HOW TO (POTENTIALLY) WIN A MILLION DOLLARS: AN INTRODUCTION TO THE RIEMANN ZETA FUNCTION

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Abstract. Proving the Riemann Hypothesis, which stems from the Riemann zeta function, is one of seven Millennium Problems from the Clay Institute, with a one million dollar prize. It is especially important because it potentially reveals much about the distribution of primes. We created an expository explanation of $\zeta(s)$ meant for a general audience while still including proofs and important prerequisites. We first introduced techniques such as Gamma functions, and Bernoulli numbers and polynomials. We then discussed the proof for computing $\zeta(s)$ for even numbers using Euler's technique, focusing specifically on proving $\zeta(2) = \frac{\pi^2}{6}$. Finally, we introduced the Riemann Hypothesis and links to the Prime Number Theorem.

Contents

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1. Introduction

1.1. **Motivation.** In the Spring of 2000, mathematicians met at the College de France in Paris to discuss seven of the hardest problems in mathematics. This meeting took place on the hundredth anniversary of Hilbert's address, in which he outlined 23 unsolved mathematical problems.

These seven problems went on to be the most notorious in mathematics, especially as the Clay Mathematics Institute offered a million dollar prize to whoever could solve each problem.

They included the Birch and Swinnerton-Dyer conjecture, the Hodge conjecture, Navier-stokes equations, **P** vs **NP**, the Yang-Mills existence and mass gap, the Poincare conjecture, and the Riemann hypothesis.

Twenty-six years after the initial announcement, only one of the Millennium problems has been solved: the Poincare conjecture. Hilbert's eighth problem - the **Riemann Hypothesis** has remained unsolved for over a century.

The Riemann hypothesis has applications ranging from quantum physics to RSA encryption. Proving the Riemann hypothesis would lead to subsequent progress in many other fields of math. It could even help solve another problem - the **P** vs **NP** problem.

Most notably, it has applications in prime numbers. Proving the Riemann hypothesis could lead to faster algorithms to find larger prime numbers, finding a pattern in the distribution of primes.

1.2. **History.** Humans have been obsessed with numbers since the beginning of oral history. [1] In 2500 BC, the Sumerians developed a number system using base 60. Around 600BC, Pythagoras formally defined the terms **prime**, **odd**, **even**, and **composite**. [1]

However, it wasn't until 300BC that mathematicians became concerned with prime numbers. In 300 BC, Euclid published his *Elements*, which included a proof that there were an infinite number of primes.

After a brief hiatus when it came to discovering new facts about primes, Euler developed his product formula in 1737 along with proving that the infinite sum of inverse of primes was divergent:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\cdots$$

The product formula was not rigorously proven until Kronecker in 1876. In 1792, Gauss came up with the prime number formula:

$$\pi(x) \sim \frac{x}{\ln(x)} \text{ as } x \to \infty$$

.

He later refined it, and more work was done until Riemann introduced his hypothesis in 1859. In 1900, Hilbert declared it one of twenty-three important unsolved problems.

Notably, there has been some progress completed including with the Weil conjectures (especially the fourth), which are interlinked with the Riemann Hypothesis proven by Deligne in 1977.

Finally, in 2000, the Clay Mathematics Institute declared it one of seven millennium problems, posting a one million dollar bounty to anyone who could provide a solution. To this day, it still remains unsolved.

2. Preliminaries and Notation

This section sets notation and defines a few key terms used later in the paper.

For the zeta function, we will be using ζ. Any use of p denotes a prime. $\sum$ indicates a summation, while $\prod$ indicates a product. $\log(x)$ denotes the natural log of x.

2.1. Primes.

An important application of the Riemann zeta function is with primes, and we list here the foundations of prime numbers.

Definition 1. An integer $p > 1$ is called a prime number, or a prime, in the case there is no divisor d of p satisfying $1 < d < p$. If an integer $a > 1$ is not a prime, it is called a composite number. [5]

Accordingly, if we use $\mathbb{Z}$ to represent the set of integers:

$$\mathbb{Z} = \{\ldots - 3, -2, -1, 0, 1, 2, 3\ldots\}$$

We use $\mathbb{P}$ to represent the set of primes:

$$\mathbb{P} = \{2, 3, 5, 7, 11, \cdots\}$$

Notably, while the set of integers includes 1, it is not considered prime by modern mathematical definition.

We present also the fundamental theorem of arithmetic:

Theorem 1. *Every integer n greater than 1 can be expressed as a product of primes (with perhaps only one factor). The factoring of any integer $n > 1$ into primes is unique apart from the order of the prime factors.* [5]

Finally, we conclude this section with Euclid's theorem on prime numbers:

Theorem 2. *The number of primes is infinite. That is, there is no end to the sequence of primes.* [5]

2.2. Bernoulli Numbers and Polynomials.

The concept of Bernoulli numbers is indeed an interesting and valuable discovery in number theory. To begin, let's take look at how we can calculate the sums of the following series:

- $\sum_{k=1}^{n} k$
- $\sum_{k=1}^{n} k^2$
- $\sum_{k=1}^{n} k^3$

Essentially each equation is asking us what is the sum from k to n if we plug them into our function. For instance in the first example we have just the term k which is asking what is the sum of:

$$1+2+3+4+5+6+\cdots+n$$

We can ask similarly for the second and third examples they are the same thing just asking what happens when we take them to their respective powers:

$$1^2+2^2+3^2+4^2+5^2+6^2+\cdots+n^2$$
$$1^3+2^3+3^3+4^3+5^3+6^3+\cdots+n^3$$

In short, this is the process (albeit a little tedious) is a method for calculating the sums for the series when our terms happen to be k, k^2, k^3

