Colors and Patterns in Graphs: Exploring Graph Coloring and Ramsey Numbers

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Abstract

This paper explores basic ideas in graph theory, focusing on graph coloring and Ramsey theory. Graph theory studies relationships using vertices and edges, giving a simple way to model complex connections. We start with core concepts such as adjacency, paths, cycles, and connectivity. We then study vertex and edge colorings, including the chromatic number and chromatic index, and present results such as the characterization of bipartite graphs and bounds on edge colorings. These ideas are illustrated through examples, including scheduling problems. We then introduce Ramsey theory, which looks at how patterns must appear in large graphs no matter how their edges are arranged. We define Ramsey numbers as a way to measure when this kind of unavoidable structure appears in edge colorings of complete graphs.

1 Introduction

Graph theory studies relationships between objects using vertices and edges. A graph consists of a set of vertices representing objects and a set of edges representing relationships between those objects. Despite its simple definition, graph theory is a powerful and widely used tool with applications in computer science, engineering, biology, and the social sciences.

The origins of graph theory date back to the eighteenth century, when Leonhard Euler solved the famous Königsberg bridge problem, which is widely regarded as the first result in graph theory. Since then, graph theory has developed into a major area of mathematics with both deep theoretical significance and practical applications. Graphs are now used to model transportation systems, communication networks, biological interactions, and social connections. Their versatility comes from the fact that many complex systems can be represented naturally through collections of objects and relationships. [SHL07]

One important application arises in **class scheduling**. Suppose a university needs to assign time slots for a collection of classes so that no student is required to attend two classes at the same time. We can model this situation using a graph: each vertex represents a class, and we draw an edge between two vertices if there exists at least one student who is enrolled in both classes. In this graph, adjacent vertices represent classes that cannot be scheduled simultaneously.

The scheduling problem then becomes a **graph coloring problem**. Assigning time slots corresponds to assigning colors to vertices, and the requirement that no student has a scheduling conflict translates to the condition that adjacent vertices must receive different colors. The objective is to minimize the number of colors used, which corresponds to minimizing the number of time slots needed. This minimum number is called the **chromatic number** of the graph.

Graph coloring is only one example of how graph theory transforms real-world constraints into mathematical problems. Similar ideas appear in register allocation in computer science, frequency assignment in telecommunications, and map coloring in geography. In each case, the central challenge is to assign limited resources while avoiding conflicts between related objects. These applications demonstrate how abstract mathematical structures can provide efficient solutions to practical problems.

In this paper, we focus on graph coloring as one of the central topics in graph theory. We begin by introducing basic definitions and terminology necessary to understand graphs. We then develop the theory of vertex coloring and edge coloring, including important results such as the characterization of bipartite graphs and bounds on the chromatic index. These results illustrate how simple constraints on graphs lead to rich and meaningful mathematical structure.

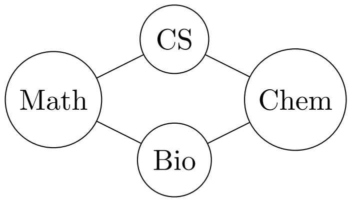

Figure 1: A class conflict graph: vertices represent classes, and edges indicate shared students

2 Background

We begin by fixing notation and introducing the basic objects used throughout graph theory.

Definition 2.1. [CZ12] A **graph** is an ordered pair

$$G = (V, E),$$

where V is a finite nonempty set of objects called **vertices** and E is a set of 2-element subsets of V called **edges**.

Definition 2.2. [CZ12] Two vertices u and v are said to be **adjacent** in G if uv is an edge of G.

Adjacency is the basic notion of connection in a graph.

Definition 2.3. [CZ12] The number of vertices in G is called the **order** of G.

Definition 2.4. [CZ12] The number of edges in G is called the **size** of G.

Order and size give a first description of the scale of a graph, but do not describe how vertices are connected.

Definition 2.5. [CZ12] The **degree** of a vertex v, denoted $\deg(v)$, is the number of edges incident with v.

Degree captures local connectivity at a vertex.

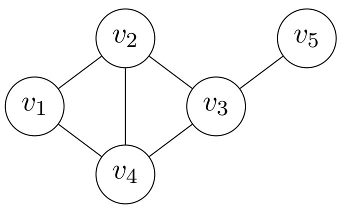

Figure 2: A graph illustrating adjacency, order, size, and degree

In this graph, there are five vertices and six edges, so the order is 5 and the size is 6. In this example, $\deg(v_2) = 3$ since three edges meet at v_2.

We now move from local properties to global notions of connectivity between vertices.

