How to Determine Origami Templates

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1 Introduction

This paper explores how to determine origami templates, such as complex tessellations, skeletal structures, and geometric constructions. These templates can be expressed through mathematical diagrams to highlight their mechanisms. The simplest patterns consist of a singular vertex, but as the structures become increasingly complex, more limitations are created. The constructions of these figures are restricted by their material, rigidity, flatness, and fold angles. From the given statements, new theorems arise through geometric reasoning, expanding on the original patterns. As a result, a deeper understanding of the mathematical framework is developed, providing the basis for a wide range of origami structures.

1.1 Terminology

Dihedral angles: Dihedral angles are interior angles formed between two intersecting planes.
Creases: Creases are determined by black lines and signify that there must be some type of fold incident to that line.
Overlap of Faces, Lines, and Points: A folding f makes two geometric objects A and B overlap if there exists $p \in A$ and $q \in B$ such that $f(p) = f(q)$.
Mountain/Valley Folds: In origami, the two main types of folds are the mountain and valley folds. The mountain fold is a fold where the sides of the paper are folded downward to resemble a mountain. To formally define this, we start with a line l that splits the plane X into subsets of the plane A and B. If we preserve the orientation of plane A and rotate plane B around line l such that plane A is above plane B, the fold at line l will be a mountain fold. This is usually denoted by a solid red line within a crease pattern
The valley fold creates a dip in the paper to resemble a valley. Given the same conditions as stated above, if we preserve the orientation of plane A and rotate plane B about line l such that plane A is below plane B, the fold at line l will be a valley fold. It is typically denoted by a dotted blue line. These two folds types are interchangeable if the paper is flipped over.

Faces: When a structure is folded, the flat surfaces created by the edges of the folds are called faces. Importantly, these faces are not curved and are bounded by the edges of each crease.
Rigid Folding: Within origami, there are subcategories of folding, each of which has its own restrictions. Rigid folding occurs when a structure is created without bending, twisting, or curving the material at hand. Imagine the faces of the origami structure are steel plates. By denoting the faces as steel plates, it becomes obvious that the structure cannot be twisted as steel cannot be manipulated in this manner. Additionally, with rigid folding, the dihedral angles of the structure must satisfy certain constraints based on linear algebra. Classifying structures as rigidly foldable provides extra constraints on how the structure can be created and how it can be defined.

2 Flat Foldability:

In origami, some structures can be completely flattened, so that the model is folded onto a single plane. The characteristic of whether or not a structure follows such constraint is called flat foldability. If a structure is flat foldable, it can be represented on a single plane without any additional creases or self-intersections.

3 Constraints

Fold patterns can be used to create a variety of distinct structures within origami. However, not all arrangements of creases and folds can be performed to create a physically attainable model. For example, a diagram cannot include a fold that creates an intersection with another face of the paper. Since the two faces cannot go through each other, this would be impossible. To ensure that origami structures fold as intended, various geometric constraints can be introduced to demonstrate the behavior of the structures.

Defining such principles assists in further discovering more complex structures. By doing so, origami templates can be viewed as geometric structures, deepening the understanding of the origami folding process. These constraints allow for better control over the behavior of models and set up a foundation for rigorous proofs of origami structures.

3.1 Local Min Lemma

The Local Min Lemma, also called the "Big-Little-Big" Lemma, states that for a single-vertex origami pattern to fold flat, every **local min sector angle** has to border one mountain and one valley fold. A local min sector angle is a sector angle that is adjacent to two larger sector angles.

Formally, it claims that for a valid folding of a crease pattern to exist, where θ_i is the local minimum angle, adjacent folds e_i and e_{i-1} around θ_i cannot have the same mountain-valley folding assignment. That is, if e_i is a mountain fold,

e_{i-1} must be a valley fold, and vice versa. This is true because for a paper to fold flat, the paper must not intersect. When folding, if one sector extends further than the crease creating the other sector, there will be a self-intersection within the model. This contradicts the characteristics of paper and origami.

