

# Group Theory and Coxeter Groups

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## Contents

<b>1</b>	<b>Abstract</b>	<b>2</b>
<b>2</b>	<b>Introduction</b>	<b>2</b>
2.1	Definition of a Group . . . . .	2
2.2	The Dihedral Group . . . . .	4
2.3	Homomorphisms and Isomorphisms . . . . .	7
<b>3</b>	<b>Generators and Relations</b>	<b>7</b>
3.1	Definitions . . . . .	7
3.2	Cyclic Groups . . . . .	8
3.3	Free Groups . . . . .	9
3.4	Presentation of Groups . . . . .	10
<b>4</b>	<b>Coxeter Groups and Diagrams</b>	<b>11</b>
4.1	Definitions . . . . .	11
4.2	Diagrams . . . . .	12
4.3	Connected Components . . . . .	13
<b>5</b>	<b>Linear Algebra</b>	<b>14</b>
5.1	Vector Spaces . . . . .	15
5.2	Bilinear Forms and Orthogonality . . . . .	16
5.3	Representations . . . . .	18
5.4	Representations of a Coxeter Group . . . . .	19
<b>6</b>	<b>Classifying Finite Coxeter Groups</b>	<b>20</b>
6.1	Finiteness and the associated Bilinear Form . . . . .	20
6.2	The Diagrams of all Finite Coxeter Groups . . . . .	20

# 1 Abstract

This paper introduces Coxeter groups through the broader language of group theory, beginning with groups, subgroups, order, homomorphisms, and isomorphisms. The dihedral group serves as a motivating example, which algebraically describes rotations and reflections, and is later reinterpreted as a Coxeter group. We then study generators and relations as tools for describing groups abstractly while noting the limitations of group presentations in general. The Adian-Rabin Theorem shows that group presentations are too limited to determine many group properties algorithmically, but Coxeter groups are special in that their presentations allow finite Coxeter groups to be classified. Coxeter groups are introduced as involutions whose pairwise products have specific orders, and Coxeter diagrams provide a visual way to encode these relations. To connect this algebraic definition with geometry, we develop the necessary linear algebra, including vector spaces, bilinear forms, orthogonality, reflections, and representations. Finally, we will present the finite Coxeter groups and their diagrams.

## 2 Introduction

### 2.1 Definition of a Group

**Definition 2.1.** A group is an ordered pair  $(G, *)$  of a set  $G$  and a binary operation  $*$  such that

1. there is an *identity* element  $e$  in  $G$  such that  $a * e = e * a = a$  for all  $a$  in  $G$
2. for all  $a$  in  $G$ , there is an *inverse* element  $a^{-1}$  such that  $a * a^{-1} = a^{-1} * a = e$
3. the associative property holds, i.e.,  $a * (b * c) = (a * b) * c$  for any  $a, b, c$  in  $G$

*Remark 2.1.* When the operation is obvious from context, we omit the operation symbol “ $*$ ” and simply say “ $G$  is a group.” In this case, for  $a, b$  elements of  $G$ , we can write  $ab$  instead of  $a * b$ .

*Remark 2.2.* The commutative property is not required, i.e.,  $ab$  is not necessarily equal to  $ba$  for general  $a, b$  in  $G$ . Groups that follow the commutative property are said to be *abelian*. An example of an abelian group is the integers under addition; an example of a nonabelian group is the dihedral group, which will be covered in section 2.2.

**Definition 2.2.** For a finite group  $G$ , we say that the *order of a group*  $G$  (written as  $|G|$ ) is the number of elements in  $G$ . [5]

**Definition 2.3.** The *order of an element*  $a$  in  $G$  (written as  $|a|$ ) is the smallest positive integer  $n$  such that  $a^n = e$  (if such an  $n$  exists). In this case,  $a$  has *finite order*. If no such  $n$  exists, then  $a$  has *infinite order*. In particular,  $|e| = 1$  because  $e^1 = e$ .

**Claim 2.1.** *The integers under addition— $(\mathbb{Z}, +)$ —form a group.*

*Proof.* The identity element of this group is 0 because  $a + 0 = 0 + a = a$  for all  $a \in \mathbb{Z}$ . For each  $a \in \mathbb{Z}$ , the inverse element is its additive inverse  $-a$  because  $a + (-a) = (-a) + a = 0$ . Finally, addition is associative:  $a + (b + c) = (a + b) + c$  for all  $a, b, c \in \mathbb{Z}$ . Because  $\mathbb{Z}$  under addition satisfies the three group axioms,  $(\mathbb{Z}, +)$  is a group.  $\square$

Since there are infinitely many integers, the order of this group is  $\infty$ . The order of the identity (0) is 1, and the order of any nonzero integer is  $\infty$ .

**Definition 2.4.** A nonempty subset  $H$  of a group  $G$  is said to be a *subgroup* of  $G$  if  $x, y$  elements of  $H$  implies  $x^{-1}, xy$  are in  $H$  (i.e.  $H$  is closed under the group operation and taking inverses). We write this relationship as  $H \leq G$ .

**Example.**  $(\mathbb{Z}, +)$  has many subgroups [5], one of which is the set of even integers  $2\mathbb{Z} \neq \emptyset$ . This is because the sum of two even integers is still an even integer, and the additive inverse of any even integer is still an even integer. Hence  $2\mathbb{Z} \leq \mathbb{Z}$ .

**Definition 2.5.** Let  $n$  be a fixed positive integer. For  $a, b \in \mathbb{Z}$ , we write

$$a \equiv b \pmod{n}$$

if  $a$  and  $b$  have the same remainder when divided by  $n$  or, equivalently,  $b - a$  is an integer multiple of  $n$ .

**Definition 2.6.** The *residue class* of  $a$  modulo  $n$ , written  $\bar{a}$ , is the set of all integers congruent ( $\equiv$ ) to  $a$  modulo  $n$ :

$$\bar{a} = \{a + kn \mid k \in \mathbb{Z}\} = \{a, a \pm n, a \pm 2n, a \pm 3n, \dots\}.$$

**Definition 2.7.** The set of all residue classes modulo  $n$  is called the *integers modulo  $n$*  (or the *integers mod  $n$* ) and is denoted by  $\mathbb{Z}/n\mathbb{Z}$ .

Since  $0, 1, \dots, n - 1$  are the  $n$  possible remainders after division by  $n$ , there are exactly  $n$  residue classes mod  $n$ :

$$\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}.$$

Addition in  $\mathbb{Z}/n\mathbb{Z}$  is defined by

$$\bar{a} + \bar{b} = \overline{a + b}.$$

That is, we add representatives and then take the residue class of the result.

**Example.** Suppose  $n = 5$ . Then

$$\mathbb{Z}/5\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}.$$

For example,

$$\bar{3} + \bar{4} = \overline{3 + 4} = \bar{7} = \bar{2}.$$

The answer does not depend on the representatives you choose. Since  $\bar{3} = \bar{8}$  and  $\bar{4} = \bar{9}$  in  $\mathbb{Z}/5\mathbb{Z}$ , we also get

$$\bar{8} + \bar{9} = \overline{8 + 9} = \overline{17} = \bar{2}.$$

**Claim 2.2.** *The integers modulo  $n$  under addition— $(\mathbb{Z}/n\mathbb{Z}, +)$ —form a group.*

*Proof.* The identity element of this group is  $\bar{0}$  because  $\bar{a} + \bar{0} = \bar{0} + \bar{a} = \bar{a}$ . For each  $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$ , the inverse element is  $\overline{-a}$  because  $\bar{a} + \overline{-a} = \overline{-a + a} = \bar{0}$ . Finally, addition is associative:  $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$  for any  $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}/n\mathbb{Z}$ . Because  $\mathbb{Z}/n\mathbb{Z}$  under addition satisfies the three group axioms,  $(\mathbb{Z}/n\mathbb{Z}, +)$  is a group.  $\square$

There are  $n$  distinct elements  $(\bar{0}, \bar{1}, \dots, \overline{n-1})$  in  $\mathbb{Z}/n\mathbb{Z}$ , so the order of this group is  $n$ . All elements in  $\mathbb{Z}/n\mathbb{Z}$  have finite order (of at most  $n$ ), since for any  $a \in \mathbb{Z}/n\mathbb{Z}$ ,  $\underbrace{\bar{a} + \dots + \bar{a}}_{n \text{ times}} \equiv \bar{0}$ .