However nicely enough, mathematicians have managed to derive explicit formulas that make computation much quicker instead of adding each term individually, therefore from our examples we get that:

- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Because these formulas help mathematicians evaluate the sums of series much more quickly and efficiently than our elementary methods, a mathematician named Bernoulli wondered whether we could find a pattern in these formulas for the sums of a series, up to the k^j-th term or in other words "generalize":

As a matter of fact, Bernoulli was onto something, but as a matter of fact he did not manage to discover a formula for his hypothesis. Rather in the early 1730s it was one of his most famous students by the name of Leonhard Euler who managed to derive a equation modeling the relationship between the sums of any power. As a tribute he honored Bernoulli by naming his discovery after him in what we know today as the Bernoulli numbers noted as B_j. Euler found that we could compute Bernoulli's number to any power through this equation :

$$0=\sum_{j=0}^{m}\binom{m+1}{j}B_j$$

In words this translates to the sum of m and the binomial multiplied by our Bernoulli number B_j equals zero. For example we can compute the first couple of Bernoulli numbers by plugging in the following into the recursive formula [1]

Consider $m=1$:

$$\binom{2}{0}B_0+\binom{2}{1}B_1=0$$
$$1\cdot 1+2B_1=0$$
$$B_1=-\frac{1}{2}$$

---

[1] A recursive formula is simply just taking the starting term and using that term to find the next term (i.e if we have a sequence of numbers $5, 10, 15, 20\cdots$ we know that the pattern to finding the next term is just adding five to the term preceding it therefore the fifth term would be $20+5=25$)

Case $m = 2$:

$$\begin{aligned}
\binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 &= 0 \\
1 + 3\left(-\frac{1}{2}\right) + 3B_2 &= 0 \\
1 - \frac{3}{2} + 3B_2 &= 0 \\
-\frac{1}{2} + 3B_2 &= 0 \\
B_2 &= \frac{1}{6}
\end{aligned}$$

Case $m = 3$:

$$\begin{aligned}
\binom{4}{0}B_0 + \binom{4}{1}B_1 + \binom{4}{2}B_2 + \binom{4}{3}B_3 &= 0 \\
1 + 4\left(-\frac{1}{2}\right) + 6\left(\frac{1}{6}\right) + 4B_3 &= 0 \\
1 - 2 + 1 + 4B_3 &= 0 \\
0 + 4B_3 &= 0 \\
B_3 &= 0
\end{aligned}$$

We can repeat this process to find infinitely many more numbers.

Additionally, Bernoulli numbers can also be generated with this Taylor series approximation:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

Subsequently they can also be represented by this expression where each of the B_k terms are the coefficients of the Bernoulli Formula:

$$\frac{x}{e^x - 1} = B_0 + B_1 x + B_2\frac{x^2}{2!} + B_3\frac{x^3}{3!} + \cdots$$

Index (n)	**Bernoulli Number** (B_n)
0	1
1	$-\frac{1}{2}$
2	$\frac{1}{6}$
3	0
4	$-\frac{1}{30}$
5	0
6	$\frac{1}{42}$
7	0

Table 1. First eight Bernoulli numbers

Notice a nice pattern of each odd $B_k > 1$ results in a Bernoulli number (coefficient) of 0. This is a really neat matter that we will touch back on in section 5 of the paper when we go over odd values of the Riemann Zeta function!

2.3. **Gamma Function.** Next up, we will examine the gamma function. But first we will need to understand what factorials are and how to compute them. Defined as $n!$ factorials are the product of all positive integers less than or equal to a given non-negative integer. Factorials can be computed using this following formula:

$$n! = n(n-1)(n-2)\cdots 1$$

In simpler terms if we want to find what 5! is all we need to do is take the product of 5 itself and all integers 1 less than it all the way to 1:

$$5! = 5(5-1)(5-2)(5-3)(5-4) = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

But what happens when we try to compute the factorial of some non-integer (i.e 2.5 or 6.767)? Taking a look at 2.5, one might initially consider: $2.5 \times 1.5 \times 1\times$ or 2.5×1 which both seem like perfectly reasonable answers, yielding two completely different values. But how do we know for sure what the value of the factorial is? This is the problem that the Gamma function solves, as it is formally defined by this equation:

Definition 2. The Gamma Function

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx \text{ for } s > 0.$$

Furthermore, the (complete) gamma function can also be stated as:

Lemma 1. *The (complete) gamma function* $\Gamma(n)$ *is defined to be an extension of the factorial to complex and real number arguments. It is related to the factorial by* Γ*(n)=(n-1)!*

Therefore in the terms of factorials $\Gamma(n)$ can also be rewritten as $n! = \Gamma(n+1)$
Understanding $\sqrt{\pi}$:
One of the most important facts about the Gamma function is that $\Gamma(0.5) = \sqrt{\pi}$. [2]

Proof. $\Gamma(0.5)$
We want to show that:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Recall the definition of the gamma function.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0.$$

Therefore for, $x = \frac{1}{2}$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\, dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt.$$

Substitute for integration $t = u^2$ so that $dt = 2u\, du$ and $\sqrt{t} = u$:

$$\int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt = \int_0^\infty \frac{e^{-u^2}}{u} \cdot 2u\, du = 2\int_0^\infty e^{-u^2}\, du.$$

Interestingly notice how, the result from integrating parallels that of the Gaussian integral from the standard distribution which gives a hint of what's to come

$$\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi} \quad \Rightarrow \quad \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

Therefore

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

□

Using these properties we now have the tools to compute 2.5! and 6.767! via the complete Gamma function.
Consider (2.5!)