3.2 Maekawa's Theorem

Maekawa's Theorem states that for each single-vertex origami pattern to be flat foldable, the difference between the number of mountain and valley folds is 2.

Formally, Maekawa's Theorem declares that given a valid folding pattern, where M symbolizes the number of edges labeled as a mountain fold and V is the number of edges labeled as a valley fold, then $|M - V| = 2$.

In a single-vertex, flat-foldable, origami pattern, the following equation holds:

$$M = V + 2$$

Where M is the number of mountain folds and V is the number of valley folds. However, the variables M and V can switch, as you can change the types of folds by flipping over the paper.

3.3 Polygon Proof

Given a circular diagram with center vertex v with mountain and valley folds as seen below in Figure 3.1 (a), the side view after partially folding 3.1 Figure (a) will result in 3.1 Figure (b), which is a polygon. The dihedral angles of the folds create the angles in this polygon.

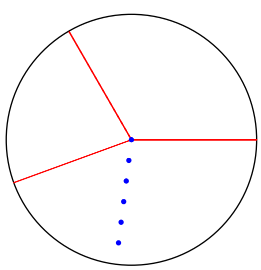

Figure 3.1 (a)

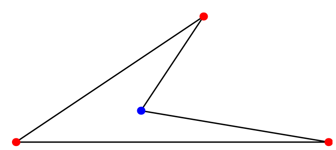

Figure 3.1 (b)

The sum of internal angles of a n vertex polygon is

$$(n-2)180$$

If you fold the paper from Figure 3.1 (b) completely flat, the mountain folds (red dots) will approach $0°$, and the valley folds (blue dot) will approach $360°$.

The number of vertices, or $M + V$, is equal to the number of sides (n). Therefore, substituting into the sum of internal angles formula yields:

$$0 \times M + V \times 360 = (M + V - 2)180$$

Simplifying this equation results in $M - V = 2$. This means that the difference between the amount of mountain folds and the difference between the amount of valley folds is 2.

3.4 Even Degree Lemma

The Even Degree Lemma states that for an origami template to be flat foldable, it is required that there be an even number of creases coinciding at the vertex. We can prove this by simply substituting from Maekawa's theorem. Given that F is the total number of creases, we know that:

$$F = M + V$$

$$M - V = 2$$

By solving for the variable M from Maekawa's theorem, the new equation is

$$M = 2 + V$$

Substituting this into the first equation yields

$$F = 2 + V + V$$

Because V is an integer, F has to be even, meaning that the total number of creases coinciding at the vertex of a flat foldable origami pattern is even.

3.5 Kawasaki's

Kawasaki's Theorem states that a pattern is flat-foldable if the sum of the alternating angles is equal to 0. That is to say:

$$a_1 - a_2 + a_3 - \ldots + a_n = 0$$

3.6 Degrees of Freedom

Degrees of Freedom(DOF) are the number of angles required to define the position of a shape. For example, 1-DOF means that if you define one angle, all the other angles will be set based on this.

4 Pattern Templates

Within origami, there are special templates following conditions given by the previously defined conditions.

4.1 Miura Unit and Tesselation

A Miura unit, depicted in Figure 4.1(a) below, is a special origami template that is able to rigidly fold flat. It consists of 3 mountain folds and 1 valley fold, or 1 mountain fold and 3 valley folds, depending on orientation. Additionally, the Miura unit has one degree-4 vertex. Degree-4 means that there are 4 edges that connect to the vertex. By putting two Miura units adjacent to each other, a third Miura unit will be created within them. By placing four Miura units next to each other, a Miura unit tesselation is formed, as seen in Figure 4.1(b).