## 2.2 The Dihedral Group

One of the main reasons groups are useful is that they allow us to describe symmetry in a precise way. The dihedral group is an important example of this idea. It models symmetries of a regular polygon: rotations and reflections.

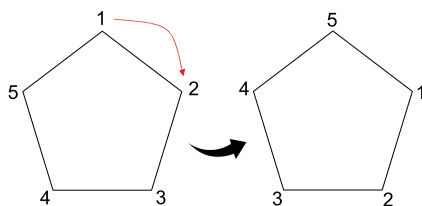
**Definition 2.8.** A *symmetry* of the  $n$ -gon is a rigid motion that maps the  $n$ -gon onto itself.

To start, a regular polygon has rotational symmetries: if we rotate it about its center by certain angles, the polygon looks exactly the same as it did before. For example, rotating a regular  $n$ -gon by  $\frac{2\pi}{n}$  radians sends each vertex to the position of its neighboring vertex, so the polygon is indistinguishable from its original position.

To describe this rotational symmetry more precisely, we label the vertices of this  $n$ -gon  $1, 2, \dots, n$  in clockwise order. Let  $r$  be the clockwise rotation by  $\frac{2\pi}{n}$  radians about the center of the  $n$ -gon. Then  $r$  sends the vertices as follows:

$$1 \mapsto 2, \quad 2 \mapsto 3, \quad \dots, \quad n \mapsto 1.$$

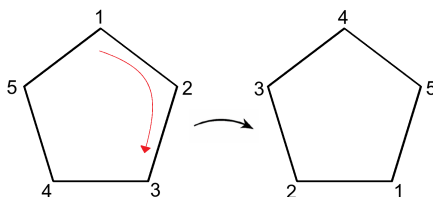
For  $n = 5$ ,  $r$  looks like this:



**Definition 2.9.** The *composition* of two symmetries  $x$  and  $y$ , written  $x \circ y$ , is the symmetry obtained by performing  $y$  first and then  $x$ .

*Remark 2.3.* Because the operation is clear from context, we often omit the symbol  $\circ$  and write  $xy$ .

**Example.** If we apply  $r$  twice, we get the symmetry  $r^2 = r \circ r$ . In other words,  $r^2$  means first applying  $r$ , then applying  $r$  again. For instance, applying  $r^2$  to a regular pentagon ( $n = 5$ ) looks like this:

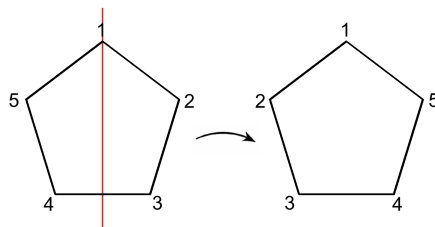


More generally, rotations by multiples of  $\frac{2\pi}{n}$  are also symmetries. Applying  $r$  a total of  $k$  times gives the symmetry  $r^k$ , which sends

$$1 \mapsto (1 + k), \quad 2 \mapsto (2 + k), \quad \dots, \quad n \mapsto n + k$$

with the labels interpreted modulo  $n$  (so that  $n + 1 \equiv 1$ ).

In addition to rotations, there is also reflectional symmetry. Let  $s$  be a reflection across the axis passing through vertex 1 and the center of the polygon. For  $n = 5$ , it looks like this:



**Definition 2.10.** We denote the set of all symmetries of the regular  $n$ -gon by  $D_n$ .

**Claim 2.3.**  $D_n$  is a group (under composition)—we call this group the dihedral group.

*Proof.* The identity element of this group is the “do-nothing” symmetry, which does not alter any vertex labels. We denote this element by  $e$ . Next, for every symmetry  $x$  element of  $D_n$ , there must be a corresponding “undo” symmetry  $x^{-1}$  in  $D_n$  (e.g. “unrotate” or “unflip”). This is because every rigid transformation has an inverse. Finally, composition is associative, so  $D_n$  must satisfy associativity. Hence  $D_n$  satisfies the group axioms, so  $D_n$  is a group.  $\square$

**Claim 2.4.**  $r$  has order  $n$ .

*Proof.* Applying  $r^n$  is a rotation by  $2\pi$ , so it is the same as “doing nothing.” In other words,  $r^n = e$ . Now suppose  $k \in \{1, \dots, n - 1\}$ . Then  $r^k$  is a rotation by  $\frac{2\pi k}{n}$ , which is strictly between 0 and  $2\pi$ . Therefore, it does not return the polygon to its original position;  $r^k \neq e$ . Hence  $n$  is the smallest positive integer such that  $r^n = e$ , and therefore  $|r| = n$ .  $\square$

**Claim 2.5.**  $s$  has order 2.

*Proof.* Since  $s$  is a reflection,  $s \neq e$ , so  $|s| > 1$ . Applying the same reflection twice returns every vertex to its original position, so  $s^2 = e$ . Therefore, the smallest positive integer  $k$  that makes  $s^k = e$  is 2. Hence  $|s| = 2$ .  $\square$

**Claim 2.6.** The dihedral group has order  $2n$ . Moreover, every element of  $D_n$  can be written as compositions of  $r$  and  $s$ .

*Proof.* A symmetry of a regular  $n$ -gon is completely determined by where it sends vertices 1 and 2. Vertex 1 can be sent to any of the  $n$  vertices. Once the location of vertex 1 is chosen, vertex 2 must be sent to either vertex that is adjacent to vertex 1: either the one immediately clockwise or the one immediately counterclockwise. Therefore, there are at most  $n \times 2 = 2n$  symmetries.

These  $2n$  possibilities are all achieved by rotations and reflections. If vertex 2 remains on the same side of vertex 1 as before, then the symmetry is a rotation, having the form  $r^k$  for some  $k \in \{0, 1, \dots, n-1\}$ . If vertex 2 ends up at the opposite vertex of 1, then the symmetry is a reflection followed by a rotation, so it has the form  $r^k s$  for some  $k \in \{0, 1, \dots, n-1\}$ .  $\square$

**Claim 2.7.**  $r^k s = sr^{-k}$  for all positive integer  $k$ .

*Proof.* Recall that  $r$  sends each vertex one step clockwise, so  $r^k$  sends each vertex  $k$  steps clockwise:

$$r(i) = i + 1, \quad r^k(i) = i + k$$

with labels interpreted modulo  $n$ . Also, the reflection  $s$  moves each vertex across the axis:

$$s(i) = (n + 2) - i.$$

Now consider the image of an arbitrary vertex  $i$  under both  $r^k s$  and  $sr^{-k}$ . On one hand,

$$r^k s(i) = r^k(s(i)) = r^k(n + 2 - i) = n + 2 - i + k.$$

On the other hand,

$$sr^{-k}(i) = s(r^{-k}(i)) = s(i - k) = n + 2 - (i - k) = n + 2 - i + k.$$

Thus  $r^k s$  and  $sr^{-k}$  send every vertex  $i$  to the same vertex  $n + 2 - i + k$ . Hence

$$r^k s = sr^{-k}.$$

$\square$

The relation  $rs = sr^{-1}$  is useful because it lets us rewrite any composition of  $r$  and  $s$  into a standard form. Whenever an  $s$  appears to the right of a power of  $r$ , we can move it to the left using

$$r^k s = sr^{-k}.$$

After doing this repeatedly, all the  $s$  can be moved to the left. Since  $s^2 = e$ , pairs of  $s$  must cancel, leaving either no  $s$  or exactly one  $s$ . The remaining powers of  $r$  can then be combined and reduced modulo  $n$  since  $r^n = e$ .

Thus every element of  $D_n$  can be uniquely written in one of the forms

$$r^k \quad \text{or} \quad sr^k \quad (0 \leq k < n).$$

Therefore,

$$D_n = \{e, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\},$$

and these elements satisfy the equations

$$r^n = s^2 = 1, \quad rs = sr^{-1}.$$

We can write these facts as  $D_n = \langle s, r \mid s^2 = e, r^n = e, rs = sr^{-1} \rangle$ . This tells us that every element of  $D_n$  can be written as the composition of multiples of  $s, r$ , with these the only relations that  $s$  and  $r$  satisfy.

The dihedral groups are a special case of Coxeter groups, which will be introduced later.

