v) = \sigma_t(v)$ for all $v \in V(G) \setminus M(\sigma)$.

We will show that σ'_{t+1} satisfies the above as well. We have $\deg_H(v) = \deg_G(v)$ for all $v \in V(G) \setminus M(\sigma)$, so $F^{\sigma'}_t(v) = F^\sigma_t(v)$. Similarly, $F^{\sigma'}_t(u) = F^{\sigma^m}_t(u)$ for all $u \in V(A_m) \setminus \{m\}$ for some $m \in M(\sigma)$. Finally, for all $m \in M(\sigma)$,

$$\begin{aligned} F^{\sigma^m}_t(m) = 0 \implies & \\ \sigma'_t(m) &\le k_m(\deg_{A_m}(m) - 1) + \deg_G(m) + \delta^\sigma_t(m) - a_m \\ &= k_m \deg_{A_m}(m) + \deg_G(m) + \delta^\sigma_t(m) - b_m - 1 \\ &\le \deg_H(m) - 1, \end{aligned}$$

and

$$
\begin{aligned}
F_t^{\sigma^m}(m) = 1 \implies \\
&\sigma'_t(m) \geq k_m \deg_{A_m}(m) + \deg_G(m) + \delta_t^\sigma(m) - a_m \\
&\qquad \geq \deg_H(m),
\end{aligned}
$$

so $F_t^{\sigma'}(m) = F_t^{\sigma^m}(m) = F_t^\sigma(m)$.

We know $F_t^{\sigma'}(v) = F_t^\sigma(v)$ for all $v \in V(H)$, so clearly $\sigma'_{t+1}(v) = \sigma_{t+1}(v)$ for all $v \in V(G) \setminus M(\sigma)$ and $\sigma'_{t+1}(u) = \sigma^m_{t+1}(u)$ for all $u \in V(A_m) \setminus \{m\}$ for all $m \in M(\sigma)$. We have

$$
\begin{aligned}
\sigma'_{t+1}(m) &= k_m \sigma_t^m(m) + \deg_G(m) + \delta_t^\sigma(m) - a_m \ + \Phi_t^{\sigma'}(v) - F_t^{\sigma'}(v) \deg_H(v) \\
&= k_m \sigma_t^m(m) + \deg_G(m) + \delta_t^\sigma(m) - a_m + \Phi_t^\sigma(v) - F_t^\sigma(v) \deg_G(v) \ + \\
&\qquad k_m \Phi_t^{\sigma^m}(v) - k_m F_t^{\sigma^m}(v) \deg_{A_m}(v) \\
&= k_m(\sigma_t^m(m) + \Phi_t^{\sigma^m}(v) - F_t^{\sigma'}(v) \deg_{A_m}(v)) + \deg_G(m) \ + \\
&\qquad (\delta_t^\sigma(m) + \Phi_t^\sigma(v) - F_t^\sigma(v) \deg_G(v)) - a_m \\
&= k_m \sigma_{t+1}^m(m) + \deg_G(m) + \delta_{t+1}^\sigma(m) - a_m.
\end{aligned}
$$

for all $m \in M(\sigma)$.

We construct σ'_0 such that it satisfies (1), (2), and (3). By induction, for all $t \in \mathbb{N}$, σ'_t also satisfies (1), (2), and (3), which means that $F_t^{\sigma'}(v) = F_t^\sigma(v)$ for all $v \in V(G)$. $\square$

6. Clumpy Firing Patterns

In this section we take 0 and 1 to represent the boolean values *false* and *true*, respectively. If B is a boolean value, then "$B = 1$", "B is true", and simply "B" are equivalent, as are "$B = 0$", "B is false", and "not B".

Recall that a firing pattern is *clumpy* iff it contains two consecutive 0s and two consecutive 1s.

Conjecture 6.1 (Nonclumpiness conjecture)**.** *No vertex can have a clumpy periodic firing pattern in an ordinary parallel chip-firing game.*

In particular, it appears that vertices with clumpy periodic firing patterns need more "support" from some of their neighbors than they are able to supply to their other neighbors. The following lemma demonstrates why this would make clumpy periodic firing patterns impossible.