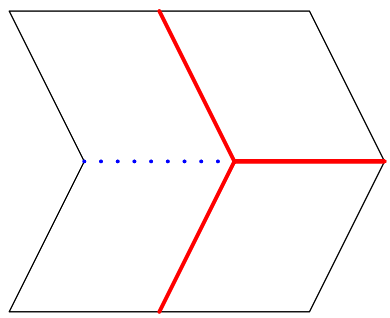

Figure 4.1 (a)

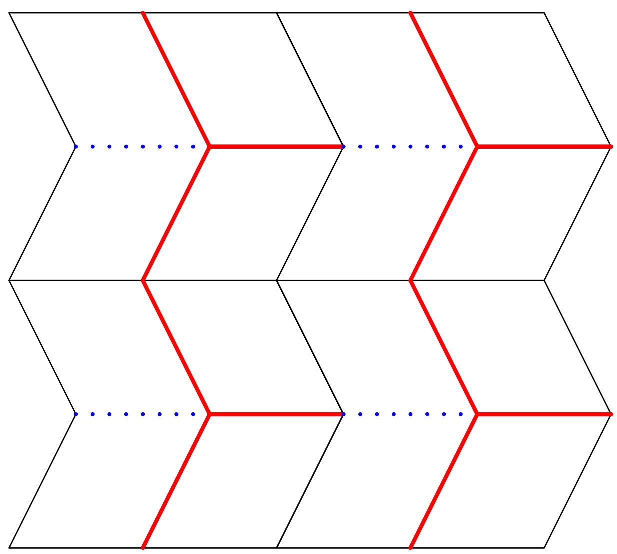

Figure 4.1 (b)

4.2 Miura Unit Proof

To prove the Miura Unit and its tesselation, the first condition that must be verified is the Degrees of Freedom of the Miura Unit. Focusing on a single Miura Unit with faces A, B, C, and D, split the unit in half. Let γ and α be the dihedral angles between B and C, and A and D, respectively as seen in Figure 4.1. Along the side splitting the Miura Unit, let b', v', and d' be the vertices on the faces B and C, and let b, v and d be the vertices on the faces A and D. Let the angle between b', v', and d' be designated as Δ_γ and the angle between b, v, and d be designated as Δ_α. In order for b', v', and d' to coincide with b, v and d respectively, angles Δ_γ and Δ_α must be equivalent.

As a result, Δ_γ serves as a function of γ, where there must exist an inverse function between γ and Δ_γ. For this function and its inverse to exist, there must be a one-to-one set of values for γ and Δ_γ. That is to say, for every value of γ, there is a unique value for Δ_γ and vice versa. A similar relationship can be applied to α and Δ_α as well, where function with an inverse must be created to define the relationship between the two angles. From this, the conclusion can be drawn that by having one of the two angles within these pairs, the complete pair can be determined.

By combining these two conclusions, it's evident that by determining a single angle, all other angles of the Miura Unit are fixed. For example, if the angle Δ_α is determined, angles Δ_γ and α must also be determined, given that Δ_α and Δ_γ must be equal, as well as the one-to-one relationship between Δ_α and α. By deriving the value of Δ_γ, this also defines the value for the angle γ, which is the final angle within the Miura Unit. As a result, since only one angle is necessary to determine the remaining angles in the Miura Unit, the Miura Unit must have 1-DOF.

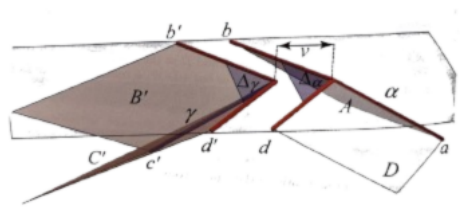

Figure 4.2

4.3 Square Twist 1 and 2

There are two versions of the Square Twist structure: one that can rigidly flat fold and one that cannot fold rigidly. The version that cannot rigidly flat fold, referred to as Square Twist 1, is depicted in Figure 4.3(b), with the specific crease pattern and mountain-valley assignments that cause the model to bend. Although it has the motion of a 1-DOF conceptually, in reality, due to the arrangements of the mountain-valley folds, the folds do not line up so that the faces can fold behind each other without intersecting. As a result, it cannot fold rigidly like the other Square Twist. Conversely, the version that cannot fold rigidly is shown in Figure 4.3(b). In this Square Twist, which is designated as Square Twist 2, the different mountain-valley crease pattern changes its fold angles and allows it to rigidly fold flat.