## 2.3 Homomorphisms and Isomorphisms

**Definition 2.11.** Let  $(G, \star)$  and  $(H, \diamond)$  be groups. A function  $\varphi : G \rightarrow H$  is called a *homomorphism* if it preserves the group operation. That is,

$$\varphi(x \star y) = \varphi(x) \diamond \varphi(y) \quad \forall a, b \text{ in } G.$$

*Remark 2.4.* When group operations for  $G$  and  $H$  are omitted, we simply write the homomorphism condition as

$$\varphi(xy) = \varphi(x)\varphi(y).$$

**Definition 2.12.** A function  $\varphi : G \rightarrow H$  is *injective* if for any  $g_1, g_2$  in  $G$ ,  $g_1 \neq g_2$  implies  $\varphi(g_1) \neq \varphi(g_2)$ .

**Definition 2.13.** A function  $\varphi : G \rightarrow H$  is *surjective* if for any  $h$  element of  $H$ , there exists some  $g$  in  $G$  such that  $\varphi(g) = h$ .

**Definition 2.14.** A function  $\varphi : G \rightarrow H$  is *bijective* if it is both injective and surjective.

**Definition 2.15.** A function  $\varphi : G \rightarrow H$  is called an *isomorphism* if

1.  $\varphi$  is a homomorphism and
2.  $\varphi$  is a bijection.

If such isomorphism exists, then  $G$  and  $H$  are said to be *isomorphic*, written  $G \cong H$ .

Isomorphisms provide the notion of when two groups have an essentially identical structure.

## 3 Generators and Relations

The elements  $r$  and  $s$  in the dihedral group are examples of a general phenomenon of groups where all elements of the dihedral group can be obtained through combinations of  $r$  and  $s$ . In this sense, the group can be understood through the behavior of these elements.

### 3.1 Definitions

**Definition 3.1.** The *generators* of a group  $G$  is the set  $S$  of elements of  $G$  created such that every element in  $G$  can be written as a finite product of elements of  $S$  and their inverses. We call  $S$  a *generating set*. [5]

**Definition 3.2.** The *relations* in a group  $G$  are all of the equations that its generators satisfy.

**Definition 3.3.** A *word* is any finite product of generators and their inverses. We can write words as  $s_1^{a_1} s_2^{a_2} \cdots s_n^{a_n}$  for  $s_i$  contained in the set of the generators, and  $a_i$  element of  $\mathbb{N}$ , i.e.,  $s_i \in S \cup S^{-1} \cup \{e\}$  for set  $S$  of the generators.

## 3.2 Cyclic Groups

The fundamental arithmetic and algebra structuring mathematics can be viewed from the perspective of group theory. Many operations, such as addition and arithmetic modulo  $n$ , can be represented as a specific type of groups called cyclic groups.

**Definition 3.4.** A *cyclic group* is a group that can be generated by a single element. [4]

That is, for cyclic group  $C$ , there exists an element  $x$  in  $C$  such that  $C = \{x^n \mid n \in \mathbb{Z}\}$ . We can represent  $C$  being *generated by  $x$*  through writing  $C = \langle x \rangle$ . A cyclic group can have multiple generators. For example, the cyclic group of the set of integers  $\mathbb{Z}$  with addition can be generated by both 1 and -1. Also, as we denote the elements of a cyclic group with  $x^n$ , we can see that by the laws for exponents, cyclic groups are abelian.

Cyclic groups are classified as either *finite* or *infinite*. For a finite cyclic group with generator  $x$ , there must be some  $n \in \mathbb{Z}$  such that  $x^n = e$ . That is,  $C = \{e, x, x^2, \dots, x^{n-1}\}$ . Alternatively, in infinite cyclic groups, all of the powers  $x^k$  have distinct values for every integer  $k$ , so there is no  $n \in \mathbb{Z}$  such that  $x^n = e$ .

**Claim 3.1.** *The subgroup of  $D_n$  consisting of the rotations of the  $n$ -gon is a finite cyclic group.*

*Proof.* For  $D_n = \langle r, s \mid r^n = s^2 = e, rs = sr^{-1} \rangle$ , let  $H$  be the subgroup consisting of the rotations of the  $n$ -gon. This is cyclic because

Additionally, we claim that  $|H| = n$ , so this is finite. Note that the order of the generator  $r$  is  $n$ , so there are at least  $n$  elements, namely,  $e, r, r^2, \dots, r^{n-1}$ , which are all distinct. Furthermore, any power  $r^p$  for any  $p \in \mathbb{N}$  can be written as  $r^k$  for  $k = 0, 1, \dots, n-1$ . We see this through using the Division Algorithm, given that we can write  $p = qn + k$  where  $q, k \in \mathbb{Z}$ ,  $0 \leq k < n$ , so then

$$r^{qn+k} = (r^n)^q r^k = (e)^q r^k = r^k.$$

□

For example, consider  $H$  the subgroup of rotations of the regular pentagon. Then  $H = \{e, r, r^2, r^3, r^4\}$ , and  $r^{24}$  can be written as  $r^{5 \cdot 4 + 4} = r^4$ .

**Lemma 3.2** (The Well Ordering Principle). If  $S$  is any nonempty subset of  $\mathbb{Z}^+$ , there exist some element  $m$  in  $S$  such that  $m \leq s$  for all  $s$  in  $S$ , and  $m$  is called the *minimal element* of  $S$ .

We will use this principle to prove the following theorem.

**Theorem 3.3.** *Every subgroup of a cyclic group is cyclic.* [4]

*Proof.* Let  $G$  be a cyclic group with generator  $g$ . Let  $H \leq G$ . If  $H = \{e\}$ , then  $H$  is generated by the identity element, so it is cyclic. Then, assume that  $H$  is non-trivial. We want to show that  $H$  is generated by some  $g^n$ .

Since  $H \neq \{e\}$ , there must exist some  $k \neq 0$  such that  $g^k$  is an element in  $H$ . If  $k < 0$ , then since  $H$  is closed under inverses,  $g^{-k}$  belongs in  $H$ , so  $H$  will always contain some positive power of  $g$ . Let

$$K = \{b \mid b \in \mathbb{Z}^+ \text{ and } g^b \in H\}$$

By the Well Ordering Principle, there is a minimum element of  $K$ , which we will call  $n$ . Now, since  $G = \langle g \rangle$ , all elements  $h$  in  $H$  can be written as  $h = g^a$  with  $a \in \mathbb{Z}$ . We want to show that all  $h$

can be written as a power of the smallest positive element  $g^n$  in  $H$ , i.e.,  $n \mid a$ . By the Division Algorithm,  $a = nq + r$  where  $0 \leq r < n$ . It follows that

$$h = g^{nq+r} = (g^n)^q g^r, \text{ so } g^r = h(g^n)^{-q}$$

Since  $g^n$  is an element of  $H$  and  $H$  is closed under inverses and products,  $(g^n)^{-q}$  is also in  $H$ . Then, both  $h$  and  $(g^n)^{-q}$  are in  $H$ , so  $g^r$  must also be contained in  $H$ . However,  $n$  is the smallest element of  $K$  and  $r < n$ , so  $r$  must equal 0. Thus, we have  $a = nq$ , so  $h = (g^n)^q$  is contained in  $\langle g^n \rangle$ . This proves that  $g^n$  generates  $H$ , so  $H$  is cyclic. □

### 3.3 Free Groups

Every group can be obtained from a free group by specifying the relations between its generators. Thus, studying free groups leads to a better understanding of the structure of groups in general.

**Definition 3.5.** A *free group*  $F(S)$  is a group generated by set  $S$  where there are no relations satisfied by the elements in  $S$  except for the group axioms. [5]

Explicitly, the elements of the free group would be written as finite products of elements of  $S$  and their inverses. For example, some elements of the free group generated by  $S = \{r, s\}$  are  $r, s, sr^{-1}, rsr$ , and so on. We call these expressions reduced words and define the group operation as concatenation followed by reduction.

**Definition 3.6.** A *reduced word* is a word in which there are no pair of terms of the form  $ss^{-1}$  or  $s^{-1}s$  for  $s \in S \cup S^{-1}$ .

**Definition 3.7.** *Concatenation* of words is the operation of forming a new word by placing one word directly after another.

For example, the concatenation of  $aba^{-1}$  with  $ab^{-1}c^2$  is  $aba^{-1}ab^{-1}c^2$ . However, concatenation alone cannot define the group operation on free groups, since concatenation of two reduced words does not necessarily produce a reduced word, as seen in the above example. Hence, concatenation needs to be followed by reduction.