Definition 2.6. [CZ12] A u-v **walk** in G is a sequence of vertices starting at u and ending at v such that consecutive vertices are adjacent.

Walks allow repetition of vertices and edges, so they describe general movement through a graph. For example, in Figure 2,

$$v_1, v_2, v_4, v_1, v_2$$

is a walk since each consecutive pair of vertices is connected by an edge.

Definition 2.7. [CZ12] A u-v **trail** is a u-v walk in which no edge is used more than once.

Trails are more restrictive because edges cannot be repeated. In Figure 2,

$$v_1, v_2, v_3, v_4, v_1$$

is a trail since every edge is used exactly once.

Definition 2.8. [CZ12] A u-v **path** is a u-v walk in which no vertices are repeated.

Paths represent the simplest form of connection between two vertices because neither vertices nor edges repeat. For instance, in Figure 2,

$$v_1, v_2, v_3, v_5$$

is a path from v_1 to v_5.

Definition 2.9. [CZ12] A **circuit** in a graph G is a closed trail of length at least 3.

Definition 2.10. [CZ12] A circuit with no repeated vertices except the first and last is a **cycle**.

Definition 2.11. [CZ12] A k**-cycle** is a cycle of length k.

Cycles are fundamental structures that appear throughout graph theory.

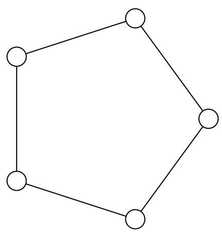

Figure 3: An odd cycle C_5

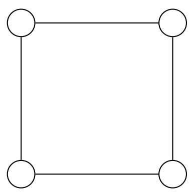

Figure 4: An even cycle C_4

We now define when vertices are connected through paths.

Definition 2.12. [CZ12] Two vertices u and v are connected if there exists a u-v path in G.

Connectivity divides a graph into components.

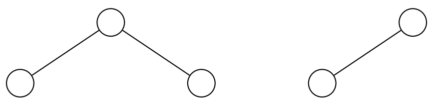

Figure 5: A graph with two connected components

Vertices in the same component are connected by paths, while vertices in different components are not connected.

3 Graph Colorings

Graph coloring assigns labels to vertices or edges under constraints.

3.1 Vertex Coloring

Definition 3.1. [CZ12] A **proper vertex coloring** assigns colors to the vertices of a graph so that any two adjacent vertices receive different colors.

Proper vertex colorings help us organize graphs while avoiding conflicts between connected vertices.

Definition 3.2. [CZ12] The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colors needed in a proper coloring of G.

The chromatic number measures how complicated a graph is from a coloring perspective.

Instead of asking for the exact chromatic number, we sometimes only want to know whether a graph can be colored using at most a certain number of colors.

Definition 3.3. [CZ12] A graph is k**-colorable** if $\chi(G) \leq k$.

In other words, a graph is k-colorable if there exists a proper coloring using at most k colors. For instance, bipartite graphs are exactly the graphs that are 2-colorable.

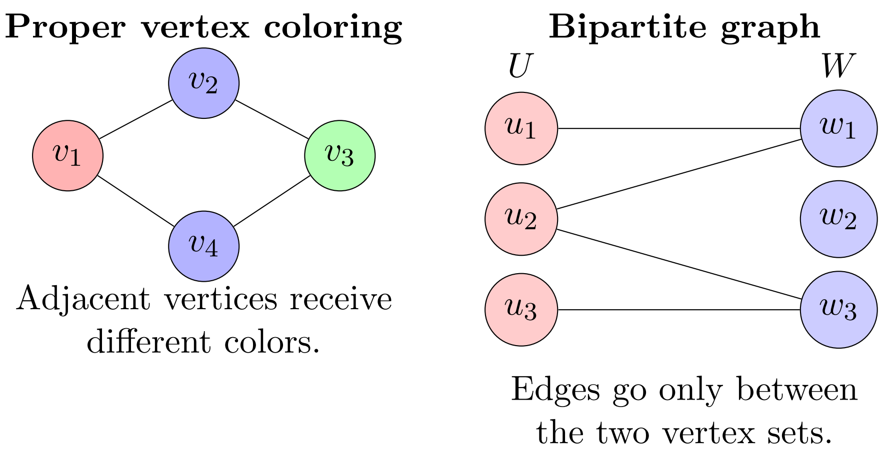

Figure 6: A proper vertex coloring and a bipartite graph

Interestingly, certain families of graphs always share the same chromatic number because of their underlying structure. One important example is the class of bipartite graphs, which can always be colored using only two colors.