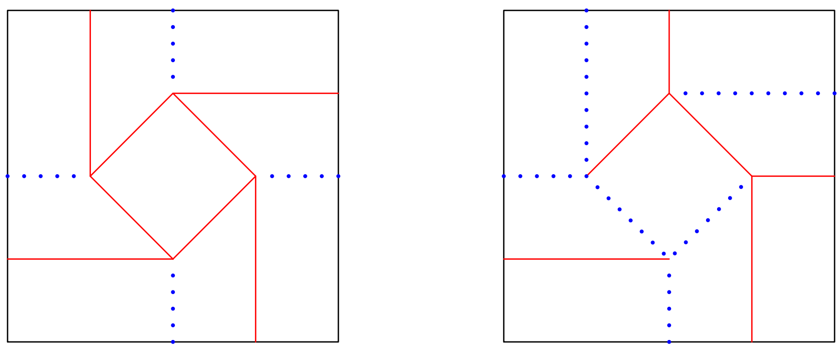

Figure 4.3 (a) Figure 4.3 (b)

5 Skeletal Structure

Building off of flat folding, another aim of origami is to create a desired shape or final model. To do so, designers often create bases comprised of simpler crease patterns that can be expanded into the final figure. In particular, one type of base is a uniaxial base, in which, when collapsed, all its flaps must lie on a single axis. Focusing on uniaxial bases, a key component in determining the uniaxial base of a diagram is the skeletal structure.

Skeletal structures are the "framework" used to represent the uniaxial base, and can be expressed using trees. Trees are connected graphs of vertices with no cycles. That is, there is no sequence of vertices that starts and ends at the same vertex within the graph. There is only one unique path connecting any two vertices within a tree. These properties can be applied to skeletal structures, which helps construct the skeletal structures for shapes, such as convex polygons. They are created by shrinking the sides of the given shape at the same rate parallel to the original sides so that as they shrink, they eventually meet. At these intersections, the connected lines become a single line, which continues to shrink until all edges of the skeletal structure are connected.

5.1 Determining the Skeletal Structure of Convex Polygons

Convex polygons have special skeletal structures called **straight skeletons**. The straight skeleton is created in the same process as a normal skeletal structure. For any convex polygon with n sides, the number of edges in its straight skeleton is exactly $2n-3$. This helps predict the framework for convex polygons, which becomes increasingly useful for more complex figures.

5.1.1 Proof for the Number of Edges

Let n be the number of vertices of the convex polygon. The minimum number of edges in the skeletal structure of this diagram must be n. This occurs when all the vertices of the polygon converge at the center vertex. In this case, one edge is created from each vertex, resulting in n total skeletal edges.

To find the maximum number of edges in the skeletal structure of this polygon, for each pair of edges, the vertices must converge at a point. This pattern repeats until one point is left, which becomes the center vertex.

For any skeletal structure, it can be represented by a tree diagram as seen below. Let I be the number of internal vertices. For a convex polygon, each internal vertex has degree-3. So:

$$n = I + 2$$

where n is the number of vertices in the polygon, which is equivalent to the number of endpoints in the tree diagram representation of the skeletal structure. Therefore,

$$I = n - 2$$

In any tree representing a skeletal structure of a convex polygon:

$$e = v - 1$$

where e is the total number of edges in the skeletal structure and v is the total number of vertices in the structure. The total number of vertices is found by adding the internal vertices to the endpoints of the tree diagram, which is the same as the number of vertices in the polygon. Therefore:

$$v = I + n$$

$$v = 2n - 2$$

As a result:

$$e = (2n - 2) - 1$$

$$e = 2n - 3$$

Therefore, the number of edges in a skeletal diagram of a convex polygon with n vertices must be from n to $2n - 3$ edges.