Lemma 6.2. *Let G be a graph, $P : V(G) \to \{0, 1\}$ be a property, and $Q : V(G)^2 \to \{0, 1\}$ and $R : V(G)^2 \to \{0, 1\}$ be relations such that:*

(1) *For all $\{v, w\} \in E(G)$, $P(v) \wedge P(w) \wedge Q(v, w) \implies R(w, v)$.*
(2) *For all $v \in V(G)$,*

$$P(v) \implies |\{w \in N(v) \mid P(w) \wedge Q(v, w)\}| > |\{w \in N(v) \mid P(w) \wedge R(v, w)\}|.$$

Then $P(v)$ is false for all $v \in V(G)$.

Proof. Let $S = \{v \in V(G) \mid P(v)\}$ and $S(v) = S \cap N(v)$. Consider

$$n = \sum_{v \in S} \sum_{w \in S(v)} (Q(v,w) - R(v,w)).$$

By (1), $n \leq 0$, but by (2), $n > 0$ if S is nonempty, so $P(v)$ is false for all $v \in V(G)$. □

We can think of $Q(v,w)$ as meaning "w helps v satisfy P" and $R(v,w)$ as meaning "w hurts v in satisfying P". From Lemma 6.2 we can prove a special case of the nonclumpiness conjecture.

Theorem 6.3. *Given game σ on G, all $v \in V(G)$ there are no $t, k \in \mathbb{N}$ such that σ_t is periodic, $1 \leq k \leq p(\sigma) - 1$, and $T_0^\sigma(v) \cap [t, t + p(\sigma) - 1] = [t, t + k]$.*

Proof. Without loss of generality, let σ_0 be periodic. Let

$$Z_v(t,k) = \begin{cases} 1 & \text{if } 1 \leq k \leq p(\sigma) - 3 \text{ and } T_0^\sigma(v) \cap [t, t+p(\sigma)-1] = [t, t+k] \\ 0 & \text{otherwise} \end{cases}$$

$$P(v) = \begin{cases} 1 & \text{if there exist } t, k \in \mathbb{N} \text{ such that } Z_v(t,k) \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

$$Q(v,w) = \begin{cases} 1 & \text{if } \sum_{i=0}^{p(\sigma)-1} F_i^\sigma(w)(1 - F_{i+1}^\sigma(v)) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$R(v,w) = \begin{cases} 1 & \text{if } \sum_{i=0}^{p(\sigma)-1} F_i^\sigma(w)(1 - F_{i+1}^\sigma(v)) \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that for some $v, w \in V(G)$, $P(v)$, $P(w)$, and $Q(v,w)$ are true. This is the case iff there exist $t, k \in \mathbb{N}$ such that $Z_v(t+1, k+1)$ and $Z_w(t,k)$ are true. Therefore, $F_{t-1}^\sigma(v) = F_t^\sigma(v) = 1$ and $F_t^\sigma(w) = F_{t+1}^\sigma(v) = 0$, so

$$F_{t-1}^\sigma(v)(1 - F_t^\sigma(w)) + F_t^\sigma(v)(1 - F_{t+1}^\sigma(w)) = 2,$$

which implies $R(w,v)$. (Note σ_{t-1} is well defined as the unique periodic position that leads to σ_t.)

Suppose that $Z_v(t,k)$ is true for some $v \in V(G)$ and $t, k \in \mathbb{N}$. We know

$$\begin{aligned} \deg(v) - 1 &\geq \sigma_{t+k}(v) \\ &= \sigma_{t-1}(v) - F_{t-1}^\sigma(v) \deg(v) + \sum_{i=t-1}^{t+k-1} \Phi_i^\sigma(v) \\ &\geq \sum_{i=t-1}^{t+k-1} \Phi_i^\sigma(v). \end{aligned}$$

Let $n_i = |\{w \in N(v) \mid |T_1^\sigma(w) \cap [t-1, t+k-1]| = i\}|$ for all $i \in \mathbb{N}$. Then

$$\sum_{i=0}^{\infty} n_i - 1 = \deg(v) - 1 \geq \sum_{i=t-1}^{t+k-1} \Phi_i^\sigma(v) = \sum_{i=0}^{\infty} i n_i$$

$$|\{w \in N(v) \mid Q(v,w)\}| = n_0 > \sum_{i=2}^{\infty} (i-1) n_i \geq |\{w \in N(v) \mid R(v,w)\}|.$$