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[2] This is a pretty complex proof, that we will not go over all the steps for, since our main goal is just to give readers a basic description of the Gamma function, the rest was just for people who were interested... So don't worry if you get a little lost!

$$\begin{aligned}
2.5! &= \Gamma(2.5+1) \\
&= \Gamma(3.5) \\
&= 2.5 \times \Gamma(2.5) \\
&= 2.5 \times 1.5 \times \Gamma(1.5) \\
&= \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma(0.5) \\
&= \frac{15}{8}\sqrt{\pi} \\
&\approx 3.32335
\end{aligned}$$

Now consider (6.767!)

$$\begin{aligned}
6.767! &= \Gamma(6.767+1) \\
&= \Gamma(7.767) \\
&= \int_0^\infty t^{6.767} e^{-t}\, dt \\
&= 6.767 \times 5.767 \times 4.767 \times 3.767 \times 2.767 \times 1.767 \times \Gamma(1.767) \\
&\approx 3162.66004
\end{aligned}$$

To reiterate, the Gamma function is a really useful function because it help mathematicians fill in the gaps to express any value of factorials, such as non-integers and values that also extend to the complex plane which becomes increasingly useful in Section 5 when trying to understand and decipher the Riemann Hypothesis.

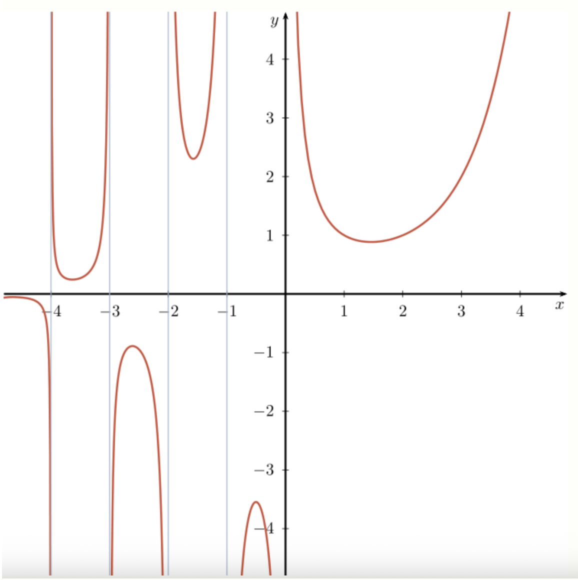

FIGURE 1. A nice Visual of the Gamma Function on the Cartesian Plane

[8]

3. CONVERGENCE AND DIVERGENCE

This section will tackle series, convergence, and divergence for Euler's zeta series, which is:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for all real positive values } s > 1$$

We present a few fundamental definitions and theorems:

Definition 3. A series is an infinite sum.[6]

This leads to a secondary definition:

Definition 4. Series $\sum_{a_n}$ is *convergent* and *converges* to S as $n \to \infty$ for all $\epsilon > 0$ there exists an N such that

$$\left| \sum_{k=1}^{n} a_k - S \right| < \epsilon$$

for all $n > N$. If no such S exists, the series diverges. [6]

3.1. **Harmonic Series.** Take the following sum, where we use 1 for s, yielding the following form:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

As a result, we get the following sequence:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\dots$$

This sequence is portrayed on the following graph:

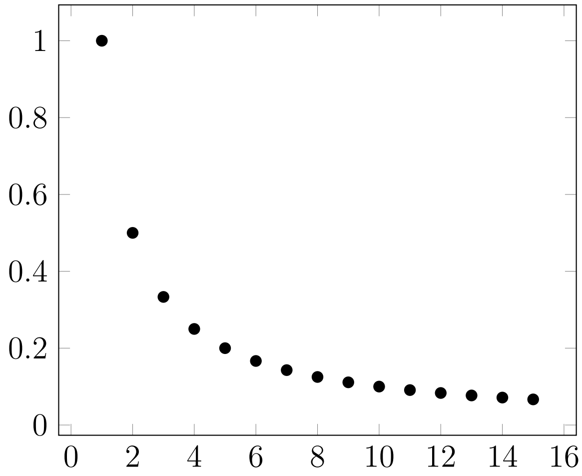

FIGURE 2. Harmonic Series

Intuitively, this series seems to converge, as additional terms grow increasingly smaller as $n \to \infty$. However, it would diverge. To illustrate this example, we present the following lemma:

Lemma 2. *Given infinite series*

$$\sum_{k=1}^{\infty} a_k \quad \textit{and} \quad \sum_{k=1}^{\infty} b_k$$

of nonnegative terms $(a_k \geq 0; b_k \geq 0)$*, with* $a_k \geq b_k$*. Then we have*

- *If* $\sum a_k$ *converges, then* $\sum b_k$ *converges.*
- *If* $\sum b_k$ *diverges, then* $\sum a_k$ *diverges.*[6]

With this lemma, we can finally produce our proof

Proof. Let $n = k$. The sum of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{n}$ can be expressed as:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\dots$$

This can be further grouped into:

$$1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7})\dots$$

Let this be notated as $\sum_{k=1}^{\infty} a_k$.

Accordingly, the sum of the series $\sum_{k=1}^{\infty} \frac{1}{2}$ can be expressed as:

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\dots$$

Let this series be notated as $\sum_{k=1}^{\infty} b_k$

We see that this sum $\sum_{k=1}^{\infty} b_k$ very clearly diverges as $x \to \infty$.

Under the grouping, each term of the harmonic series is greater or equal to each term of the series b_k. Thus, all prerequisites for the lemma 2 are met.