5.2 Diagrams

Here is the skeletal structure of a hexagon with the maximum number of skeletal edges: As you can see, there are alternating sides of long and short lengths. This allows for the skeletal structures for the short lengths to converge and create a degree-3 vertex. These 3 extended lines meet at the center, forming another degree-3 vertex.

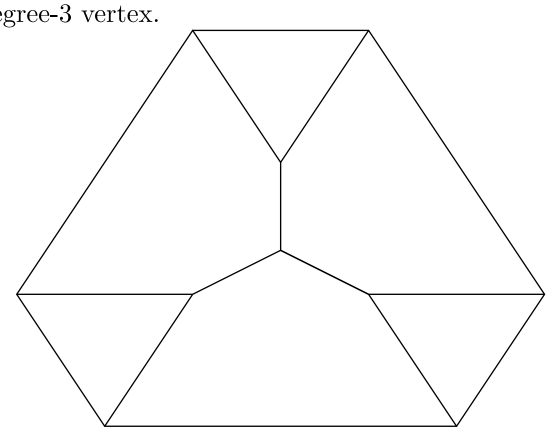

Figure 5.1 (a)

Here is a tree diagram for the hexagon that represents the skeletal structures. As you can see, the point in the center divides into more skeletal edges using degree-3 vertices.

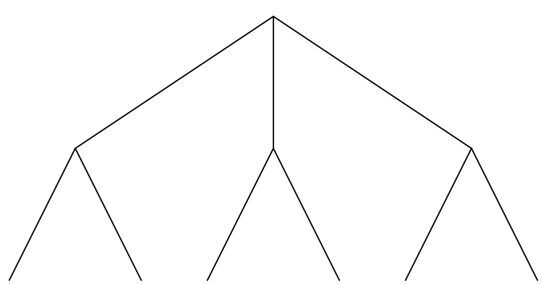

Figure 5.1 (b)

5.3 Molecules

Skeletal structures are a key concept in creating flat folding designs. To break down more complex designs, fundamental models, known as molecules, are introduced. Molecules refer to patterns that allow the creases in a structure to coincide when its base is collapsed.

In molecules, some important creases are the **hinge** and **ridge creases**. Hinge creases are the creases that fold inwards to the center of the structure, which allow the stick figure to fold onto one axis. On the other hand, the ridge creases are the lines or edges of the actual stick figure.

These molecules allow us to break down the larger crease patterns into simpler pieces. By first understanding how these respective crease patterns fold, and combining these molecules together, we will be able to conceptualize how the final shape folds.

5.3.1 Rabbit Ear Molecule

For all triangular patterns, the Rabbit Ear Molecule is used to ensure that the triangle coincides. The triangle folds into a skeletal structure comprised of the intersection of the three angle bisectors of the triangle. As a result, the center vertex of the molecule is the incenter of the triangle, as seen below. Following this, the three angle bisectors form the hinge creases in the structure, while the ridge creases result from the perpendicular radii from the inscribed circle of the triangle to the edges.

Additionally, these three perpendicular radii are also tangent to the circles centered at the vertices of the triangle, demonstrating that the circle tangency points of the structure and hinge creases meet at the same point. To satisfy the flat folding requirement of a uniaxial base, Maekawa's Theorem can also be used to determine that only one of the three valley folds is used, producing three possible flat folding options for the Rabbit Ear Molecule.

As seen below in Figure 5.2, the perpendicular lines have the same length r because they are all the radii of the inscribed circle. As a result, by folding the hinge creases into mountain folds, three triangles will be created with the equal height r with the sides of the original triangle being the bases. This allows the triangles formed by the mountain folds to be reflected onto each other over the ridge folds along a singular axis since the heights from the base to the opposite vertex are congruent. This results in a flat folded version of the triangular base, which satisfies the molecule.

