Representations of Cherednik Algebras Associated to Complex Reflection Groups in Positive Characteristic

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January 1, 2012

Abstract

We consider irreducible lowest-weight representations of Cherednik algebras associated to certain classes of complex reflection groups in characteristic p. In particular, we study maximal submodules of Verma modules associated to these algebras. Various results and conjectures are presented concerning generators of these maximal submodules, which are found by computing singular polynomials of Dunkl operators. This work represents progress toward the general problem of determining Hilbert series of irreducible lowest-weight representations of arbitrary Cherednik algebras in characteristic p.

1 Introduction

Double affine Hecke algebras were first introduced by Ivan Cherednik in 1994 to study the Macdonald conjectures, which have since been proven. Cherednik algebras, a certain degeneration of these algebras, were later studied in 2002 by Etingof–Ginzburg in [EG]. Since then, the representation theory of Cherednik algebras has become a topic of study in itself. One of the main problems is to understand the dimensions, and in particular, the Hilbert series, of their lowest-weight irreducible representations. While this problem is very difficult to study in general, much is known about a large number of cases. For example, Hilbert series for lowest-weight irreducible representations of Cherednik algebras associated to S_n in characteristic zero are calculated in [Gor].

However, the positive characteristic case is not as well-studied, though progress has recently been made in the case of rank 1 groups in [Lat], as well the cases of the matrix groups $\mathrm{GL}_n(\mathbf{F}_q)$ and $\mathrm{SL}_n(\mathbf{F}_q)$ in [BC]. Furthermore, the case of the symmetric group S_n in characteristic p with $p > n$ is studied geometrically in [BFG]. The representation theory of Cherednik algebras in positive characteristic is of particular interest for several reasons. For example, the irreducible lowest-weight representations are always finite dimensional, a phenomenon which does not occur in characteristic zero. Also, many of the tools used to study the characteristic zero case do not carry over to positive characteristic, so new techniques are needed. Finally, because the structures of representations of (complex) reflection groups change in characteristic p, the resulting changes in the structures of representations of their associated Cherednik algebras are also of interest.

In this paper, we consider representations of Cherednik algebras associated with complex reflection groups in characteristic p. In Section 3, we consider the algebra associated to the symmetric group S_3 and its trivial representation when our value of the parameter c is taken to be in the field $\mathbf{F}_p$, the only cases in which the Hilbert series of the irreducible lowest-weight representation differs from that of the case when c is generic. Following [CE] and [BO], we show that the irreducible quotient $L_c = M_c/J_c$ is a complete intersection and give generators for J_c for most values of c. In Section 4, we consider the dihedral groups $G(m, m, 2)$, giving a complete answer for one dimensional representations τ as well as certain two-dimensional representations τ when m is odd. Finally, in Section 5, we return to the group S_n and its trivial representation, considering another special case: when $p|n$. Here, we give a recursive algorithm for constructing minimal degree generators of the ideal J_c, and present partial calculations of these generators.

2 Definitions and Previous Results

Definition 2.1. Let $\mathfrak{h}$ be a vector space, and let $s \in \mathbf{GL}(\mathfrak{h})$ be a finite order operator on $\mathfrak{h}$. s is a **reflection** if $\operatorname{rank}(1-s) = 1$. A subgroup $G \subset \mathbf{GL}(\mathfrak{h})$ generated by reflections is a **reflection group**.

Definition 2.2. Let $G \subset \mathbf{GL}(\mathfrak{h})$ be a reflection group where $\mathfrak{h}$ is a vector space over a field K. Let $\mathcal{S}$ be the set of reflections in G. For each $s \in \mathcal{S}$, we pick a vector $\alpha_s \in \mathfrak{h}^*$ that spans the image of $1-s$, where we use the induced action of G on $\mathfrak{h}^*$, and let $\alpha_s^\vee \in \mathfrak{h}$ be defined by the property

$$(1-s)x = (\alpha_s^\vee, x)\alpha_s.$$

Let $\hbar$ be a parameter and c_s be a parameter for each $s \in \mathcal{S}$, where we require that $c_s = c_{s'}$ if s and s' are conjugate. Let $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ be the tensor algebra. The **Cherednik algebra** $H_{\hbar,c}(G, \mathfrak{h})$ is the quotient of $(K[G] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)) \otimes_K K(\hbar, \{c_s\})$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = \hbar(y, x) - \sum_{s \in \mathcal{S}} c_s(y, \alpha_s)(x, \alpha_s^\vee)s$$

where $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$. Here, $(y, x) \in K$ denotes x evaluated at y. and $[,]$ denotes the commutator in $H_{\hbar,c}$.

We take $\hbar = 1$ throughout this paper; note that as long as $\hbar \neq 0$ we have $H_{\hbar,c} \cong H_{1,c/\hbar} \cong H_{1,c}$.

Definition 2.3. Let τ be a representation of G. We let $\operatorname{Sym}(\mathfrak{h})$ act as 0 on τ and construct the **Verma module**

$$M_c(G, \mathfrak{h}, \tau) = H_{\hbar,c}(G, \mathfrak{h}) \otimes_{K[G] \ltimes \operatorname{Sym}(\mathfrak{h})} \tau.$$

These give lowest-weight representations of $H_{\hbar,c}$.

Theorem 2.4 ([EM, 3.12])**.** *Let $\bar{c}\colon S \to K$ be defined so that $\bar{c}_s = c_{s^{-1}}$. There exists a unique contravariant form $\beta_c\colon M_c(G, \mathfrak{h}, \tau) \times M_{\bar{c}}(G, \mathfrak{h}^*, \tau^*) \to K$ satisfying the following properties:*

i. For all $x \in \mathfrak{h}^*$, we have $\beta_c(xf, h) = \beta_c(f, xh)$.

ii. For all $y \in \mathfrak{h}$, we have $\beta_c(f, yh) = \beta_c(yf, h)$.

iii. For all $v \in \tau$ and $w \in \tau^*$, we have $\beta_c(v, w) = w(v)$.

Furthermore, M_c has a unique maximal submodule J_c, which may be realized as the kernel of β.