Resultingly, since $\sum_{k=1}^{\infty} b_k$ diverges, under the second part of the lemma, $\sum_{k=1}^{\infty} a_k$ would also diverge.

□

We present a graph of partial sums, which demonstrates divergence, verifying our proof:

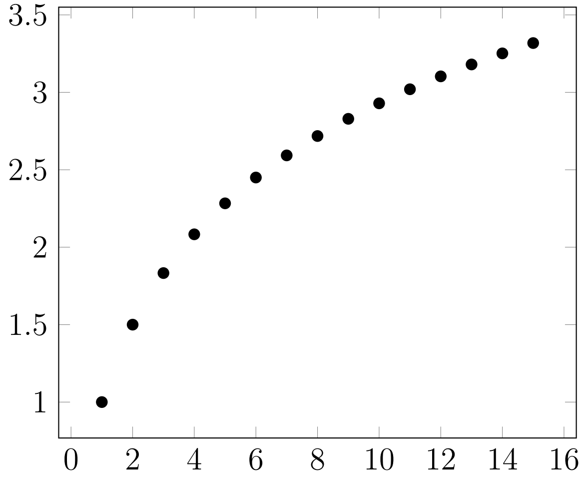

Figure 3. Partial Sums of Harmonic Series

4. Computing ζ(even)

Up to now, we should have a good understanding of the behaviors of $\zeta(1)$ such it diverges and grow without bound. Conversely we also know that $\zeta(2)$ or more generally any $\zeta(s)$ where $s \geq 2$ will eventually converge at some limit as illustrated by the harmonic series. Now one may wonder at what value does $\zeta(2)$ actually converge to?

This question eventually became known as the Basel Problem[3] which was subsequently proved by Euler in 1735:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Proof. The Basel Problem $\zeta(2)$

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[3] This is because it originated in Basel, Switzerland

According to the Taylor Series, Euler knew that $\sin x$ can be expressed as an infinite sum:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Next, dividing both sides by x we can get a new function:

$$f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

With our new function $f(x)$ the roots of this function will correspond to $\pm\pi, \pm 2\pi, \pm 3\pi \cdots$ because:

$$\frac{\sin \pi}{1} = 0$$
$$\frac{\sin 2\pi}{2} = 0$$
$$\frac{\sin 3\pi}{3} = 0...$$

Now, we want to factor our infinite polynomial. By the Fundamental theorem of Algebra we factor the non-linear terms into their linear factors via the form:

Theorem 3. *Fundamental Theorem of Algebra*

$$P(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{i=1}^{n} (z - c_i)$$

In other words, the theorem is saying that we can just take any polynomial whether it is to the 3rd degree or to an infinite degree and factor it to linear terms like how we can factor the quadratic $x^2 - 4$ into $(x-2)(x+2)$.

What Euler got was:

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdots = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

If we rearrange this equation with with π^2 and x^2 we get:

$$= \frac{x^2}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)$$

Which also equals!:

$$= \frac{x^2}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots\right)$$

Now all is left to do was equating x^2 from our product expansion to the x^2 term from the original Taylor series which is $-\frac{1}{3!} = \frac{1}{6}$. So finally equating the coefficients we can do some simple rearranging which lends:

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6} \tag{1}$$

completing our proof.

□

Since Euler's proof helps us find the exact value of $\zeta(2)$ it poses the question if a similar process can be used to find any given value for any even power?

Indeed it can be generalized with this formula which Euler derived building off his success with the Basel problem:

Theorem 4. *Euler's formula for even values of* ζ

$$\zeta(2k)=\frac{(-1)^{k+1}B_{2k}(2\pi)^{2k}}{2(2k)!} \tag{2}$$

Proof. We begin with the Euler's product formula for $\sin x$

$$\sin x = x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)$$

Now let's take the natural log of both sides which breaks down the product into a sum:

$$\ln(\sin(x))=\ln(x)+\sum_{k=1}^{\infty}\ln\left(1-\frac{x^2}{k^2\pi^2}\right)$$

Next, we want to substitute $-\frac{iu}{2}=x$ which gives us:

$$\ln(\sin(-\frac{iu}{2}))=\ln(-\frac{iu}{2})+\sum_{k=1}^{\infty}\ln\left(1-\frac{(-\frac{iu}{2})^2}{k^2\pi^2}\right)$$

If we take a look at the fractional term for the sum we see that we can simplify:

$$1-\frac{(-\frac{iu}{2})^2}{k^2\pi^2}$$

$$1-\frac{(-\frac{i^2u^2}{2^2})}{k^2\pi^2}$$

$$1-\frac{(-\frac{1u^2}{4})}{k^2\pi^2}=(1+\frac{u^2}{4k^2\pi^2})$$

$$\ln(\sin(-\frac{iu}{2}))=\ln(-\frac{iu}{2})+\sum_{k=1}^{\infty}\ln\left(1+\frac{u^2}{4k^2\pi^2})\right)$$

The next step perhaps the most challenging part of this proof, but we will break it down into manageable steps

First let's differentiate the function with respect to u and take the derivative of each term which is:

$$\frac{du}{dx}\ln(\sin(-\frac{iu}{2}))$$

$$\frac{du}{dx}\ln(-\frac{iu}{2})$$

$$\frac{du}{dx}\sum_{k=1}^{\infty}\ln\left(1+\frac{u^2}{4k^2\pi^2})\right)$$

Focusing on the first term we need to use the chain rule:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Take the derivative of ln which is

$$\frac{1}{x}$$

$$\frac{1}{(\sin -(\frac{iu}{2}))}$$

And because this is the chain rule we plug $-\frac{iu}{2}$ underneath and proceed with the next term which is $\sin(x)$