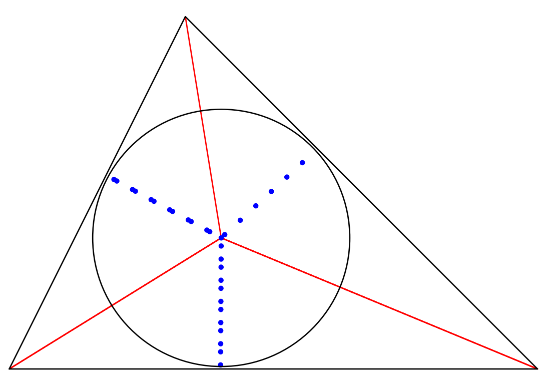

Figure 5.2

5.3.2 Four-Circle Quadrilateral Molecule

To create the molecule known as the Four-Circle Quadrilateral Molecule, like Figure 5.3, tangent circles are formed centered at the corners of a quadrilateral. Importantly, the quadrilateral is not required to be regular, meaning that it can have distinct side lengths and angles, but it must be a tangential quadrilateral. A tangential quadrilateral is a convex quadrilateral whose sides are all tangent to the incircle. Forming the ridge creases of this molecule, the four angle bisectors of the quadrilateral intersect at the incenter of the quadrilateral, similar to the center vertex in the Rabbit Ear Molecule. Its skeletal structure consists of a degree-4 vertex and the four ridge creases. The perpendiculars from the incenter to the sides create the hinge creases, tangent to the circles centered at the corners of the quadrilateral.

To find tangential quadrilaterals, **Pitot's Theorem** can be used to verify the qualifications of a given quadrilateral. Pitot's Theorem states that the sum of the lengths of opposite sides of a tangential quadrilateral are equal, and inversely, a convex quadrilateral is tangential if the sum of the opposite side lengths are equal. In reference to the Four-Circle Quadrilateral, by representing the side lengths using the radii of the four circles, A, B, C, and D, Pitot's Theorem can be written as

$$(A + B) + (C + D) = (A + D) + (B + C)$$

which is equivalent, validating Pitot's Theorem.

We know that the perpendicular lines have the same length, as they are all radii of the incenter. Therefore, when following the fold pattern, the perimeter of the quadrilateral will coincide on one line.

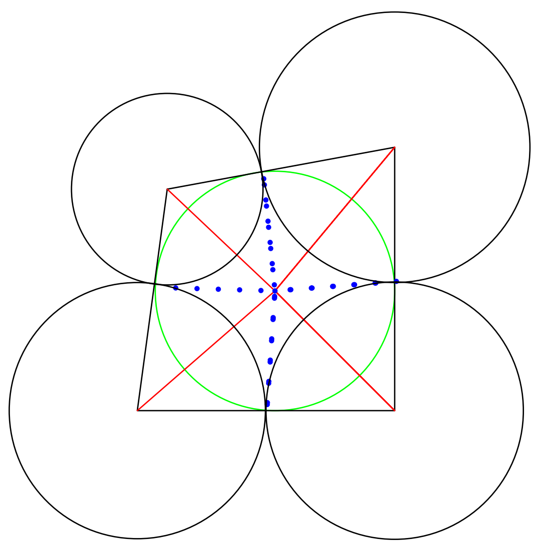

Figure 5.3

5.3.3 Waterbomb

The Waterbomb Molecule is a special case of the Four-Circle Quadrilateral Molecule where the quadrilateral is a square. The skeletal structure of the Waterbomb Molecule consists of the center vertex of degree-4 and four equal-length edges. These edges are the ridge edges of the molecule, and also serve as the angle bisectors of the square. The four circles centered at the corners of the square are congruent, each with a radius of half the side length of the square. The hinge creases are represented by the tangents between the sides of the square and its incircle, which is also incident with the horizontal and vertical midlines of the square.