Definition 2.5. Let $x_{i_1}x_{i_2}\cdots x_{i_m} \otimes v \in \operatorname{Sym}(H^*) \otimes \tau$, where $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are dual bases of H^* and H, respectively. We define the **Dunkl operator** D_{y_i} by

$$D_{y_i}(x) = \frac{\partial}{\partial x_i}(x) \otimes v - \sum_{s \in S} c_s \frac{2\langle y_i, \alpha_s\rangle}{1-\lambda_s} \cdot \frac{x - s(x)}{\alpha_s} \otimes s(v),$$

The Dunkl operator is the main tool in dealing with elements of J_c for the following reason:

Theorem 2.6 ([EM, Proposition 3.2])**.** *In M_c, $y_i x_{i_1} x_{i_2} \cdots x_{i_m} = D_{y_i}(x_{i_1} x_{i_2} \cdots x_{i_m})$, that is, the action of y_i on $Sym(H^*)$ is given by the Dunkl operator.*

Remark 2.7. Note that $\beta(D_k P, Q) = \beta(P, x_k Q)$. Thus, if $D_k P = 0$, we have $P \in J_c$: in this case P is said to be **singular**. More generally, if $D_k P \in J_c$, then $P \in J_c$. However, if P is a minimal degree element of J_c, we must have $D_k P = 0$.

Proposition 2.8. *Given a representation τ of the symmetric group Σ_n, let $h^{\text{gen}}(t)$ be the Hilbert function of $L_c(\tau)$ for c generic. If $c \notin \mathbf{F}_p$, then we have $h^c(t) = h^{\text{gen}}(t)$ where $h^c(t)$ is the Hilbert function of $L_c(\tau)$.*

Proof. Fix a $\mathbf{Z}$-form of τ, i.e., so all $g \in \Sigma_n$ act by integer-valued matrices. Let R be the ring $\mathbf{Z}[c]$ localized at the prime ideal generated by p. Hence R is a principal ideal domain. For generic c, let $M_c(\tau)$ be the corresponding Verma module over the Cherednik algebra (defined over R). Let β_c^d be a matrix representing the contravariant form on $M_c(\tau)$ in degree d. The determinant of β_c^d depends on the R-bases chosen, but the number of times that p divides the determinant does not, so let $o_{R,p}(\beta_c^d)$ be this number. We can characterize $o_{R,p}(\beta_c^d)$ in a different way. For each $i \geq 1$, let $K(i) = \{v \mid \beta_c^d v \text{ is divisible by } p^i\}$. Write $K(i)_p = K(i) \otimes_{\mathbf{Z}} \mathbf{Z}/p$. Then $o_{R,p}(\beta_c^d) = \sum_{i \geq 1} \dim_{\mathbf{F}_p} K(i)_p$, which can be shown by considering a Smith normal form of β_d^c over R.

If c is transcendental over $\mathbf{F}_p$, the there is nothing to prove, so we assume that c is algebraic over $\mathbf{F}_p$. Let $a(x) \in \mathbf{F}_p[x]$ be the minimal polynomial of c over $\mathbf{F}_p$. Lift this to an integer polynomial $\tilde{a}(x) \in \mathbf{Z}[x]$, i.e., the reduction modulo p of $\tilde{a}(x)$ is $a(x)$. Let $\tilde{c}$ be a root of $\tilde{a}(x)$. Then $\mathbf{Z}[x]/(\tilde{a}(x)) \otimes_{\mathbf{Z}} \mathbf{Z}/p$ is a finite extension $\mathbf{F}_q$ of $\mathbf{F}_p$ containing c. Let R' be the localization of $\mathbf{Z}[x]/(\tilde{a}(x))$ at the prime ideal generated by p.

Consider the Verma module $M_{\tilde{c}}(\tau)$ over the Cherednik algebra defined over R'. Let $\beta_{\tilde{c}}^d$ be a matrix representing the contravariant form on $M_{\tilde{c}}(\tau)$ in degree d. We can define $o_{R',p}(\beta_{\tilde{c}}^d)$ and $K'(i)$ as before. Note that

$$\dim_{\mathbf{F}_q} K'(i)_p \geq \dim_{\mathbf{F}_p} K(i)_p \tag{2.9}$$

and saying that the dth coefficient of $h^c(t)$ is strictly less than the dth coefficient of $h^{\text{gen}}(t)$ is equivalent to saying that (2.9) is a strict inequality for $i = 1$, and hence equivalent to the strict inequality $o_{R',p}(\beta_{\tilde{c}}^d) > o_{R,p}(\beta_c^d)$.

So suppose that this inequality holds. The determinant of β_c^d is of the form $N_d \prod_j (a_j c - b_j)$ where $N_d \in \mathbf{Z}$ and $a_j, b_j \in \mathbf{Z}$ are such that $a_j \neq 0$ and $\gcd(a_j, b_j) = 1$ [E, Corollary 3.3], and hence $\det(\beta_{\tilde{c}}^d) = N_d \prod_j (a_j \tilde{c} - b_j)$. Hence there is some j such that p divides $(a_j \tilde{c} - b_j)$. Reducing this relation modulo p, this implies that $c \in \mathbf{F}_p$ since we cannot have both a_j and b_j divisible by p. □

Remark 2.10. The above proof shows that in general, a function c is a special value in characteristic p only if it is the reduction mod p of a special value c coming from characteristic 0. □

3 S_n with special values of c

Throughout this section, we consider $M_c(S_n, \mathfrak{h}, \tau)$ with τ trivial. Let S_n act by permutation of indices on the basis $\{x_1, x_2, \ldots, x_n\}$. Note that reflections in S_n are simply transpositions, which all have the same cycle type, requiring c to be constant. We consider $c \in \mathbf{F}_p$, which are the only values of c that can give different Hilbert series from $L_c(S_n, \mathfrak{h}, \tau)$ where c is taken to be generic, by 2.8.