**Definition 3.8.** *Reduction* is the operation of deleting adjacent pairs of terms that are inverses of each other.

Reducing our earlier example,  $aba^{-1}ab^{-1}c^2$ , we would first cancel the adjacent pair  $a^{-1}a$ , yielding  $abb^{-1}c^2$ . Similarly, we delete  $bb^{-1}$ , resulting in  $ac^2$ . Hence, reduced words can also be viewed as words that are left unchanged by reduction.

Now, we can define the binary operation on  $F(S)$  more formally. First, let  $s_1^{\sigma_1} s_2^{\sigma_2} \dots s_n^{\sigma_n}$  denote a reduced word where  $s_1, s_2, \dots, s_n$  are not necessarily distinct elements of  $S \cup S^{-1}$  and  $\sigma_1, \sigma_2, \dots, \sigma_n$  have values of either 1 or  $-1$ . In this way, terms of form  $s_j^k$  for  $k \geq 1$  can be written as  $\underbrace{s_j^1 \cdots s_j^1}_{k \text{ times}}$ , and

similarly when  $\sigma_j \leq -1$ . This specific notation would help us represent the different ways reduction can happen after the concatenation of two reduced words.

Consider the group operation on the reduced words  $R = r_1^{\rho_1} r_2^{\rho_2} \dots r_m^{\rho_m}$  and  $S = s_1^{\sigma_1} s_2^{\sigma_2} \dots s_n^{\sigma_n}$ . First, assume that  $m \leq n$ . Let  $1 \leq i \leq m+1$  be the smallest integer such that  $s_i^{\sigma_i} \neq r_{m-i+1}^{-\rho_{m-i+1}}$ . Then the

product  $RS = (r_1^{\rho_1} r_2^{\rho_2} \dots r_m^{\rho_m})(s_1^{\sigma_1} s_2^{\sigma_2} \dots s_n^{\sigma_n})$  is defined under the following cases: partial cancellation of both words, full cancellation of  $R$  but partial cancellation of  $S$ , and the complete cancellation of both words.

First, the partial cancellation of  $R$  and  $S$  in  $(r_1^{\rho_1} r_2^{\rho_2} \dots r_m^{\rho_m})(s_1^{\sigma_1} s_2^{\sigma_2} \dots s_n^{\sigma_n})$  would look like  $r_1 \dots r_k^{\rho_k} s_j^{\sigma_j} \dots s_n^{\sigma_n}$  for some  $k$  and  $j$ . Since we defined  $i$  as the smallest integer such that  $s_i^{\sigma_i}$  is the first term that cannot be canceled, this case would occur when  $i$  is less than  $m$ . Also, we know that  $s_i^{\sigma_i}$  must start the sequence of  $s_n^{\sigma_n}$  terms in the result, i.e.,  $j = i$ . Now, notice that we must have canceled the same amount of  $s_n^{\sigma_n}$  terms as  $r_n^{\rho_n}$  terms. This means that we have canceled out  $i - 1$  terms, making the last  $r_m^{\rho_m}$  term equal to  $r_{m-i+1}^{\rho_{m-i+1}}$ . In summary, if  $i \leq m$ , then  $RS = r_1^{\rho_1} \dots r_{m-i+1}^{\rho_{m-i+1}} s_i^{\sigma_i} \dots s_n^{\sigma_n}$ .

Next, the second case occurs when  $i = m + 1$  and is not greater than  $n$ . This signifies that all of the  $r_m^{\rho_m}$  terms will be canceled out, leaving only  $s_n^{\sigma_n}$  terms. Since we canceled out  $m$  terms from  $R$ , then the first  $s_n^{\sigma_n}$  term would be  $s_{m+1}^{\sigma_{m+1}}$ . Thus, when  $i = m + 1 \leq n$ ,  $RS$  reduces to  $s_{m+1}^{\sigma_{m+1}} \dots s_n^{\sigma_n}$ . Finally, the last case is when both  $R$  and  $S$  cancel fully, leaving the identity. This implies that  $R = S^{-1}$ . This case occurs when  $n = m$  and  $i = m + 1$ .

Formally, we can define the product of two reduced words to be:

$$(r_1^{\rho_1} r_2^{\rho_2} \dots r_m^{\rho_m})(s_1^{\sigma_1} s_2^{\sigma_2} \dots s_n^{\sigma_n}) = \begin{cases} r_1^{\rho_1} \dots r_{m-i+1}^{\rho_{m-i+1}} s_i^{\sigma_i} \dots s_n^{\sigma_n}, & \text{if } i \leq m \\ s_{m+1}^{\sigma_{m+1}} \dots s_n^{\sigma_n}, & i = m + 1 \leq n \\ 1, & i = m + 1 \text{ and } m = n \end{cases}$$

Similarly, when  $m \geq n$ :

$$(r_1^{\rho_1} r_2^{\rho_2} \dots r_m^{\rho_m})(s_1^{\sigma_1} s_2^{\sigma_2} \dots s_n^{\sigma_n}) = \begin{cases} r_1^{\rho_1} \dots r_{m-i+1}^{\rho_{m-i+1}} s_i^{\sigma_i} \dots s_n^{\sigma_n}, & \text{if } i \leq n \\ r_1^{\rho_1} \dots r_{m-n}^{\rho_{m-n}}, & i = n + 1 \leq m \\ 1, & i = n + 1 \text{ and } n = m \end{cases}$$

**Definition 3.9.** Let  $F(S)$  be the free group on  $S$ . Then the set  $S$  is called a set of *free generators* or a *free basis* of  $F$ .

**Definition 3.10.** The *rank* of the free group  $F(S)$  is  $|S|$ , the cardinality of generating set  $S$ .

*Remark 3.1.* The integers with addition can be viewed as the free group on one generator, which is 1. Since  $S = \{1\}$ , the rank of this free group would be 1.

### 3.4 Presentation of Groups

So far, many of the groups that we have encountered have additional conceptual meaning beyond its group structure. While this is useful for understanding their possible applications and perspectives, some groups are inherently more abstract, thus requiring more abstract representations. One way to address this is to view groups through their generators and relations, also known as the group presentation. We have used presentations when defining the dihedral group earlier, and they will help us define Coxeter groups.

**Definition 3.11.** A *group presentation* is an expression  $G = \langle S \mid R \rangle$ , where  $S$  is a set of generators and  $R$  is a set of relations, such that  $G$  is the group given by these. [5]

*Remark 3.2.* As mentioned earlier, the presentation of the dihedral group is  $D_n = \langle s, r \mid s^2 = e, r^n = e, rs = sr^{-1} \rangle$ .

**Definition 3.12.** A group  $G$  is *finitely generated* if there is a presentation  $\langle S \mid R \rangle$  such that  $S$  is a finite set.

**Definition 3.13.** Then,  $G$  is *finitely presented* if  $R$  is also a finite set. [7]

**Proposition 3.4.** All groups have presentations that are finite if the group is finite.

**Theorem 3.5** (Adian-Rabin Theorem). *There does not exist an algorithm that takes in a group presentation and determines whether or not the group defined by this presentation has a specific property.* [10]

Group presentations are surprisingly limiting in the amount of information they carry about a group. As the theorem states, we are unable to determine even finiteness or if it is the identity group.

## 4 Coxeter Groups and Diagrams

In the second half of the paper, we will focus on a special type of groups—the Coxeter group. From the Adian-Rabin Theorem, we know that we can't determine many properties of most groups solely from their presentation. However, Coxeter groups are defined in a way such that we can identify finite Coxeter groups purely from their presentation. We will discuss more on these finite groups when we classify them later. Moreover, Coxeter groups represent an interesting intersection between algebra and geometry. Namely, Coxeter groups can be viewed as reflection groups on vector spaces.

### 4.1 Definitions

**Definition 4.1.** Let  $W$  be a group generated by set  $S$ . For generators  $s_i, s_j$  in  $S$ , let  $m_{ij}$  denote the order of the product  $s_i s_j$ .  $W$  is a *Coxeter group* if it has the presentation

$$\langle s_1, s_2, \dots, s_n \in S \mid (s_i s_j)^{m_{ij}} = e \rangle,$$

where  $m_{ii} = 1$ ,  $m_{ij} = m_{ji}$ , and  $m_{ij} > 1$  for  $i \neq j$ . [8]

That is, the square of every generator is the identity, and for any two distinct generators, the order of their product is greater than 1 and depends only on the pair, not the way they are arranged.