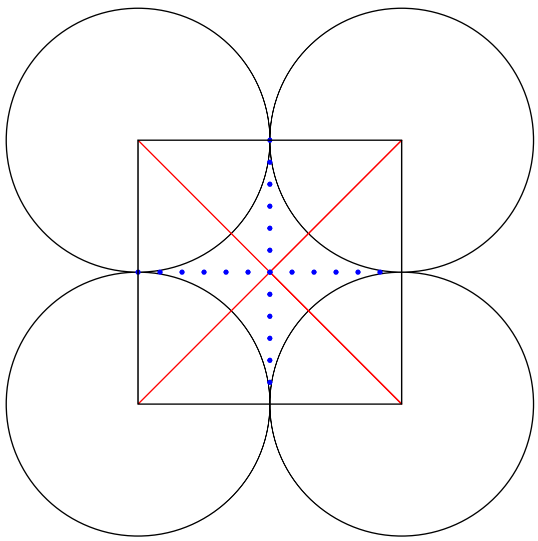

Figure 5.4

5.3.4 Sawhorse Molecule

The final molecule examined in this paper is the Sawhorse Molecule. The Sawhorse Molecule consists of an irregular quadrilateral defined by its skeletal structure. Unlike the other skeletal structures examined in previous molecules, the skeletal structure for Sawhorse Molecule is not centered at a single vertex. Instead, the skeletal structure is made up of two degree-3 vertices, with one edge connecting the two vertices. The edge connecting the two vertices is called an internal edge, which is an edge connecting two vertices that are not at the end of the skeletal structure. The remaining edges of the skeletal structure are represented by the angle bisectors of the quadrilateral, similar to previous molecules.

In the Sawhorse Molecule, the circles centered around the corners of the quadrilateral are separated along two of the sides by a section of the quadrilateral called the **river**, which has the same width as the skeletal structure's internal edge. It is also perpendicular to two opposite sides of the quadrilateral. Along the other pair of sides, the circles are still tangent to one another, which is where the two hinge creases of the Sawhorse Molecule meet. The Sawhorse Molecule only has these two hinge creases, which are represented by the blue dotted line in Figure 5.5 below.

The Sawhorse Molecule is also an example of the **Circle/River Method**, which uses, as the name suggests, circles and rivers to determine more complex molecules. This is done by overlapping and extending lines within a projected plane, causing select vertices to intersect at infinity. This allows us to combine molecules and understand where the perpendicular valley folds lie within the structure.

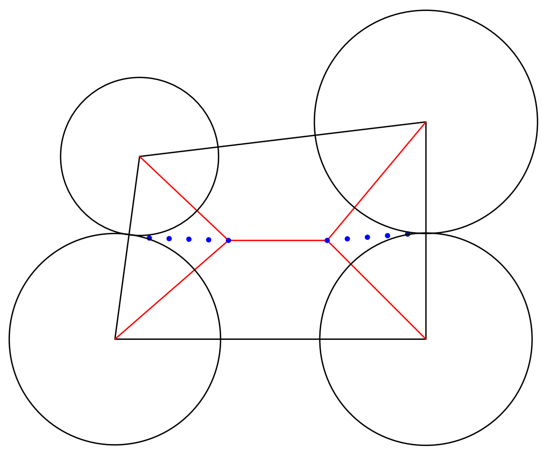

Figure 5.5

6 Concluding Section

By utilizing different methods and structural orientations, we are able to create a wide variety of origami shapes and designs. This can be done through mathematical understanding of the interactions and correlations between angles, as well as the orientations of planes within the structure. These origami designs include complex tesselation, skeletal structures, and geometric constructions, each with its own unique properties and respective constraints. We utilize theorems and diagrams used to analyze origami behavior to demonstrate the connection between geometric constructions and origami, providing strong real-world applications and representations of these concepts. This process of defining structures and their respective limitations allows us to understand and model increasingly complicated structures using mathematical concepts.

Through these origami principles, new discoveries can be constructed for research regarding spatial and geometric constrictions. Origami provides an avenue for problem solving, which applies geometry to explain physical constraints on how materials can be manipulated. As a result, ideas that once existed solely in thought can be transferred into tangible reality without defying the limitations of physical matter.