Theorem 3.1. *When $c = 0$, $h_{L_c}(t) = \left(\dfrac{1 - t^p}{1 - t}\right)^n$ for all n.*

Proof. Because $c = 0$, the Dunkl operators are partial derivatives, and thus kill $x_1^p, \dots, x_n^p$. Letting J_c' be the ideal generated by these elements, we have $h_{M_c/J_c'}(t) = \left(\frac{1-t^p}{1-t}\right)^n$. Now, it is clear by induction that for any monomial X with degree less than p in each of the x_i, we have $\beta(X, X) \neq 0$, so that the coefficient on the t^d term of $h_{L_c}(t)$ is at least that of $\left(\frac{1-t^p}{1-t}\right)^n = h_{M_c/J_c'}(t)$. However, noting that $J_c' \subseteq J_c$, we immediately obtain $h_{L_c}(t) = \left(\frac{1-t^p}{1-t}\right)^n$ (and $J_c' = J_c$). □

Theorem 3.2. *When $c = 1/n$, we get $h_{L_c}(t) = \frac{1-t^p}{1-t}$.*

Proof. Note that $D_i(x_i) = 1 - c(n-1) = c$ and $D_i(x_j) = c$ whenever $i \neq j$, so $x_1 - x_j \in J_c$ for $j = 2, 3, \dots, n$. Furthermore, $x_1^p \in J_c$. Also, for $0 \leq d \leq p-1$, we may check by induction that $\beta(x_1^d, x_1^d) = 0$. It follows that for $d < p$, the d-th graded component of L_c is spanned by x_i^d, and that for $d \geq p$ the component is trivial, implying the result. □

We now turn our attention to the case in which $n = 3$, which is of particular interest.

Theorem 3.3. *When $n = 3$ and $p > 3$, express c as a positive integer with $0 < c < p$. In the following three cases, M_c/J_c is a complete intersection, where the degrees of the generators of J_c are noted below:*

1. $0 < c < p/3$: $p, p+3c, p+3c$
2. $p/3 < c < p/2$: $3c-p, 3c-p, p$
3. $2p/3 < c < p$: $p-3c, p-3c, p$

Proof. Consider the polynomial $P(t) = (1-tx_1)^{c'}(1-tx_2)^{c'}(1-tx_3)^{c'}$, and let G be the coefficient of the $t^{3c'+1}$ term of P. Then, we first show that $\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_3}$ are killed by Dunkl operators in characteristic zero where we take $c = c'$. We will assume $3c'$, but c' is not an integer (in particular $c' \neq 0$): the Taylor series of each factor modulo p will be defined since $p > 3$.

Without loss of generality, consider $\frac{\partial G}{\partial x_1}$. Since partial derivatives with respect to the x_i do nothing to t, it suffices to check that the $t^{3c'+1}$ coefficient of $\frac{\partial P}{\partial x_1}$ is killed by Dunkl operators. We have

$$\frac{\partial P}{\partial x_1} = -c't(1-tx_1)^{c'-1}(1-tx_2)^{c'}(1-tx_3)^{c'}.$$

Now,

$$\begin{aligned}\frac{1}{-c't}D_1\frac{\partial G}{\partial x_1} =& -t(c'-1)(1-tx_1)^{c'-2}(1-tx_2)^{c'}(1-tx_3)^{c'}\\ &- tc'(1-tx_1)^{c'-1}(1-tx_2)^{c'-1}(1-tx_3)^{c'} - tc'(1-tx_1)^{c'-1}(1-tx_2)^{c'}(1-tx_3)^{c'-1}.\end{aligned}$$

Dropping another factor of the indeterminate t and changing sign, we wish to check that the $t^{3c'-1}$ coefficient of

$$\begin{gathered}(c'-1)(1-tx_1)^{c'-2}(1-tx_2)^{c'}(1-tx_3)^{c'}\\ +c'(1-tx_1)^{c'-1}(1-tx_2)^{c'-1}(1-tx_3)^{c'} + c'(1-tx_1)^{c'-1}(1-tx_2)^{c'}(1-tx_3)^{c'-1}\end{gathered}$$

is zero. Consider the coefficient of $x_1^i x_2^j x_3^k t^{3c'-1}$, where we note that we must have $i+j+k=3c'-1$ (otherwise this coefficient is trivially zero). Up to a sign, this is

$$(c'-1)\binom{c'-2}{i}\binom{c'}{j}\binom{c'}{k}+c'\binom{c'-1}{i}\binom{c'-1}{j}\binom{c'}{k}+c'\binom{c'-1}{i}\binom{c'}{j}\binom{c'-1}{k},$$

which, upon multiplication by the appropriate factors becomes simply

$$(c'-1-i)+(c'-j)+(c'-k)=0.$$

Now, in characteristic p, we construct singular polynomials of the desired degrees in the three cases listed above. In Case 1, consider $\frac{\partial G}{\partial x_1}$ and $\frac{\partial G}{\partial x_2}$, with $c'=c+p/3$, and in the other two cases, take the same polynomials with $c'=c-p/3$; it is trivial to check that these give the correct degrees. Scale these polynomials by the appropriate rational factor so that they are non-zero in characteristic p: it is now clear that the resulting polynomials G_1, G_2 are singular in characteristic p. Furthermore, let $G_3 = x_1^p + x_2^p + x_3^p$, which is both killed by partial derivatives and an S_3-invariant, so also singular.

Note that in each case, G_1, G_2, G_3 are linearly independent. Let J_c' be the ideal that they generate. Then, we see that, using the fact that M_c/J_c' is a complete intersection, $h_{M_c/J_c'}(t)$ agrees with $h_{L_c}(t)$ (as computed in [BO]), indeed we must have $J_c' = J_c$. This completes the proof. □

Remark 3.4. We conjecture that when $p/2 < c < 2p/3$, M_c/J_c is again a complete intersection, with generators of degrees $6c-3p, p, p$. It is easy to check that the degree $6c-3p$ generator is $(x_1-x_2)^{2c-p}(x_2-x_3)^{2c-p}(x_3-x_1)^{2c-p}$ and that one of the degree p generators is $x_1^p+x_2^p+x_3^p$. Furthermore, it is known that in the case of $c=1/2$, the second degree p generator is $\sum_{\text{sym}} \frac{x_1^p(x_1-x_2)}{x_1-x_3}$; however the form of this third generator is unclear in general.