Because $Z_v(t,k) \wedge Q(v,w) \implies Z_w(t-1,k)$ for all $w \in V(G)$, we have

$$|\{w \in N(v) \mid P(w) \wedge Q(v,w)\}| > |\{w \in N(v) \mid P(w) \wedge R(v,w)\}|.$$

By Lemma 6.2, $P(v)$ is false for all $v \in V(G)$. □

7. Interior of Firing and Waiting Sets of Vertices

Let G be a graph and $U \subseteq V(G)$. The set $\{v \in U \mid N(v) \subseteq U\}$ is the *interior* of U.

Conjecture 7.1 (Single interior conjecture)**.** *Given game σ on G with periodic σ_0, for all $t \in \mathbb{N}$ and $f = 0$ or 1 let $W_t^f = \{v \in V(G) \mid F_t^\sigma(v) = f\}$. There exists $f = 0$ or 1 such that the interior of W_t^f is empty for all $t \in \mathbb{N}$.*

Theorem 7.2. *The nonclumpiness conjecture (Conjecture 6.1) is equivalent to the single interior conjecture (Conjecture 7.1).*

Proof. Suppose that the single interior conjecture is false. Then there exist $v_0, v_1 \in V(G)$ and $t_0, t_1 \in \mathbb{N}$ such that

$$\begin{aligned} F_{t_0}^\sigma(v_0) &= 0 & F_{t_1}^\sigma(v_1) &= 1 \\ \Phi_{t_0}^\sigma(v_0) &= 0 & \Phi_{t_1}^\sigma(v_1) &= \deg(v_1). \end{aligned}$$

This means that v_0 will wait twice in a row and v_1 will fire twice in a row. By Lemma 3.1, either v_0 fires at least half the time or v_1 waits at least half the time, so at least one of v_0 and v_1 will both wait twice in a row and fire twice in a row, a counterexample to the nonclumpiness conjecture.

Suppose that the nonclumpiness conjecture is false. Consider a motorized game σ on the two-vertex path graph $G = (\{l, m\}, \{\{l, m\}\})$ with $M(\sigma) = \{m\}$. Suppose further that m has a clumpy periodic firing pattern. For all $t \in \mathbb{N}$ and $f = 0$ or 1, if $F_t^\sigma(m) = F_{t+1}^\sigma(m) = f$ then $F_{t+1}^\sigma(l) = f$, so l is in the interior of either the waiting or firing vertex set. By Theorem 5.3, there is a supergraph of G which preserves the degree of l such that this situation occurs in an ordinary game, which is a counterexample to the single interior conjecture. □

8. Discussion and Directions for Future Work

We have characterized 2-periodic positions and applied the characterization to prove that the total number of chips in a game on a tree determines the eventual period. We have introduced motors, studied motorized games on trees, and shown that motor-like behavior can be constructed in ordinary games. Finally, we have proposed the nonclumpiness conjecture, proven a special case of it, and shown it to be equivalent to another conjecture.

Motors allow the simulation of some aspects of the *dollar game*, a variant of the general chip-firing game discussed by Biggs [4]. In the dollar game, exactly one vertex, the "government", may have a negative number of chips and fires iff no other vertices can fire. We can construct a corresponding motorized parallel chip-firing game in which we replace the government with a motor that waits a sufficiently large number of steps between each firing such that it never fires in the same step as another vertex. Biggs showed that every dollar game tends towards a critical position regardless of the order of vertex firings; therefore, this particular motorized parallel chip-firing game also tends towards the same critical position. Motors may help reveal the extent to which the parallel chip-firing game can simulate additional aspects of the dollar game and other general chip-firing games.

The nonclumpiness conjecture will be the focus of further research. If it remains unproven, it may be helpful to study its implications or study a modified parallel chip-firing game in which clumpy firing patterns are simply disallowed. Corollary 5.2 would likely restrict without loss of generality the graphs necessary to consider when studying such games to those with no leaves.