Since the derivative of $\sin(x) = \cos(x)$ we can now rewrite the *sin* term and plug in $-\frac{iu}{2}$ into our expression:

$$\cos(-\frac{iu}{2})$$

Lastly we need to take the derivative of $-\frac{iu}{2}$itself which if we apply the power rule:

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

Thus:

$$\frac{du}{dx}(-\frac{iu}{2}) = -\frac{i}{2}u^{1-1} = -\frac{i}{2}u^0 = -\frac{i}{2}$$

Therefore we can now rewrite our equation as:

$$\frac{1}{(\sin -(\frac{iu}{2}))}\cos(-\frac{iu}{2}) - \frac{i}{2}$$

This takes us halfway there, now we must differentiate the other side

$$\ln(-\frac{iu}{2}) + \sum_{k=1}^{\infty}\ln\left(1+\frac{u^2}{4k^2\pi^2})\right)$$

Beginning with $\ln -\frac{iu}{2}$ we will need to apply the chain rule again, this is really similar to what we just did above the derivative of ln which is just:

$$\frac{1}{-\frac{iu}{2}}$$

And then moving on to our second term $-\frac{iu}{2}$the derivative of that is just $-\frac{i}{2}$ so we can rewrite our new expression to be:

$$\frac{1}{-\frac{iu}{2}} - \frac{i}{2}$$

Finally we are left to differentiate our infinite sum. One of the requirements for a infinite sum to be differentiable is that it must converge or in other words not explode to infinity. Since we earlier proved that any value in the zeta function where the integer $s > 1$ converges this means that we can differentiate this infinite series of which is referring to zeta values greater than 2.

The first step in our approach should be to use the chain rule again because we are dealing with multiple functions with respect to u.

$$\sum_{k=1}^{\infty} \ln \left(1 + \frac{u^2}{4k^2\pi^2})\right)$$

Therefore for the first term ln we get:

$$\frac{1}{(1 + \frac{u^2}{4k^2\pi^2}))}$$

And for the subsequent the derivative of 1 is just 0

$$\frac{du}{dx}1 = 0$$

Lastly, the derivative of our last term can be achieved using the power rule:

$$(\frac{u^2}{4k^2\pi^2})$$

$$(\frac{1}{4k^2\pi^2})u^2$$

$$(\frac{1}{4k^2\pi^2})2u^{2-1}$$

$$= (\frac{1}{4k^2\pi^2})2u$$

Now we have all the terms, so let's rewrite our new equation from both sides.

$$\frac{1}{(\sin -(\frac{iu}{2}))} \cos(-\frac{iu}{2})(-\frac{i}{2}) = \frac{1}{-\frac{iu}{2}}(-\frac{i}{2}) + \sum_{k=1}^{\infty}(\frac{1}{4k^2\pi^2})2u$$

Simplifying the right hand side by multiplying cos × sin we get:

$$\left(-\frac{i}{2}\right) \frac{\cos\left(-\frac{iu}{2}\right)}{\sin\left(-\frac{iu}{2}\right)}$$

For the left hand side we can we can simplify the fraction first:

$$\frac{1}{-\frac{iu}{2}}(-\frac{i}{2})$$

$$\frac{-\frac{i}{2}}{-\frac{iu}{2}}$$

Then we can cancel out the negatives and multiply by the reciprocal on the denominator

$$\frac{i}{2}\frac{2}{iu}$$

Since $\frac{2}{2} = 1$ and $\frac{i}{i} = 1$ then we can simplify this to:

$$\frac{1}{u}$$

For the infinite sum all we have to do is multiply the $2u$ which will lend:

$$\sum_{k=1}^{\infty} \frac{2u}{4n^2\pi^2 + u^2}$$

And last but not least for this part we can rewrite everything as:

$$\left(-\frac{i}{2}\right)\frac{\cos\left(-\frac{iu}{2}\right)}{\sin\left(-\frac{iu}{2}\right)} = \frac{1}{u} + \sum_{k=1}^{\infty} \frac{2u}{4n^2\pi^2 + u^2} \tag{4.0}$$

We now have a nice trigonometric identity on the left hand side, but in order to understand the next steps we will first need to understand Euler's formula.

Definition 5. Euler's Formula $e^{ix} = \cos(x) + i\sin(x)$

With Euler's formula we can subsequently prove that:

$$\cos(-\frac{iu}{2}) = z$$

$$\sin(-\frac{iu}{2}) = z$$

This helps us simplify our trigonometric identity that we derived earlier to:

$$\left(-\frac{i}{2}\right)\frac{\cos(z)}{\sin(z)} = \frac{1}{u} + \sum_{k=1}^{\infty} \frac{2u}{4n^2\pi^2 + u^2}$$

Furthermore, by deriving each trigonometric function by part, we get these following identities:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{and} \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Therefore our trig function becomes :

$$\left(-\frac{i}{2}\right)\frac{\left(\frac{e^{-u/2}+e^{u/2}}{2}\right)}{\left(\frac{-e^{-u/2}+e^{u/2}}{2i}\right)}$$