Conjecture 3.5. *Consider $J_c(S_n, \mathfrak{h}, \tau)$ with τ trivial and n arbitrary, and $p > n$. Consider $c \in \mathbf{F}_p$ expressed as an integer with $1 \le c \le p-1$. Consider the set S of distinct rationals of the form $\frac{ap}{b}$ with integers a, b satisfying $0 \le a \le b < p$. Suppose that there is no element of S between c and $c+1$. Then, the sum of the degrees of the elements of a minimal set of generators of $J_{c+1}(S_n, \mathfrak{h}, \tau)$ is exactly $n!$ more than the analogous sum for $J_c(S_n, \mathfrak{h}, \tau)$.*

Remark 3.6. The above has been verified for $n=3$ and, in a small number of cases, $n=4$.

Conjecture 3.7. *$M_c(S_n, \mathfrak{h}, \tau)/J_c(S_n, \mathfrak{h}, \tau)$ is a Gorenstein algebra for all n whenever τ is trivial, both when $c \in \mathbf{F}_p$ and $c \notin \mathbf{F}_p$. More generally, for all τ, $h_{L_c(S_n,\mathfrak{h},\tau)}(t)$ is palindromic.*

Remark 3.8. In general, when τ is trivial, we do not expect M_c/L_c to be a complete intersection. However, in the case that $n=3$, because S_3 has rank 2, the above conjecture immediately implies that M_c/J_c is a complete intersection.

4 Dihedral Groups $G(m, m, 2)$

In this section, we construct singular polynomials of Dunkl operators for the rank 2 dihedral group $D_m = G(m, m, 2)$. Observe that we must have $p \nmid m$. Suppose our field K contains all mth roots of unity, and let ζ be a fixed primitive m-th root of unity. The reflections in D_m may be realized as acting by the 2×2 matrices

$$\begin{bmatrix} & \zeta^{-k} \\ \zeta^k & \end{bmatrix}$$

for $0 \leq k < m$. When m is odd, all such reflections lie in the same conjugacy class. However, when m is even, there are two conjugacy classes or reflections, given by those with i odd and i even.

Theorem 4.1. *When τ is trivial and c is generic, $x_1^p x_2^p, x_1^{pm} + x_2^{pm} \in J_c$.*

Proof. We re-write these polynomials as $(x_1x_2)^p$ and $(x_1^m + x_2^m)^p$. Both are killed by partial derivatives, and it is trivial to check that x_1x_2 and $x_1^m + x_2^m$ are D_m-invariants, so these polynomials are both singular. □

We now consider τ with $\dim(\tau) > 1$: otherwise $L_c(\tau) \cong L_c(\text{triv})$. The irreducible representations of D_m are two-dimensional, and are denoted ρ_a, $1 \leq a < m/2$, where the action of the reflections is given by

$$\begin{bmatrix} & \zeta^{-k} \\ \zeta^k & \end{bmatrix} \mapsto \begin{bmatrix} & \zeta^{-ak} \\ \zeta^{ak} & \end{bmatrix}$$

Let $\{e_1, e_2\}$ be a basis for τ.

Theorem 4.2. *Let $\tau = \rho_a$, with $a > p$, and suppose m is odd. Then, $x_1^p \otimes e_1, x_1^p \otimes e_2, x_2^p \otimes e_1, x_2^p \otimes e_2 \in J_c$.*

Proof. The partial derivatives of these vectors are zero, so the Dunkl operator D_1 acts by

$$-\sum_{k=0}^{m-1} c_s \frac{1}{x_1 - \zeta^k x_2}(1-s) \otimes s.$$

Since m is odd, c_s is a constant c. We now compute

$$\begin{aligned} D_1(x_1^p \otimes e_1) &= -\sum_{k=0}^{m-1} c \frac{1}{x_1 - \zeta^k x_2}(x_1^p - (\zeta^k x_2)^p) \otimes \zeta^{ak} e_2 \\ &= -\sum_{k=0}^{m-1} c\zeta^{ak} \left(\sum_{\ell=0}^{p-1} \zeta^{\ell k} x_1^{p-1-\ell} x_2^{\ell} \right) \otimes e_2. \end{aligned}$$

Consider the $x_1^{p-1-\ell} x_2^{\ell}$ coefficient in the first component: we wish to show that it is zero. This coefficient is

$$-\sum_{k=0}^{m-1} c\zeta^{ak} \cdot \zeta^{k\ell} = -\sum_{k=0}^{m-1} c(\zeta^{a+\ell})^k,$$

Note that $0 < a + \ell < a + p < a + a < m$. Thus, $m \nmid (a+\ell)$, and $\zeta^{(a+\ell)} \neq 1$. Our sum is thus the sum of d-th roots of unity for some $d > 1$, where $d|m$, each d-th root of unity appearing m/d times; the sum is thus equal to zero. In a similar way, we compute

$$D_1(x_1^p \otimes e_2) = -\sum_{k=0}^{m-1} c\zeta^{-ak} \left(\sum_{\ell=0}^{p-1} \zeta^{k\ell} x_1^{p-1-\ell} x_2^{\ell} \right) \otimes e_1,$$

and we wish to check that

$$\sum_{k=0}^{m-1} c(\zeta^{-a+\ell})^k = 0.$$

Again, we have $\zeta^{-a+\ell} \neq 1$ because $-a+\ell < -p+\ell < 0$, and furthermore it is clear that $|-a+\ell| \leq a+\ell < m$. Next,

$$D_1(x_2^p \otimes e_1) = -\sum_{k=0}^{m-1} c \frac{-\zeta^{-k}}{x_2 - \zeta^{-k}x_1}(x_2^p - (\zeta^{-k}x_1)^p) \otimes \zeta^{ak} e_2$$
$$= \sum_{k=0}^{m-1} c\zeta^{ak-k} \left(\sum_{\ell=0}^{p-1} \zeta^{-k\ell} x_2^{p-1-\ell} x_1^{\ell} \right) \otimes e_2,$$

and we need to check that

$$\sum_{k=0}^{m-1} c(\zeta^{a-\ell-1})^k = 0,$$

which follows from the fact that $m > a > a-\ell-1 > p-(p-1)-1 = 0$. Finally,

$$D_1(x_2^p \otimes e_2) = 0$$

is equivalent to

$$\sum_{k=0}^{m-1} c(\zeta^{-a-\ell-1})^k = 0,$$

which in turn follows from $0 > -a-\ell-1 \geq -a-(p-1)-1 > -2a > -m$. In exactly the same way, D_2 kills all four vectors, so we're done. □

Remark 4.3. The above proof fails for $a = p$ because in the third case, we can have $a-\ell-1 = p-(p-1)-1 = 0$.