The primary attraction of the nonclumpiness conjecture is its potential to reduce the game to one of interacting "gliders". For example, consider the situation in Corollary 5.2. (See Figure 4.) Intuitively, we can think of this corollary as stating that whenever a sparse motor attached to a tree fires in a periodic position, it creates a wave of gliders that travels away from the motor. Each glider in this example consists of a head vertex h with $\deg(h)$ chips and a tail vertex with 0 chips. Every non-glider vertex v has $\deg(v)-1$ chips. The fact that the gliders have to be synchronized in terms of their distances from the motor characterizes the possible periodic positions on a motorized tree, provided the motor is sparse. However, when a motor with a clumpy periodic firing pattern is permitted, gliders no longer suffice to describe the dynamics.

There is certainly a duality between firing patterns (which are cross sections of an entire game in space) and positions (which are cross sections in time), if only because the game is deterministic. Theorem 7.1 is perhaps a step towards characterizing that duality; the situation described in the single interior conjecture can be thought of as "spatially clumpy". A specific case of this duality was investigated by Dall'Asta, who showed that every period of length greater than 2 on a cycle can be described by gliders [9]. (See Figure 5.)

It is not clear whether the method employed in Lemma 4.1 to characterize 2-periodic positions will be readily generalizable to characterize n-periodic positions for $n \geq 3$. However, we may be able to make progress by considering periodic positions on motorized trees. It might be possible to "stitch together" more complicated graphs from trees. We hope that motors prove

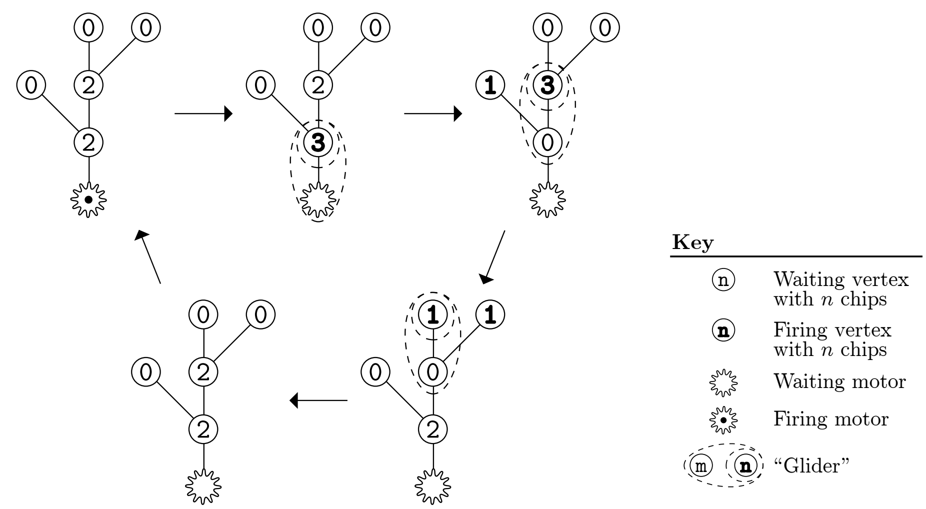

Figure 4. A motor triggers a branching wave of gliders, one of which is shown.

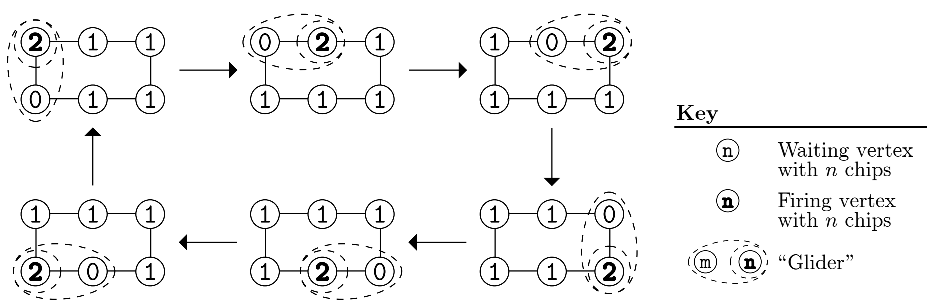

Figure 5. A game on a 6-cycle in which a glider orbits once each period.

to be a useful tool for determining the conditions under which particular period lengths are possible and for characterizing periodic positions.

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