Let's simplify the denominator's on our new equation. Notice that the 2 from $\frac{i}{2}$and the 2 from $\cos(z)$ cancel out. That leaves us with:

$$\frac{-i}{2i}$$

And because $\frac{-i}{i} = -1$ we get $\frac{1}{2}$ and we can simplify the equation to we get:

$$= \frac{1}{2}\frac{(e^{-u/2} + e^{u/2})}{(-e^{-u/2} + e^{u/2})}$$

Now let's factor out a $1e^u$

$$= \frac{1}{2} \cdot \frac{(e^{-u/2})(e^u - 1 + 2)}{(e^{-u/2})(-1 + e^u)}$$

$$= \frac{1}{2} + \frac{1}{e^u - 1}.$$

Now let's substitute this equation back into our original equality:

$$\frac{1}{2} + \frac{1}{e^u - 1} = \frac{1}{u} + \sum_{k=1}^{\infty} \frac{2u}{4n^2\pi^2 + u^2}$$

Then let's multiply u from $\frac{1}{u}$ and subtract by one yielding:

$$\frac{u}{2} + \frac{u}{e^u - 1} - 1 = \sum_{k=1}^{\infty} \frac{2u}{4n^2\pi^2 + u^2} \tag{4.1}$$

Recall that:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

Therefore

$$\frac{u}{2} + \frac{u}{e^u - 1} - 1 = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} + \frac{u}{2} - 1$$

Now if we cancel out like terms we see that

$$\frac{u}{e^u - 1} = \sum_{k=2}^{\infty} \frac{B_k x^k}{k!}$$

And when we change the counter from k=2 to k=1 we get

$$= \sum_{k=1}^{\infty} \frac{B_{2k} u^{2k}}{(2k)!} \tag{4.3}$$

Then bringing everything back together we can write everything in the terms of:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2u^2}{4k^2\pi^2+u^2} &= \sum_{k=1}^{\infty} \frac{2u^2}{(2k\pi)^2} \left(\frac{1}{1+\left(\frac{u}{2k\pi}\right)^2} \right) \\
&= \sum_{k=1}^{\infty} \frac{2u^2}{(2k\pi)^2} \sum_{j=0}^{\infty} (-1)^j \left(\frac{u}{2\pi k}\right)^{2j} \\
&= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} 2(-1)^j \left(\frac{u}{2\pi k}\right)^{2j+2} \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2(-1)^{j-1} \left(\frac{u}{2\pi k}\right)^{2j} \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{2(-1)^{j-1}}{(2\pi)^{2j}} \left(\frac{1}{k^{2j}}\right) u^{2j} \\
&= \sum_{j=1}^{\infty} \frac{2(-1)^{j-1}}{(2\pi)^{2j}} \sum_{k=1}^{\infty} \left(\frac{1}{k^{2j}}\right) u^{2j} \\
&= \sum_{j=1}^{\infty} \frac{2(-1)^{j-1}}{(2\pi)^{2j}} \zeta(2j) u^{2j}. \qquad (4.4)
\end{aligned}$$

We can change the index from j to k, and substitute (4.2) and (4.3) back into equation (4.1) which yields:

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} u^{2k} = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{(2\pi)^{2k}} \zeta(2k) u^{2k}.$$

Finally, we can equate the corresponding coefficients:

$$\frac{B_{2k}}{(2k)!} = \frac{2(-1)^{k-1}}{(2\pi)^{2k}} \zeta(2k)$$

$$\Downarrow$$

$$\frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!} = \zeta(2k).$$

Therefore completing our proof. □

5. How $\zeta(s)$ Relates to Primes

5.1. **Prime Number Theorem.** In 1792 or 1793, Gauss found himself working on a problem on the distribution of prime numbers. He had a table of prime number counts that he continually added to, and he observed that the density of prime numbers appeared to be on average:

Theorem 5. $\pi(x) \sim \frac{x}{\ln x}$

x	$\pi(x)$	$\pi(x) - \frac{x}{\log x}$	$\frac{\pi(x)}{x / \log x}$
10	4	−0.3	0.921
10^2	25	3.3	1.151
10^3	168	23	1.161
10^4	1 229	143	1.132
10^5	9 592	906	1.104
10^6	78 498	6 116	1.084
10^7	664 579	44 158	1.071
10^8	5 761 455	332 774	1.061
10^9	50 847 534	2 592 592	1.054
10^{10}	455 052 511	20 758 029	1.048
10^{11}	4 118 054 813	169 923 159	1.043

FIGURE 4. Values of $\pi(x)$ vs. $\mathrm{Li}(x)$

This theorem is known as the **Prime Number Theorem**, and it describes the approximate distributions of primes.

Gauss later refined it to:

$$\pi(x) \sim \mathrm{Li}(x) = \int_2^x \frac{dt}{\ln x} \quad as \quad x \to \infty$$

However, one important thing to note is that Gauss's Prime Number Theorem does not use the Zeta function. As mathematicians began to observe the Riemann hypothesis, they came up with two more prime number theorems.

Keep in mind that both of these prime number theorems will use Mangoldt's function, denoted as:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and some integral } k > 0 \\ 0 & \text{if otherwise} \end{cases}$$

Theorem 6. *Suppose there exists $\theta < 1$ such that*

$$Z \subset \{s \in \mathbb{C} \mid 1 - \theta \leq \Re(s) \leq \theta\}.$$

Then as $x \to \infty$

$$\sum_{n \leq X} \Lambda(n) = X + O(X^\theta (\log X)^2).$$

Theorem 7. *Suppose there exists an $\alpha < 1$ such that $x \to \infty$*

$$\sum_{n \leq X} \Lambda(n) = X + O(X^\alpha)$$

then

$$Z \subset \{s \in \mathbb{C} \mid \Re(s) \leq \alpha$$

However, these two theorems have not been useful simply because there have not been θ or α found which satisfies both of them. A proof of this theorem does not require the Riemann hypothesis to be true, but if it was true, it would improve the proof.