Theorem 4.4. *Let $\tau = \rho_p$, and suppose m is odd. Then, $x_1^p \otimes e_1, x_2^p \otimes e_2, x_1^{3p} \otimes e_2, x_2^{3p} \otimes e_1 \in J_c$.*

Proof. We can check that the first two generators are killed by Dunkl operators using the same logic as in the previous theorem, noting that in these two cases the necessary strict inequalities still hold. To check that the other two vectors are in J_c, it is enough to show that that applying Dunkl operators gives multiples of $x_1^p \otimes e_2, x_2^p \otimes e_1 \in J_c$.

We see that

$$D_1(x_1^{3p} \otimes e_2) = -\sum_{k=0}^{m-1} c\zeta^{-pk} \left(\sum_{\ell=0}^{3p-1} \zeta^{k\ell} x_1^{3p-1-\ell} x_2^{\ell} \right) \otimes e_1,$$

and the coefficient on the $x_1^{3p-1-\ell} x_2^{\ell}$ term in the first component is

$$\sum_{k=0}^{m-1} c(\zeta^{-p+\ell})^k.$$

We claim that $D_1(x_1^{3p} \otimes e_2)$ is a multiple of $x_1^p \otimes e_1 \in J_c$. To check this, it suffices to show that if the above coefficient is non-zero, then $3p-1-\ell \geq p$. Clearly, $-p+\ell \geq -p > -m$, and also $-p+l \leq -p+3p-1 = 2p-1 < m$. Now, if the coefficient on $x_1^{3p-1-\ell} x_2^{\ell}$ is non-zero, we have $\zeta^{-p+\ell} = 1$, and $m|(-p+\ell)$. Therefore, we must have $\ell = p$, and indeed $3p-1-l = 2p-1 \geq p$. It follows that $x_1^{3p} \otimes e_2 \in J_c$ because $x_1^p \otimes e_1 \in J_c$.

Finally, we compute

$$D_2(x_2^{3p}\otimes e_1) = \sum_{k=0}^{m-1} c\zeta^{pk-k}\left(\sum_{\ell=0}^{3p-1}\zeta^{-k\ell}x_2^{3p-1-\ell}x_1^{\ell}\right)\otimes e_2,$$

and the $x_2^{3p-1-\ell}x_1^{\ell}$ coefficient is

$$\sum_{k=0}^{m-1} c(\zeta^{p-\ell-1})^k.$$

We claim that $D_2(x_2^{3p}\otimes e_1)$ is a multiple of $x_2^p\otimes e_2 \in J_c$. Note that $p-\ell-1 \geq p-(3p-1)-1 = -2p > -m$, and $p-\ell-1 \leq p-1 < m$. Thus, if our coefficient is non-zero, we must have $\zeta^{p-\ell-1}=1$ and $\ell = p-1$, so that $3p-1-\ell = 2p > p$. It follows that $x_2^{3p}\otimes e_1 \in J_c$ because $x_2^p \otimes e_2 \in J_c$, and the proof is complete. □

Remark 4.5. Assuming that J_c is indeed generated by the aforemetioned singular vectors in each of the three cases below (which we conjecture to be the case based on computational evidence), we get the following Hilbert series:

$$\tau \text{ trivial: } h_{L_c}(t) = \left(\frac{1-t^p}{1-t}\right)\left(\frac{1-t^{pm}}{1-t}\right)$$

$$\tau = \rho_a, a > p : h_{L_c}(t) = 2\left(\frac{1-t^p}{1-t}\right)^2$$

$$\tau = \rho_p : h_{L_c}(t) = 2\left(\frac{1-t^p}{1-t}\right)\left(\frac{1-t^{3p}}{1-t}\right)$$

5 S_n with $p|n$ and c generic

Theorem 5.1. *Consider $G = S_n$, where n is even, with τ trivial and c generic, and $p = 2$. Then, M_c/J_c is a complete intersection with J_c generated by $n-1$ elements of degree 2 and one of degree 4. Thus, $h_{L_c}(t) = (1+t)^n(1+t^2)$.*

Proof. Write $J = J_c$. Given $i < j$, define $f_{ij} = c(x_i + x_j)(\sum_k x_k) + x_i^2 + x_j^2$ and let $g = \sum_i x_i^2$. Let I be the ideal generated by $\{g, f_{1,2}, \dots, f_{1,n-1}, x_1^4\}$. We claim that $J = I$.

We first check that f_{ij}, g are killed by Dunkl operators, so that $g \in J$ and $f_{i,j} \in J$ for all i, j. Clearly, g is killed by Dunkl operators, since all of its partial derivatives are zero and it is a S_n-invariant. Now, consider D_1 applied to the $f_{i,j}$. When $i, j > 1$, we have that $D_1 f_{i,j}$ is equal to

$$c(x_i+x_j) - c\left(\frac{1}{x_1-x_i}(c(x_i-x_1)\left(\sum_k x_k\right) + x_i^2 - x_1^2) + \frac{1}{x_1-x_j}(c(x_j-x_1)\left(\sum_k x_k\right) + x_j^2 - x_1^2)\right),$$

which we see vanishes in characteristic 2. Also

$$D_1 f_{1,j} = c(x_1+x_j) + c\left(\sum_k x_k\right) - c\sum_{\ell\neq 1,j}\frac{1}{x_1-x_l}\left(c\left(\sum_k x_k\right)(x_1-x_\ell) + x_\ell^2 - x_1^2\right) = 0,$$

since n is even, so it follows that the $f_{i,j}$ are singular. Note that $\{g, f_{i,j}\}$ is linearly dependent and one possible basis is $S = \{g, f_{1,2}, \dots, f_{1,n-1}\}$. Linear independence of S follows immediately from noting that for $i = 2, \dots, n-1$, the only appearance of an x_i^2 term in a linear combination of the elements of S is in $f_{1,i}$, and the only appearance of an x_n^2 term is in g. To check that S spans our ideal, note that $(c+1)g + f_{1,2} + \cdots + f_{1,n-1} = f_{1,n}$ and $f_{1,i} + f_{1,j} = f_{i,j}$.