**Definition 4.2.** A *Coxeter matrix* is an  $n \times n$  matrix  $M = (m_{ij})_{1 \leq i, j \leq n}$  with natural number entries such that  $m_{ii} = 1$  and  $m_{ij} = m_{ji}$  is greater than 1 for all  $i \neq j$ . Thus, we can view each entry as the order of a product of generators of the Coxeter group.

The Coxeter group together with its generating set,  $(W, S)$ , is called the Coxeter system of type  $M$ .

**Claim 4.1.** *The dihedral group is a Coxeter Group.*

The dihedral group  $D_n$  has a presentation

$$\langle r, s \mid r^n = s^2 = e, rs = sr^{-1} \rangle.$$

Let  $a = s$  and  $b = sr$ . Then the condition  $rs = sr^{-1}$  becomes

$$s(rs)r = s(sr^{-1})r$$

$$\begin{aligned}(sr)^2 &= e \\ b^2 &= e,\end{aligned}$$

and the condition  $s^2 = e$  becomes  $a^2 = e$ . We can see that  $ab = (s)(sr) = s^2r = r$ , so we can write  $r^n = e$  as  $(ab)^n = e$ . Therefore, the new presentation becomes

$$D_n = \langle a, b \mid a^2 = b^2 = (ab)^n = e \rangle.$$

Here,  $D_n$  is a Coxeter group because  $a$  and  $b$  are defined such that  $a^2 = b^2 = e$ , and the products of the two generators (i.e.  $ab$  and  $ba$ ) have the same order  $n > 1$ :

$$\begin{aligned}ba &= (sr)(s) = (r^{-1}s)s = r^{-1} \\ (r^{-1})^n &= (r^n)^{-1} = e^{-1} = e\end{aligned}$$

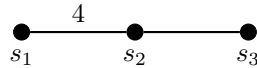
Thus,  $ba$  has order  $n$ . Since the product  $ab$  is defined to have order  $n$ ,  $ab$  and  $ba$  have the same order.

## 4.2 Diagrams

Coxeter groups are represented through *Coxeter diagrams* with generators represented as vertices and edges between generators  $s_i, s_j$  formed by the following rules:

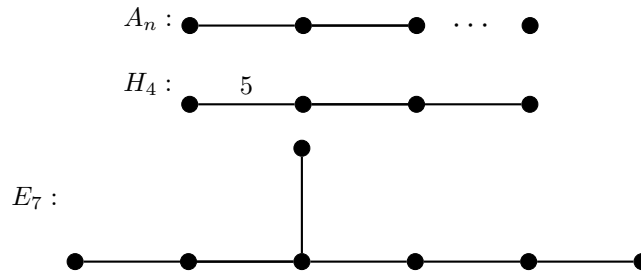
1. If  $m_{ij} = 2$ , no edge is drawn
2. If  $m_{ij} = 3$ , an unlabeled edge is drawn
3. If  $m_{ij} > 3$ , an edge is drawn and labeled with the value of  $m_{ij}$  [9]

For example, let a Coxeter group be  $\langle s_1, s_2, s_3 \mid s_i^2 = e, (s_1s_2)^4 = e, (s_2s_3)^3 = e, (s_1s_3)^2 = e \rangle$ . Then the Coxeter diagram would have 3 vertices,  $s_1, s_2$ , and  $s_3$ . Since  $m_{12} = 4$ , an edge would be drawn between vertices  $s_1$  and  $s_2$  and labeled with 4. Then,  $m_{23} = 3$ , so an unlabeled edge would be drawn between  $s_2$  and  $s_3$ . Finally,  $m_{13} = 2$ , so no edge would be drawn. The resulting Coxeter diagram looks like the following:



This group is called  $B_3$  and represents the symmetry group of the cube. We will discuss similar groups in greater detail when we classify finite Coxeter groups.

Some other examples of Coxeter diagrams are as follows:

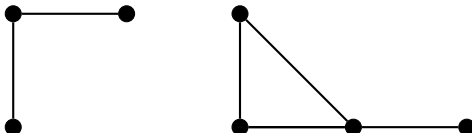


### 4.3 Connected Components

Now, we can define connected components for the Coxeter system  $(W, S)$  of type  $M$ .

**Definition 4.3.** Let  $W$  be a Coxeter group generated by set  $S = \{s_1, s_2, \dots, s_n\}$ . Then, let  $J$  be the largest possible subset of  $[n] = \{1, 2, 3, \dots, n\}$  that satisfies  $m_{jk} = 2$  for all  $j \in J$  and  $k \in [n] \setminus J$ . Then, a *connected component* is the set  $C = \{s_j \mid j \in J\}$ . [9]

Graphically, this means that in the Coxeter diagram of  $W$ , there are no edges connecting the diagram of  $C$  with any other vertices, as shown below.



If a diagram has a single connected component, it is called connected or irreducible. Likewise, the Coxeter group would be called irreducible if its diagram is connected. The examples given in the previous section are all connected diagrams.

**Definition 4.4.** The *direct product*  $G_1 \times G_2 \times \dots \times G_n$  of groups  $G_1, G_2, \dots, G_n$  with operations  $\circ_1, \circ_2, \dots, \circ_n$  respectively is defined as the group with elements of form  $(g_1, g_2, \dots, g_n)$  where  $g_i$  belongs in  $G_i$ . [5] For two elements  $(g_1, g_2, \dots, g_n)$  and  $(h_1, h_2, \dots, h_n)$  in  $G_1 \times G_2 \times \dots \times G_n$ , the group operation is defined as:

$$(g_1, g_2, \dots, g_n)(h_1, h_2, \dots, h_n) = (g_1 \circ_1 h_1, g_2 \circ_2 h_2, \dots, g_n \circ_n h_n)$$

For example, the direct product of  $D_2$  with  $D_3$  is

$$D_2 \times D_3 = \{(g, h) \mid g \in D_2 \text{ and } h \in D_3\}.$$

Then, the result of  $(s, r^2)(sr, e)$  would be  $(s \cdot sr, r^2 \cdot e) = (r, r^2)$ .

**Claim 4.2.** Let  $W$  be a Coxeter group generated by set  $S$ , and let  $S_1, S_2, \dots, S_k$  be connected components of the Coxeter diagram of  $W$  such that

$$S = S_1 \cup S_2 \cup \dots \cup S_k.$$

Let  $W_i = \langle S_i \rangle$  for  $i = 1, 2, \dots, k$ . Then,

$$W \cong W_1 \times W_2 \times \dots \times W_k.$$

[9]

In words, for any Coxeter group  $W$  with  $S$  the set of its generators, let  $S_1, S_2, \dots, S_k$  be pairwise disjoint sets such that every generator is contained in exactly one of these sets. Additionally, assume that generators in different sets commute, i.e., for  $s_i$  an element of  $S_i$  and  $s_j$  an element of  $S_j$  where  $i \neq j$ ,  $s_i s_j = s_j s_i$ . Next, define  $W_i$  as the subgroup of  $W$  generated by the elements in  $S_i$ . Then, we claim that  $W$  is isomorphic to the direct product of  $W_1, W_2, \dots, W_k$ . That is, each element  $w$  in  $W$  corresponds to a unique  $k$ -tuple  $(w_1, w_2, \dots, w_k)$  where  $w_i$  is an element of  $W_i$  for  $i \in 1, 2, \dots, k$ .

*Proof.* We will prove this theorem by first defining a map between  $W$  and  $W_1 \times W_2 \times \cdots \times W_k$ , proving that the map is bijective and a homomorphism between  $W$  and  $W_1 \times W_2 \times \cdots \times W_k$ , and finally using that to conclude that  $W$  is isomorphic to  $W_1 \times W_2 \times \cdots \times W_k$ .