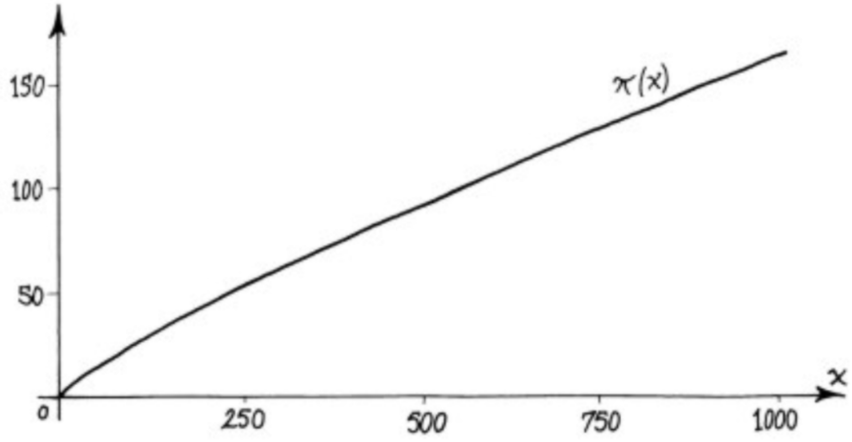

FIGURE 5. Graph of $\pi(x)$

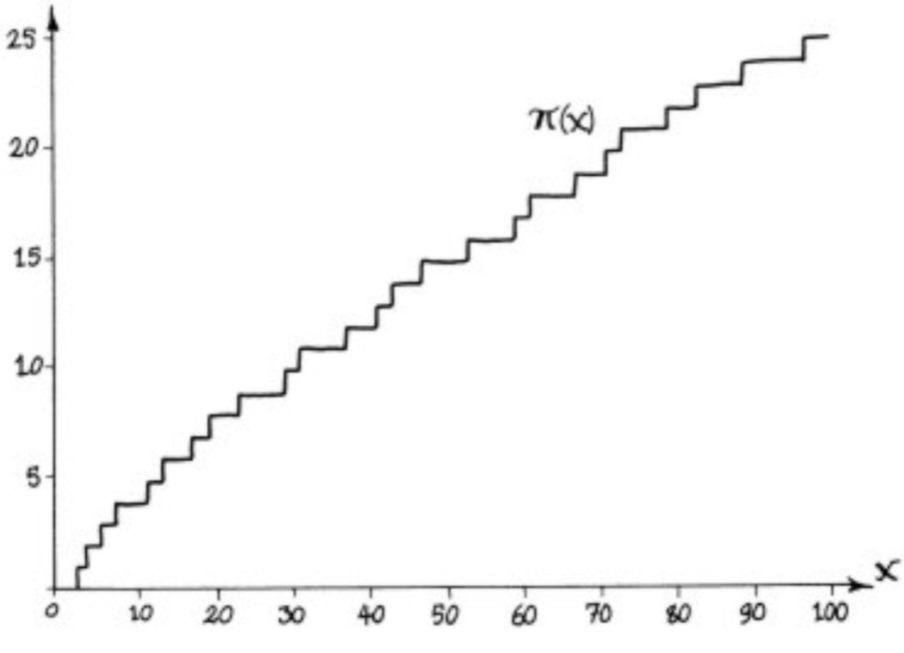

FIGURE 6. Graph of Prime Staircase Function

5.2. **Riemann Hypothesis.** Let's begin with the following definition:

Definition 6. $Z = \{s \in \mathbb{C} \mid \zeta(s) = 0 \text{ and } 0 < \Re(s) < 1\}$

With Z denoting the nontrivial zeros of the Riemann zeta function, Riemann hypothesized that:

$$Z \subset \{s \in \mathbb{C} \mid \Re(s) = 1/2\}$$

Riemann essentially used analytic continuation to extend the Riemann zeta function to the complex plane, which allowed for a link to prime numbers.

$$\pi(x) \sim \frac{x}{\ln x} \text{ as } x \to \infty$$
$$\mathrm{Li}(x) \sim \frac{x}{\ln x} \text{ as } x \to \infty$$

Thus,

$$\mathrm{Li}(x) \sim \pi(x) \text{ as } x \to \infty$$

Without the Riemann hypothesis, the error bound for the prime counting function would look something like this:

$$\pi(x) = \mathrm{Li}(x) + O(xe^{-\sqrt{\log(x)}})$$

If the Riemann hypothesis is true, a much tighter error bound is created:

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}(\log(x))^2)$$

This allows us to get a greater estimate on the distribution of primes!

5.3. **Current Approaches.**

5.3.1. *Initial Attempts.* Mathematicians de la Vallée Poussin and Hadamard proved independently in 1896 that $\zeta(s)$ does not vanish when the real part of s is equal to 1. This led to a proof of the Prime Number Theorem. [2]

Over the years, mathematicians were able to generate two equivalent functions to the Riemann hypothesis:

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{1/2+\epsilon}) \text{ for all } \epsilon > 0$$

The next equivalence used the Möbius function $\mu(n)$ such that:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 - \frac{1}{p^s})$$

to express the Riemann hypothesis as

$$M(x) = \sum_{n \le x} \mu(n)$$

It seems easier to work with $M(x)$ than $\pi(x)$, and Dutch mathematician Thomas Joannes Stieltjes let the mathematical world know that he had a proof that $M(x) = O(x^{1/2+\epsilon})$.

Hadamard even refers to this claim in his 1896 paper, but the proof was never published.