Now, note that $(x_i + x_j)^3 \in J$, since

$$(x_i + x_j)^3 = c^2(x_i + x_j)g + (x_i + x_j + c(\textstyle\sum_k x_k))f_{i,j}$$

Also, when $k \neq 1$,

$$D_k x_1^4 = -c\frac{x_1^4 - x_k^4}{x_k - x_1} = c(x_1 + x_k)^3,$$

and furthermore

$$D_1 x_1^4 = c\sum_{k\neq 1}\frac{x_1^4 - x_k^4}{x_1 - x_k} = c\sum_{k\neq 1}(x_1 + x_k)^3,$$

so it follows that $x_1^4 \in J$. So we have shown that $I \subseteq J$. Furthermore, we see that $x_i^4 \in I$ for all i since we have

$$x_1^4 + x_i^4 = (x_1 + x_i)(x_1 + x_i)^3.$$

This implies that A/I is a finite-dimensional vector space over $\mathbf{F}_2(c)$: for example, any monomial of degree at least $3n+1$ must have some variable x_i raised to at least the 4th power, so it is divisible by i and must belong to I. Hence A/I can only be nonzero for degrees at most $3n$. Since A has Krull dimension n, and I has n generators, we conclude that I is a complete intersection, and from the degrees, its Hilbert series is $h_{A/I}(t) = (t+1)^n(t^2+1)$. Since $I \subseteq J$, we know that $h_{A/I}(t) \geq h_{A/J}(t)$ coefficientwise. By [BC, Proposition 3.3], $h_{A/J}(t) = (t+1)^n h(t^2)$ for some h. So the only possibilities are $h = 1 + t$, in which case $I = J$, or $h(t) = 1$ (the case $h(t) = t$ is not allowed since $h_{A/J}(0) = 1$). If $h(t) = 1$, then J_2 contains n linearly independent polynomials.

To finish, it suffices to check that J only contains $n-1$ linearly independent polynomials of degree 2. Suppose not. When $c = 0$, J_2 is spanned by $\{x_1^2, \dots, x_n^2\}$. By considering the limit $c \to 0$, we see that J contains a generator of the form $\phi = \sum_i \alpha_i(c)x_i^2 + c(\sum_{i\neq j} a_{i,j}(c)x_i x_j)$ where $\alpha_i(c)$ and $a_{i,j}(c)$ are polynomials in c with $\alpha_1(0) = 1$ and $\alpha_j(0) = 0$ for $j > 1$, and we take $a_{i,j}(c) = a_{j,i}(c)$. However, note that

$$\begin{aligned}
D_1\phi &= c\sum_{j\neq 1}\alpha_{1,j}x_j - c\sum_{j\neq 1}\frac{1}{x_1 - x_j}\left((\alpha_1 - \alpha_j)(x_1^2 - x_j^2) + c\sum_{\ell\neq 1,j}(\alpha_{1,\ell} - \alpha_{\ell,j})x_\ell(x_1 - x_j)\right) \\
&= c\sum_{j\neq 1}\left(\alpha_{1,j}x_j - (\alpha_1 - \alpha_j)(x_1 + x_j) - c\sum_{\ell\neq 1,j}(\alpha_{1,\ell} - \alpha_{\ell,j})x_\ell\right) \\
&= c\left(\sum_{j\neq 1}\left(\alpha_{1,j} + \alpha_1 + \alpha_j + c\sum_{\ell\neq 1,j}\alpha_{\ell,j}\right)x_j + \left(\sum_j \alpha_j\right)x_1\right).
\end{aligned}$$

If $\phi \in J$, we have that the above is identically zero, and since c is indeterminate,

$$\sum_{j\neq 1}\left(\alpha_{1,j} + \alpha_1 + \alpha_j + c\sum_{\ell\neq 1,j}\alpha_{\ell,j}\right)x_j + \left(\sum_j \alpha_j\right)x_1$$

must also be identically zero. However, the coefficient on the x_1 term evaluates to 1 when $c = 0$, since $\alpha_1(0) = 1$ and $a_j(0) = 0$ for $j > 1$. Therefore, ϕ is not singular, and we have reached a contradiction. This completes the proof. □

The conjectured generalization of the preceding theorem is the following:

Conjecture 5.2. *Consider $G = S_n$, where $p|n$, with τ trivial and c generic. Then, M_c/J_c is a complete intersection with J_c generated by $n-1$ elements of degree p and one of degree p^2.*

We may write the generators of degree p as $F = F_0 + cF_1 + c^2F_2 + \cdots$, where the F_i are degree p polynomials individually killed by the Dunkl operators $D_k = \partial_k - cB_k$, where ∂_k denotes partial differentiation, and we take F to be in the polynomial ring of the x_i with the coefficient field being Laurent series in c. Applying Dunkl operators to F and setting the result equal to 0, we get:

$$\partial_k F_0 = 0$$
$$\partial_k F_m = B_k F_{m-1}$$

The first relation gives us that F_0 must be of the form $F_0 = \sum_{i=1}^{n} a_i x_i^p$, where $a_1, a_2, \ldots, a_n \in \mathbf{F}_p$.

Remark 5.3. Note that given F_{m-1}, there exists F_m such that $\partial_k F_m = B_k F_{m-1}$ for $k = 1, 2, \ldots, n$ if and only if $\partial_i B_k F_{m-1} = \partial_k B_i F_{m-1}$ for all $i = 1, 2, \ldots, n$ (to ensure equality of mixed partial derivatives).

Lemma 5.4. *J_c contains no generators of degree less than p.*

Proof. Consider the lowest-degree generators, which must be killed by Dunkl operators. We may still write a generator F in the form $F = F_0 + cF_1 + \cdots$ satisfying the same relations as above; in particular, $\partial_k F_0 = 0$ for all k. However, this is impossible unless $\deg(F)$ is a multiple of p. □

Lemma 5.5. $\sum_{i=1}^{n} a_i = 0$.