For two connected components  $S_i$  and  $S_j$  where  $i, j$  are values in  $[1, 2, \dots, k]$  and  $i \neq j$ , let  $s_i \in S_i$  and  $s_j \in S_j$ . Then, by the definition of connected components,  $(s_i s_j)^2 = e$ . Also,  $s_i^2 = s_j^2 = e$  since  $s_i, s_j$  are generators of the Coxeter group  $W$ . So we have  $s_i = s_i^{-1}$  and  $s_j = s_j^{-1}$ . Then,

$$\begin{aligned}(s_i s_j)^2 &= s_i s_j s_i s_j = e \\ s_i s_j s_i s_j &= s_i s_j s_i^{-1} s_j^{-1} = e \\ s_i s_j &= s_j s_i.\end{aligned}$$

Hence, any two generators from different connected components of the Coxeter diagram of  $W$  commute. Consequently, for any word  $a$  in  $W$ , we can repeatedly swap adjacent commuting generators until all generators in  $S_i$  are grouped together. Thus, we will be able to write  $a$  as  $a_1 a_2 \dots a_k$  where  $a_i$  contains the generators from  $S_i$  that is contained in the word  $a$  for  $i = 1, 2, \dots, k$ . Finally, we will define a map  $\pi : W \rightarrow W_1 \times W_2 \times \cdots \times W_k$  such that for the word  $a = a_1 a_2 \dots a_k$  in  $W$ , it will map to  $(a_1, a_2, \dots, a_k) \in W_1 \times W_2 \times \cdots \times W_k$ .

Now, we claim that  $\pi$  is bijective. First, let  $a, b$  be elements of  $W$  such that  $\pi(a) = \pi(b)$ . Since  $\pi(a) = (a_1, a_2, \dots, a_k)$  and  $\pi(b) = (b_1, b_2, \dots, b_k)$ , then we have  $a_1 = b_1, a_2 = b_2, \dots, a_k = b_k$ . Then,  $a = a_1 a_2 \dots a_k = b_1 b_2 \dots b_k = b$ , so  $a$  must equal  $b$ . Therefore,  $\pi$  is injective. Now, let  $(w_1, w_2, \dots, w_k)$  contained in  $W_1 \times W_2 \times \cdots \times W_k$ . Since each of the connected components is a subgroup of  $W$ , all of their generators are contained in  $W$ . Hence, let  $w$  be in  $W$  such that  $w = w_1 w_2 \dots w_k$ . Then,  $\pi(w) = (w_1, w_2, \dots, w_k)$ , so  $\pi$  is surjective. Since  $\pi$  is both surjective and injective, it is bijective.

In our next step, we will show that the map  $\pi$  is a group homomorphism. Our goal is to prove that  $\pi(ab) = \pi(a)\pi(b)$  for elements  $a, b$  of  $W$ . First, let  $a = a_1 a_2 \dots a_k$  and  $b = b_1 b_2 \dots b_k$ . So

$$\begin{aligned}\pi(a)\pi(b) &= (a_1, a_2, \dots, a_k) \cdot (b_1, b_2, \dots, b_k) \\ &= (a_1 b_1, a_2 b_2, \dots, a_k b_k).\end{aligned}$$

Then,  $ab = (a_1 a_2 \dots a_k)(b_1 b_2 \dots b_k)$ . Since generators from different connected components commute pairwise, we can rearrange  $ab$  such that  $a_i$  is next to  $b_i$ , i.e.,  $ab = a_1 b_1 a_2 b_2 \dots a_k b_k$ . Then,  $\pi(ab) = (a_1 b_1, a_2 b_2, \dots, a_k b_k)$ , which is equal to the value we found for  $\pi(a)\pi(b)$ . From this, we can conclude that  $\pi$  is a homomorphism.

Hence,  $\pi$  is bijective and a homomorphism, so it is an isomorphism. That is,  $W \cong W_1 \times W_2 \times \cdots \times W_k$ .  $\square$

## 5 Linear Algebra

Up to this point, Coxeter groups have been described through generators, relations, and diagrams. We now shift to linear algebra in order to explain why these groups are also geometric objects. The goal of the next several sections is to build a bridge from Coxeter presentations to reflections on vector spaces.

First, we introduce vector spaces and linear transformations, which give us a setting where group elements can act as geometric motions. Next, we define bilinear forms and orthogonality, which allow us to describe angles, perpendicularity, and reflections algebraically. Then, we use representations to

realize abstract Coxeter generators as actual linear reflections. Finally, this geometric interpretation will let us distinguish finite Coxeter groups using the associated bilinear form and lead into the classification of finite Coxeter groups.

## 5.1 Vector Spaces

In this section, we will work with real vector spaces.

**Definition 5.1.** A *real vector space* is a set  $V$  whose elements are called *vectors*, together with two operations: vector addition and multiplication by real numbers. [5] That is, for any  $v, w$  vectors in  $V$  and  $c \in \mathbb{R}$ , we have

$$v + w \in V \quad \text{and} \quad cv \in V.$$

These operations satisfy the following rules for all vectors  $u, v, w$  in  $V$  and  $c, d \in \mathbb{R}$ :

$u + v = v + u$	commutativity of vector addition,
$(u + v) + w = u + (v + w)$	associativity of vector addition,
$v + 0 = v$	existence of a zero vector,
$v + (-v) = 0$	existence of additive inverses,
$c(u + v) = cu + cv$	distributivity over vector addition,
$(c + d)v = cv + dv$	distributivity over scalar addition,
$c(dv) = (cd)v$	compatibility of scalar multiplication,
$1v = v$	multiplication by 1.

**Example.** The space  $\mathbb{R}^n$  is a real vector space. Its vectors are ordered lists

$$(x_1, x_2, \dots, x_n),$$

where each  $x_i \in \mathbb{R}$ . Addition and scalar multiplication are defined component-wise:

$$\begin{aligned} \text{vector addition:} & \quad (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) \\ \text{scalar multiplication:} & \quad c(x_1, \dots, x_n) = (cx_1, \dots, cx_n). \end{aligned}$$

**Definition 5.2.** A *basis* of a real vector space  $V$  is a list of vectors

$$v_1, v_2, \dots, v_n$$

such that every vector  $v$  in  $V$  can be written uniquely as a linear combination of them:

$$v = c_1v_1 + c_2v_2 + \dots + c_nv_n$$

for some  $c_1, c_2, \dots, c_n \in \mathbb{R}$ .

**Definition 5.3.** If  $V$  has a finitely-sized basis, then the *dimension* of  $V$ , written as  $\dim V$ , is the number of vectors in a basis for  $V$ .

**Example.**  $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$  has dimension 3. A standard basis for  $\mathbb{R}^3$  is

$$\vec{i} = (1, 0, 0), \quad \vec{j} = (0, 1, 0), \quad \vec{k} = (0, 0, 1).$$

That is, any vector  $v = (a, b, c)$  in  $\mathbb{R}^3$  can be written as a linear combination of  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$ :

$$v = a\vec{i} + b\vec{j} + c\vec{k}.$$

To describe symmetries algebraically, we need to view them as functions that move vectors while preserving the structure of the vector space.

**Definition 5.4.** Let  $V$  and  $W$  be real vector spaces. A function  $T : V \rightarrow W$  is called a *linear transformation* if it preserves vector addition and scalar multiplication. That is, for all vectors  $v, w$  in  $V$  and  $c \in \mathbb{R}$ ,

$$T(v + w) = T(v) + T(w)$$

and

$$T(cv) = cT(v).$$

*Remark 5.1.* Linear transformations are a type of homomorphisms; they are homomorphisms between two vector spaces.

**Definition 5.5.** The general linear group  $GL(V)$  consists of all invertible linear transformations  $T : V \rightarrow V$ , with composition as the group operation.

**Claim 5.1.** *Given a real vector space  $V$ ,  $GL(V)$  must be a group.*

*Proof.* First, the identity element of this group is the identity transformation, which maps every  $v$  to itself—this could be thought of as the “do nothing” transformation. Next,  $GL(V)$  by definition only consists of invertible transformations, so every element in  $GL(V)$  must have an inverse. Finally, composition is associative, so  $GL(V)$  satisfies associativity. Hence  $GL(V)$  satisfies the group axioms.  $\square$

## 5.2 Bilinear Forms and Orthogonality

Vector spaces come equipped with bilinear forms that allow us to turn geometric ideas into algebraic ones. They give us a way to measure the relationship between two vectors, which will let us define notions such as orthogonality and reflection in a more general setting.