Similarly, Mertens made the conjecture that

$$|M(x)| \leq \sqrt{x}$$

but it was later proven false by Odlyzko and te Riele in 1985.

5.3.2. *Later Efforts.* In the early 1900s, Hardy and Littlewood gave the proof that there were infinitely many zeros on the critical line. They also found the approximate functional equation for $\zeta(s)$:

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\delta}) \ x \to \infty$$

[4]

Later, Siegel refined it into a more precise version, called the Riemann-Siegel formula: [7]

Set $N = \lfloor (t/2\pi)^{1/2} \rfloor$(the integer part of $(t/2\pi)^{1/2}$), $p = (t/2\pi)^{1/2} - N$.

Then

$$Z(t) = 2\sum_{n=1}^{N} n^{-1/2} \cos[\vartheta(t) - t\log n] + R$$

where

$$\vartheta = \Im\left[\log \Pi\left(\frac{it}{2} - \frac{3}{4}\right)\right] - \frac{t}{2}\log\pi]$$

and

$$R \approx (-1)^{N-1}(\frac{t}{2\pi})^{-1/4}[C_0 + C_1(\frac{t}{2\pi})^{-1/2} + C_2(\frac{t}{2\pi})^{-2/2} + C_3(\frac{t}{2\pi})^{-3/2} + C_4(\frac{t}{2\pi})^{-4/2}]$$

with

$$C_0 = \psi(p) = \frac{\cos[2\pi(p^2 - p - 1/16)]}{\cos(2\pi p)},$$

$$C_1 = -\frac{1}{96\pi^2}\psi^{(3)}(p),$$

$$C_2 = \frac{1}{18432\pi^4}\psi^{(6)}(p) + \frac{1}{64\pi^2}\psi^{(2)}(p),$$

$$C_3 = -\frac{1}{5308416\pi^6}\psi^{(9)}(p) - \frac{1}{3840\pi^4}\psi^{(5)}(p) - \frac{1}{64\pi^2}\psi^{(1)}(p),$$

$$C_4 = \frac{1}{2038431744\pi^8}\psi^{(12)}(p) + \frac{11}{5898240\pi^6}\psi^{(8)}(p) + \frac{19}{24576\pi^4}\psi^{(4)}(p) + \frac{1}{128\pi^2}\psi(p).$$

This allowed mathematicians to easily verify solutions along the critical line!

Later, Hardy and Littlewood were later able to find the convexity bound, or that

$$\zeta(1/2 + it) = O(t^{1/4+\epsilon})$$

5.4. **Recent Work.**

5.4.1. *Rolle's Theorem.* Rolle's theorem, a theorem commonly used in entry-level calculus courses, says the following:

Theorem 8. *If a continuous and differentiable function starts and ends at the same y-value over an interval, it must have at least one point where its slope is zero.*

Levinson attempted a converse of Rolle's theorem, asking that if $\xi'(s)$ had a certain proportion of zeros on a line, then so did ξ and for the first and second derivative. This argument has not been proven yet.

5.4.2. *Pólya's Analysis.* Pólya studied the Fourier transform of $\Xi(t) := \xi(\frac{1}{2} + it)$, of which the Riemann hypothesis asserts that all zeroes of $\Xi(t)$ are real. It can be computed explicitly, and one idea is to study classes of reasonable functions whose Fourier transforms have all real zeros and then to prove that $\Xi(t)$ are in the class.

5.4.3. *Probabilistic Models.* In probability, researchers have examined the ξ-function and how it relates to Brownian motion:

$$2\xi(s) = E(Y^s)$$

where

$$Y := \sqrt{\frac{2}{\pi}}(\max_{t\in[0,1]} b_1 - \min_{t\in[0,1]} b_2).$$

5.4.4. *Weil's Explicit Formula.* Weil was able to prove a generalization of Riemann's formula, thereby proving that there was a duality between primes and zeros of ζ:

$$\sum_{\gamma} h(\gamma) = 2h(\frac{i}{2}) - g(0)\log(\pi) + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(\frac{1}{4} + \frac{1}{2}ir)dr - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n).$$

Using this formula, he was able to give a criterion for the Riemann Hypothesis, that it only holds for every admissible function h of a specific form:

$$\sum_{\gamma} h(\gamma) > 0$$

This was later refined into Xian Jin-Li's criterion, which says that the Riemann hypothesis is true if and only if $\lambda_n \geq 0$ for each $n = 1, 2 \ldots$ where

$$\lambda_n = \sum_{\rho}(1 - (1 - (1/\rho)^n$$

5.5. **Evidence for the Riemann Hypothesis.** Over the years, many mathematicians have collected evidence for the Riemann hypothesis.

First, van de Lune proved that the first 10 billion zeroes fall on the critical line. Other computing programs have claimed to verify the first 100 billion zeroes. Many zeros can also be proven to be on the line.

Second, for all random sequences of -1's and +1's, the summatory function up to x is bounded by $x^{1/2+\epsilon}$.

Finally, the Riemann hypothesis is symmetric, and if it was false there would be strange irregularities in the distribution of primes.

6. Additional Proofs

Here is neat and shorter proof from our method in Section 3 that shows how the harmonic series diverges.

Proof. Let's assume the series converges to some value N i.e

$$\begin{aligned} N &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\ N &\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{6} + \frac{1}{6}} + \underbrace{\frac{1}{8} + \frac{1}{8}} + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ &= \frac{1}{2} + N \end{aligned}$$

This is a contradiction because $0 \neq \frac{1}{2}$, so the series **diverges**. ☐

[3]

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- If you have any questions and/or comments please feel free to send us an email, linked below!

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