Proof. We have

$$\begin{aligned} B_1 F_0 &= \sum_{j=2}^{n} \frac{1}{x_1 - x_j}(a_1 x_1^p + a_j x_j^p - a_1 x_j^p - a_j x_1^p) \\ &= \sum_{j=2}^{n} \frac{(a_1 - a_j)(x_1^p - x_j^p)}{x_1 - x_j} \\ &= \sum_{j=2}^{n} \sum_{r=0}^{p-1} (a_1 - a_j) x_1^r x_j^{p-1-r} = \partial_1 F_1. \end{aligned}$$

However, note that in characteristic p, $\partial_1 F_1$ cannot have terms of the form $a x_1^{p-1}$. Thus,

$$\sum_{j=2}^{n} (a_1 - a_j) = -\sum_{j=1}^{n} a_j = 0 \Rightarrow \sum_{j=i}^{n} a_i = 0.$$ □

Conjecture 5.6. *Given* $F_0 = \sum_{i=1}^{n} a_i x_i^p$ *with* $\sum_{j=i}^{n} a_i = 0$, *there exist* $F_1, F_2, \dots$ *such that* $\partial_k F_m = B_k F_{m-1}$ *for all positive integers* m. *Furthermore, at each step in the recursion, each* F_m *is unique up to adding* p*-th powers, and the set of all possible* $F = \sum_{i=0}^{\infty} c^i F_i$ *forms an* $\mathbf{F}_p$*-vector space of dimension* $p-1$.

We now prove a few parts of this conjecture.

Proposition 5.7. *If, given* F_{m-1}, *there exists* F_m *for which* $\partial_k F_m = B_k F_{m-1}$, *then* F_m *is unique up to adding* p*-th powers.*

Proof. Suppose there exists some F'_m satisfying $\partial_k F'_m = B_k F_{m-1}$. Then, all partials of $F_m - F'_m$ vanish, so $F_m - F'_m$ must be the sum of p-th powers. □

Proposition 5.8. *Assume that if we take each* F_i *with* $i > 0$ *to include no* p*-th powers, we can construct* $F_1, F_2, \dots$. *Then, the space of all possible generators* F *of degree* p *has dimension* $n-1$.

Proof. Let $F(a_1, a_2, \dots, a_n) \in J_c$, where $\sum_{i=1}^{n} a_i = 0$, be the generator obtained by taking $F_0 = \sum_{i=1}^{n} a_i x_i^p$ (assuming the first part of 5.6). The space of F_0 is isomorphic to the space of $\mathbf{F}_p$-vectors whose components sum to zero, which has dimension $n-1$. It suffices to show that the $F(a_1, a_2, \dots, a_n)$ span the space of degree p generators. Suppose that we have a generator F in which we add a sum of p-th powers $F'_0 = \sum_{i=1}^{n} b_i x_i^p$ to F_k, and $k \geq 1$ is minimal. Letting $F_0 = \sum_{i=1}^{n} a_i x_i^p$, note that $F' = F - F(a_1, a_2, \dots, a_n) \in J_c$. Furthermore, $F' c^{-k} \in J_c$, and F' has the form $\sum_{i=0}^{\infty} b^i F'_i$, and we may iterate the argument on F'. It follows that F is a linear combination of the $F(a_1, a_2, \dots, a_n)$, where we may take our weights to be appropriate powers of c. □

Remark 5.9. We have now reduced 5.6 to proving the existence of the F_i, provided that we include no p-th powers in any of $F_1, F_2, \dots$. Furthermore, if we can make this construction in the field of Laurent series, the series can be replaced by rational functions since the number and degrees of the generators of J doesn't change if we enlarge the field from rational functions to Laurent series. This follows from the fact that the matrices for β have entries in the field of rational functions in c, and the rank of a matrix doesn't change if the field of coefficients is enlarged.

Proposition 5.10. *Take* $p \neq 2$. *If, at the respective steps, we include no* p*-th powers, we have*

$$F_1 = -\sum_{1 \leq i < j \leq n} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{r a_i + s a_j}{rs} x_i^r x_j^s$$

and

$$F_2 = \sum_{i<j<k} \sum_{\substack{r,s,t>0 \\ r+s+t=p}} \left(\frac{r a_i + s a_j + t a_k}{rst} \right) x_i^r x_j^s x_k^t + \sum_{i<j} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{a_i - a_j}{r} \left(\frac{1}{s} - 2 \sum_{d=1}^{s} \frac{1}{d} \right) x_i^r x_j^s.$$

Proof. We already have

$$\partial_1 F_1 = \sum_{r=0}^{p-1} (a_1 - a_j) x_1^r x_j^{p-1-r},$$

and similar relations hold for the other partial derivatives. We find that the system of differential equations is indeed satisfied by

$$F_1 = -\sum_{1 \le i < j \le n} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{ra_i + sa_j}{rs} x_i^r x_j^s,$$

and it is clear that F_1 must be unique (since it is homogeneous). Continuing,

$$\begin{aligned}
B_1 F_1 &= \sum_{j=2}^{n} \frac{1}{x_1 - x_j} \left(-\sum_{1 \le i < k \le n} \sum_{\substack{r,s>0 \\ r+s=p}} (1 - s_{ij}) \frac{ra_i + sa_k}{rs} x_i^r x_k^s \right) \\
&= -\sum_{j=2}^{n} \frac{1}{x_1 - x_j} (1 - s_{ij}) \left(\sum_{k \ne 1,j} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{ra_1 + sa_k}{rs} x_1^r x_k^s + \sum_{k \ne 1,j} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{ra_j + sa_k}{rs} x_j^r x_k^s \right) \\
&\quad - \sum_{j=2}^{n} \frac{1}{x_1 - x_j} (1 - s_{ij}) \left(\sum_{\substack{r,s>0 \\ r+s=p}} \frac{ra_1 + sa_j}{rs} x_1^r x_j^s \right).
\end{aligned}$$