**Definition 5.6.** A *bilinear form*  $B$  on a real vector space  $V$  is a function  $B : V \times V \rightarrow \mathbb{R}$  that is linear in each input [6]. In other words, the bilinear form has to satisfy

$$\begin{array}{lll} \text{Superposition for the 1st element:} & B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w) & \forall v_1, v_2, w \in V \\ \text{Superposition for the 2nd element:} & B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2) & \forall v, w_1, w_2 \in V \\ \text{Homogeneity for the 1st element:} & B(cv, w) = cB(v, w) & \forall v, w \in V, c \in \mathbb{R} \\ \text{Homogeneity for the 2nd element:} & B(v, cw) = cB(v, w) & \forall v, w \in V, c \in \mathbb{R} \end{array}$$

**Example.** The standard “dot product” on  $\mathbb{R}^n$  is a bilinear form, taking two vectors and returning a real number:

$$v \cdot w = v_1 w_1 + \cdots + v_n w_n.$$

Geometrically, the dot product in  $\mathbb{R}^n$  is related to the angle  $\theta$  between  $v, w \neq 0$  by

$$v \cdot w = \|v\| \|w\| \cos \theta$$

Thus the dot product encodes geometric information about the angle between them. In particular,  $v \cdot w = 0$  exactly when  $\cos \theta = 0$ , i.e. the two vectors are perpendicular. The dot product gives the usual notion of perpendicularity: two vectors  $v$  and  $w$  are perpendicular when  $v \cdot w = 0$ . This motivates more general definition.

**Definition 5.7.** Two vectors  $v, w$  in  $V$  are said to be *orthogonal* (written as “ $v \perp w$ ”) with respect to a given bilinear form  $\langle, \rangle$  when

$$\langle v, w \rangle = 0.$$

**Definition 5.8.** The *orthogonal space* to a subspace  $W$  of  $V$  is the subspace of vectors  $v$  that are orthogonal to every vector in  $W$ . In other words,

$$W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in W\}.$$

[1]

Such a vector  $v$  in  $V$  that is orthogonal to all other vectors in  $V$  is called a null vector.

**Definition 5.9.** The set of all null vectors is called the *nullspace*, or the orthogonal space to  $V$ ,  $N$  of the form. In other words,

$$N = \{v \mid v \perp V\} = V^\perp.$$

The notion of orthogonality will be integral in defining nondegeneracy for bilinear forms. Apart from that, we will define other classifications of bilinear forms that will help us understand the differences between finite and infinite Coxeter groups.

**Definition 5.10.** A bilinear form is *symmetric* when  $\langle v, w \rangle = \langle w, v \rangle$  for all  $v, w$  in  $V$ .

**Lemma 5.2.** If a symmetric form is not identically zero, then there exists a vector  $v$  in  $V$  such that  $\langle v, v \rangle \neq 0$ .

*Proof.* By contradiction, assume that there does not exist a vector  $v$  in  $V$  such that  $\langle v, v \rangle \neq 0$ . Then, since the form is not identically zero, there exist vectors  $x, y$  in  $V$  where  $\langle x, y \rangle \neq 0$ . Moreover, since the form is symmetric,  $\langle x, y \rangle = \langle y, x \rangle$ .

Now,

$$\begin{aligned} \langle x + y, x + y \rangle &= \langle x + y, x \rangle + \langle x + y, y \rangle \\ &= \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + \langle y, y \rangle + 2 \langle x, y \rangle \end{aligned}$$

By our choice of  $x$  and  $y$ ,  $\langle x, y \rangle \neq 0$ . Then, from our assumption it follows that neither of  $\langle x + y, x + y \rangle$ ,  $\langle x, x \rangle$ , or  $\langle y, y \rangle$  is nonzero. Hence, our equation simplifies to

$$0 = 0 + 0 + 2 \langle x, y \rangle$$

$$0 = \langle x, y \rangle$$

This is a contradiction to our initial choice of  $x$  and  $y$ . Thus, there exists at least one vector  $v$  in  $\{x, y, x + y\}$  such that  $\langle v, v \rangle \neq 0$ .  $\square$

**Definition 5.11.** The bilinear form  $B$  is *positive definite* if  $B(v, v) > 0$  for all  $v \neq 0$  and  $B(v, v) = 0$  only for  $v = 0$ . [2]

**Definition 5.12.** The bilinear form  $B$  is *positive semidefinite* if  $B(v, v) \geq 0$  for all vectors  $v$  in  $V$

**Definition 5.13.** The bilinear form on  $V$  is *nondegenerate* if its nullspace is  $\{0\}$ . This means that there is no nonzero vector that is orthogonal to the entire space. On the contrary, a form that has a nonzero vector orthogonal to the entire space is called *degenerate*.

Moreover, orthogonality gives us a precise way to describe reflections. Geometrically, a reflection has two main parts: a mirror, which stays fixed, and a perpendicular direction, which gets reversed. In linear algebra, the mirror is a hyperplane, and the perpendicular direction is described using the orthogonality coming from our bilinear form  $\langle, \rangle$ .

**Definition 5.14.** A *reflection* on a vector space  $V$  is a linear transformation that fixes a hyperplane and reverses one direction orthogonal to that hyperplane. Here, a *hyperplane* is a subspace whose dimension is one less than the dimension of  $V$ .

The fixed hyperplane is called the *mirror* of the reflection, and the direction that gets sent to its negative is called a *root direction*.

### 5.3 Representations

So far, we have studied groups in terms of their elements and how those elements compose. A representation lets us make a group more concrete by realizing its elements as linear transformations.

**Definition 5.15.** Let  $G$  be a group and  $V$  a real vector space. A *representation* of  $G$  on  $V$  is a group homomorphism  $\rho : G \rightarrow GL(V)$ , where  $GL(V)$  is the group of invertible linear transformations from  $V$  to itself.

In other words, a representation assigns each group element  $g$  of  $G$  to an invertible linear transformation  $\rho(g) \in GL(V)$ . Representations are homomorphisms, meaning this mapping respects the group operation:

$$\rho(gh) = \rho(g)\rho(h).$$

Thus, multiplying two group elements  $g, h$  of  $G$  corresponds to composing their corresponding linear transformations ( $\rho(g)$  and  $\rho(h)$ ).

**Example.** The dihedral group  $D_n$  has a very natural representation on  $\mathbb{R}^2$ . Place a regular  $n$ -gon in the plane with its center at the origin and vertex 1 on the positive  $x$ -axis. Any symmetry of the polygon moves points in the plane by either rotation or reflection, and both rotations and reflections are invertible linear transformations of  $\mathbb{R}^2$ .

Therefore, each element of  $D_n$  can be mapped to a corresponding element of  $GL(\mathbb{R}^2)$ . This gives a representation  $\rho : D_n \rightarrow GL(\mathbb{R}^2)$ . Here,  $r$  corresponds to a clockwise rotation by  $\frac{2\pi}{n}$  about the origin, i.e.,  $\rho(r)$  is a transformation via

$$\begin{bmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix},$$

and  $s$  corresponds to a reflection across the  $x$ -axis, i.e.  $\rho(s)$  is a transformation via

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

## 5.4 Representations of a Coxeter Group

We have already defined a Coxeter group abstractly using generators and relations. Now we want to realize those abstract generators as actual reflections on a vector space.

Let

$$W = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = e \rangle$$

be a Coxeter group. We will construct a representation of  $W$  by assigning each generator  $s_i$  to a linear reflection  $\rho_i$ .

Let  $V$  be a real vector space with basis  $e_1, e_2, \dots, e_n$ , where each basis vector  $e_i$  corresponds to a generator  $s_i$ . Define a bilinear form  $\langle \cdot, \cdot \rangle$  on  $V$  by setting

$$\langle e_i, e_j \rangle = -2 \cos \left( \frac{\pi}{m_{ij}} \right)$$

for basis vectors  $e_i, e_j$  then extending it bilinearly to all of  $V$ . [9]

For each  $i$ , define a linear transformation  $\rho_i : V \rightarrow V$  by

$$\rho_i(x) = x - \langle x, e_i \rangle e_i.$$

$\rho_i$  is a linear transformation since  $\langle \cdot, \cdot \rangle$  is linear in the first input (where  $x$  goes).

Since  $m_{ii} = 1$ , we have  $\langle e_i, e_i \rangle = -2 \cos \pi = 2$ . Therefore,

$$\rho_i(e_i) = e_i - \langle e_i, e_i \rangle e_i = e_i - 2e_i = -e_i.$$

Thus,  $\rho_i$  sends the root direction  $e_i$  to its negative.