Consider the first term. We have

$$\begin{aligned}
&-\sum_{j=2}^{n} \frac{1}{x_1 - x_j} (1 - s_{ij}) \left(\sum_{k \ne 1,j} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{ra_1 + sa_k}{rs} x_1^r x_k^s + \sum_{k \ne 1,j} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{ra_j + sa_k}{rs} x_j^r x_k^s \right) \\
&= \sum_{j=2}^{n} \left(-\sum_{k \ne 1,j} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{ra_1 + sa_k}{rs} x_k^s \cdot \frac{x_1^r - x_j^r}{x_1 - x_j} + \sum_{k \ne 1,j} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{ra_j + sa_k}{rs} x_k^s \cdot \frac{x_1^r - x_j^r}{x_1 - x_j} \right) \\
&= \sum_{j \ne 1} \left(-\sum_{k \ne 1,j} \sum_{\substack{s>0 \\ a+b+s=p-1}} \frac{(a+b+1)a_1 + sa_k}{(a+b+1)s} x_k^s x_1^a x_j^b + \sum_{k \ne 1,j} \sum_{\substack{s>0 \\ a+b+s=p-1}} \frac{(a+b+1)a_j + sa_k}{(a+b+1)s} x_k^s x_1^a x_j^b \right) \\
&= -\sum_{j \ne 1} \sum_{k \ne 1,j} \sum_{\substack{s>0 \\ a+b+s=p-1}} \frac{a_1 - a_j}{s} x_k^s x_1^a x_j^b \\
&= -\sum_{1<j<k} \sum_{\substack{s,t>0 \\ r+s+t=p-1}} \left(\frac{a_1 - a_j}{t} + \frac{a_1 - a_k}{s} \right) x_1^r x_j^s x_k^t - \sum_{j \ne 1} \sum_{k \ne 1,j} \sum_{\substack{t>0 \\ r+t=p-1}} \frac{a_1 - a_j}{t} x_1^r x_k^t \\
&= \sum_{1<j<k} \sum_{\substack{s,t>0 \\ r+s+t=p-1}} \left(\frac{(r+1)a_1 + sa_j + ta_k}{st} \right) x_1^r x_j^s x_k^t + \sum_{k \ne 1} \sum_{\substack{t>0 \\ r+t=p-1}} \frac{a_1 - a_k}{t} x_1^r x_k^t.
\end{aligned}$$

Now, for the other term (in the last step we re-index from j to k for convenience),

$$-\sum_{j=2}^{n} \frac{1}{x_1 - x_j}(1 - s_{ij}) \left(\sum_{\substack{r,s>0 \\ r+s=p}} x_1^r x_j^s \right) = -2 \sum_{k=2}^{n} \sum_{\substack{r>s>0 \\ r+s=p}} \frac{ra_1 + sa_k}{rs} \cdot \frac{x_1^r x_k^s - x_1^s x_k^r}{x_1 - x_k}.$$

We calculate the coefficient on the term $x_1^b x_k^d$ where $b + d = p - 1$, upon combining these two parts. The first part gives a coefficient of $(a_1 - a_k) \cdot \frac{1}{d}$. For the second part, we get a contribution of $-2 \cdot \frac{ra_1 + sa_k}{rs} = -2 \cdot \frac{a_1 - a_k}{s}$ if and only if $b, d \geq s$, which happens for $s = 1, 2, \ldots, \min(b, d)$. Our coefficient is thus

$$(a_1 - a_k) \left(\frac{1}{d} - 2 \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\min(b, d)} \right) \right).$$

When p is odd, we see that in fact both values of $\min(b, d)$ give the same value since $d = p - (b+1)$. Thus (after re-indexing again), we have

$$B_k F_1 =$$

$$\sum_{1<j<k} \sum_{\substack{s,t>0 \\ r+s+t=p-1}} \left(\frac{(r+1)a_1 + sa_j + ta_k}{st}) x_1^r x_j^s x_k^t + \sum_{j \neq 1} \sum_{\substack{s>0 \\ r+s=p-1}} (a_1 - a_j) \right) \left(\frac{1}{s} - 2 \sum_{i=1}^{s} \frac{1}{i} \right) x_1^r x_j^s,$$

so taking a x_1-antiderivative and a (modified, as before) symmetric sum gives

$$F_2 = \sum_{i<j<k} \sum_{\substack{r,s,t>0 \\ r+s+t=p}} \left(\frac{ra_i + sa_j + ta_k}{rst} \right) x_i^r x_j^s x_k^t + \sum_{i<j} \sum_{\substack{r,s>0 \\ r+s=p}} \frac{a_i - a_j}{r} \left(\frac{1}{s} - 2 \sum_{d=1}^{s} \frac{1}{d} \right) x_i^r x_j^s,$$

as desired. □

Proposition 5.11. *When $p = 3$, we can take*

$$F = \sum_i a_i x_i^3 - c \sum_{i,j} (a_i - a_j) x_i^2 x_j + c^2 \left(\sum_{i<j<k} (a_i + a_j + a_k) x_i x_j x_k + \sum_{i,j} (a_i - a_j) x_i^2 x_j - \sum_i a_i x_i^3 \right).$$

Proof. The values of F_0, F_1 agree with what has already been checked. However, in F_2, note that we add a sum of p-th powers, namely, $-\sum_i a_i x_i^3$: it can be checked directly here that $B_k F_2 = 0$ for all k. □

Remark 5.12. While (we conjecture that) we could have taken F_2 to contain no p-th powers and continued the recursive process to generate $F_3, F_4, \ldots$ infinitely, note that adding the p-th powers, in this case, terminated the process.

6 Acknowledgments

The author thanks Steven Sam for his mentorship and contributions to this paper, Pavel Etingof for suggesting this project, and Sheela Devadas for our collaboration. All were invaluable resources in all stages of this work. The author is also grateful to his friends and family for their continual support. Finally, the author thanks the Program in Research in Mathematics, Engineering, and Science (PRIMES) for High School Students, the MIT Mathematics Department, the Undergraduate Research Opportunities Program (UROP), and the John Reed fund for funding this project.

Computational data for this project was gathered in SAGE ([S+09]).

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