More generally, for a basis vector  $e_j$ ,

$$\rho_i(e_j) = e_j - \langle e_j, e_i \rangle e_i.$$

$$\rho_i(e_j) = e_j + 2 \cos \left( \frac{\pi}{m_{ij}} \right) e_i.$$

Now we generalize this to any vector  $x$  in  $V$ . Let

$$x = \sum_{j=1}^n a_j e_j.$$

Since  $\rho_i$  is linear,

$$\rho_i(x) = \rho_i \left( \sum_{j=1}^n a_j e_j \right) = \sum_{j=1}^n a_j \rho_i(e_j).$$

Using the formula for  $\rho_i(e_j)$ , we obtain

$$\rho_i(x) = \sum_{j=1}^n a_j \left( e_j + 2 \cos \left( \frac{\pi}{m_{ij}} \right) e_i \right).$$

Thus,

$$\rho_i(x) = x + \left( \sum_{j=1}^n 2a_j \cos \left( \frac{\pi}{m_{ij}} \right) \right) e_i.$$

## 6 Classifying Finite Coxeter Groups

Now, we are ready to define and classify all of the finite Coxeter groups. As mentioned earlier, Coxeter groups are interesting in that we can determine if a given Coxeter group is finite just through its representation. This is because there are a limited amount of types of finite Coxeter groups which we can recognize from their group presentation. In this way, if our given Coxeter group does not fall within one of these finite Coxeter groups, we can determine that it is infinite.

### 6.1 Finiteness and the associated Bilinear Form

First, we begin by defining the three types of Coxeter groups. This can be done in two ways: by viewing the groups through their bilinear form or by the space on which they act as reflection groups.

**Theorem 6.1.** *A Coxeter group is finite if and only if the associated bilinear form is positive definite. [9].*

Finite Coxeter groups can be viewed as the reflection group on Euclidean space.

*Remark 6.1.* Given  $W$  a finite Coxeter group, the representation  $\pi : W \rightarrow GL(V)$  of the Coxeter group on a vector space  $V$  is a group generated by reflections  $\pi(s_i)$ , each with root  $e_i^\perp$ .

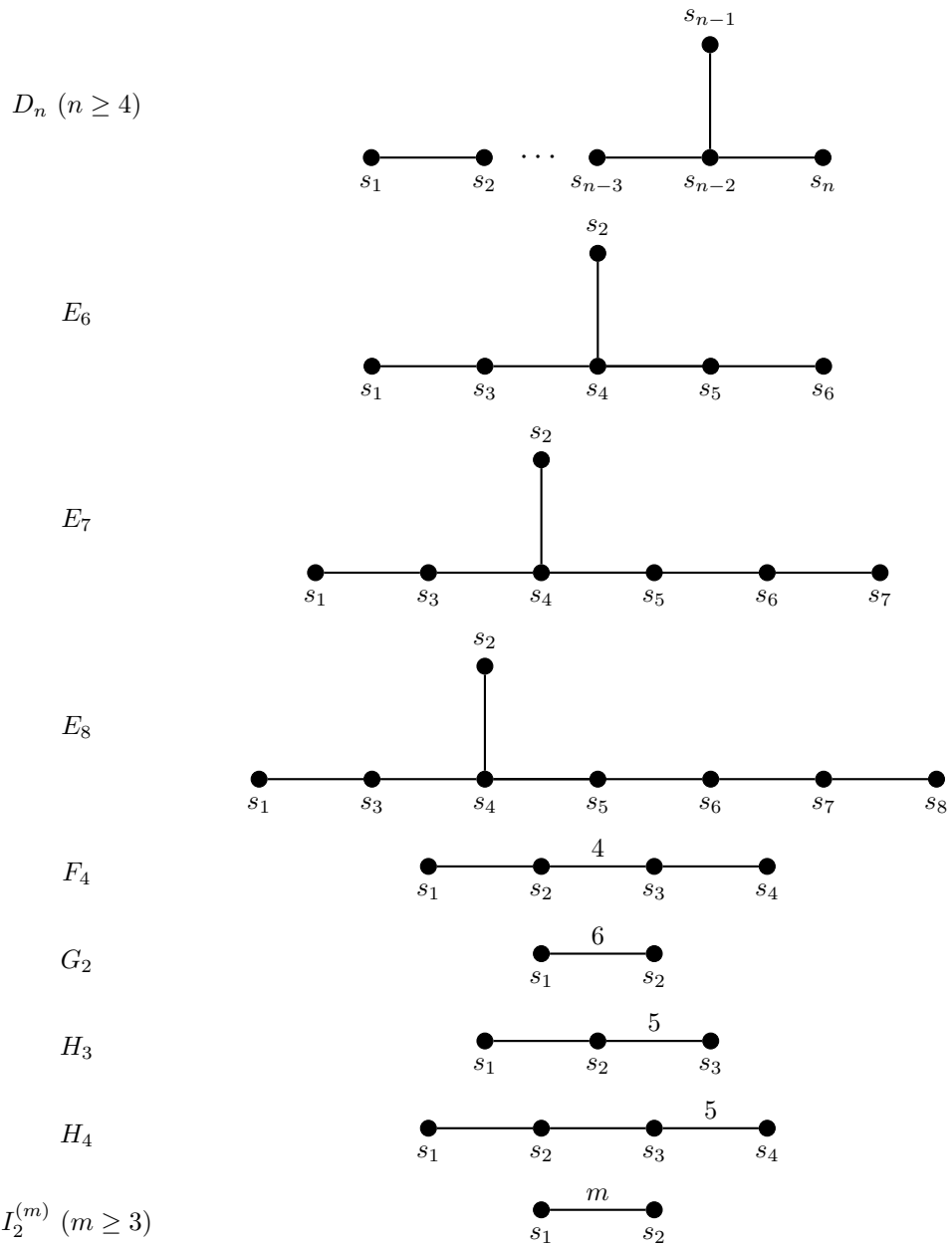
**Definition 6.1.** An Affine Coxeter Group is a Coxeter Group where the associated bilinear form  $B(v, w)$  is positive semi-definite but not positive definite ( $B(v, v) \geq 0$  for all vectors  $v$  in  $V$ , yet there exists  $v \neq 0$  such that  $B(v, v) = 0$ ).

### 6.2 The Diagrams of all Finite Coxeter Groups

Now, we will focus on the finite Coxeter groups. In 1935, H.S.M. Coxeter proved that the only finite Coxeter groups are the groups of type  $A, B, C, D, E, F, G, H, I$ . This is due to the requirement that the bilinear form of finite Coxeter groups need to be positive definite.

We have listed all of the finite Coxeter groups and their diagrams below. [3]

Name	Diagram
$A_n \ (n \geq 1)$	
$B_n = C_n \ (n \geq 3)$	



## References

- [1] Michael Artin. *Algebra*. 2nd ed. Pearson, 2010.

- [2] *Bilinear Forms*. URL: <https://www.math.uwaterloo.ca/~snew/MATH245/Chap9BilinearForms.pdf>. (accessed: 04.23.2026).
- [3] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Springer Press, 2005.
- [4] Keith Conrad. *Subgroups of cyclic groups*. URL: <https://kconrad.math.uconn.edu/blurbs/grouptheory/cyclicgp.pdf>.
- [5] David S. Dummit and Richard M. Foote. *Abstract Algebra*. 3rd ed. New York: Wiley, 2004.
- [6] A. Eremenko. *Bilinear forms*. URL: <https://www.math.purdue.edu/~eremenko/dvi/bilinear>.
- [7] Derek Holt. *Presentation of Groups*. URL: [https://warwick.ac.uk/fac/sci/maths/people/staff/fbouyer/presentation\\_of\\_group.pdf](https://warwick.ac.uk/fac/sci/maths/people/staff/fbouyer/presentation_of_group.pdf). (accessed: 04.17.2026).
- [8] James Humphreys. *Reflection Groups and Coxeter groups*. Cambridge University Press, 1990.
- [9] Shourya Mukherjee. *Classification of Finite Coxeter Groups*. URL: <https://math.uchicago.edu/~may/REU2019/REUPapers/Mukherjee.pdf>. (accessed: 04.26.2026).
- [10] Carl-Fredrik Nyberg-Brodde. *The Adian-Rabin Theorem – An English translation*. 2024. arXiv: 2208.08560 [math.GR]. URL: <https://arxiv.org/abs/2208.08560>.