Definition 3.4. [CZ12] A graph is **bipartite** if its vertex set can be partitioned into two sets such that every edge joins different sets.

Theorem 3.5. *[CZ12] A graph G has chromatic number 2 if and only if G is a nonempty bipartite graph with a nonempty edge set.*

Proof. Suppose first that G is bipartite with partite sets U and W. Assign one color to all vertices in U and a second color to all vertices in W. Since every edge of G joins a vertex in U to a vertex in W, this is a proper coloring. Hence $\chi(G) \leq 2$. Because G is nonempty, at least one edge exists, so at least two colors are required. Therefore, $\chi(G) = 2$.

Conversely, suppose that $\chi(G) = 2$. Then there exists a proper vertex coloring of G using exactly two colors. Let U be the set of vertices assigned the first color and W the set assigned the second color. Since adjacent vertices must receive different colors, no edge joins two vertices within the same set. Therefore, every edge joins a vertex in U to a vertex in W, and G is bipartite. □

Theorem 3.6. *[CZ12] A graph is bipartite if and only if it contains no odd cycle.*

Proof. Suppose first that G is bipartite with partite sets U and W. Any cycle in G must alternate between vertices in U and vertices in W. Therefore, every cycle has even length, so G contains no odd cycle.

Conversely, suppose that G contains no odd cycle. Choose a vertex v in a connected component of G. Partition the vertices in that component into two sets:

$$U = \{u : \text{the distance from } v \text{ to } u \text{ is even}\}$$

and

$$W = \{u : \text{the distance from } v \text{ to } u \text{ is odd}\}.$$

If two vertices in the same set were adjacent, then the edge between them, together with paths back to v, would create an odd cycle. Since G has no odd cycle, every edge joins a vertex in U to a vertex in W. Thus this component is bipartite. Applying the same argument to each component shows that G is bipartite. □

By the previous theorem, a graph is bipartite if and only if it contains no odd cycle. Therefore, any graph containing an odd cycle must satisfy $\chi(G) \geq 3$.

In particular, for cycles we obtain the following behavior:

- If $n \geq 4$ is even, then C_n is bipartite and hence $\chi(C_n) = 2$.
- If $n \geq 3$ is odd, then C_n is not bipartite, so $\chi(C_n) \geq 3$.

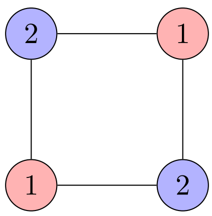

Figure 7: C_4: requires 2 colors

The cycle C_4 is even, so alternating two colors is sufficient. Thus $\chi(C_4) = 2$.

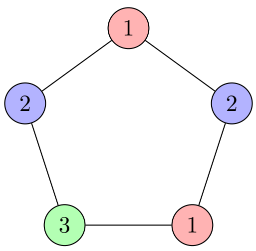

Figure 8: C_5: requires 3 colors

The cycle C_5 is an odd cycle, so two colors are not sufficient. A third color is required, so $\chi(C_5) = 3$.

Definition 3.7. [CZ12] The **maximum degree** of a graph G, denoted $\Delta(G)$, is the largest degree among all vertices of G.

Theorem 3.8. *[CZ12] For every graph G, the chromatic number satisfies*

$$\chi(G) \le \Delta(G) + 1.$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We construct a proper coloring of G inductively.

Define a coloring function $c : V(G) \to \mathbb{N}$ as follows. First assign

$$c(v_1) = 1.$$

Now assume that $c(v_1), c(v_2), \ldots, c(v_i)$ have been defined. To define $c(v_{i+1})$, assign to v_{i+1} the smallest positive integer that is not used by any of its already-colored neighbors.

Since v_{i+1} has at most $\deg(v_{i+1})$ neighbors, at most $\deg(v_{i+1})$ colors are forbidden. Therefore, among the integers

$$1, 2, \ldots, \deg(v_{i+1}) + 1,$$

at least one color is available. Hence,

$$c(v_{i+1}) \le \deg(v_{i+1}) + 1.$$

Proceeding inductively, this defines a proper coloring of all vertices of G.

Let v_j be a vertex receiving the maximum color used in this construction. Then

$$\chi(G) \le c(v_j) \le \Delta(G) + 1.$$

Thus,

$$\chi(G) \le \Delta(G) + 1,$$

as desired. □

3.2 Edge Coloring

Definition 3.9. [CZ12] An **edge coloring** of a nonempty graph G is an assignment of colors to the edges of G, one color to each edge, such that adjacent edges are assigned different colors.

Definition 3.10. [CZ12] The minimum number of colors that can be used to color the edges of G is called the **chromatic index** (or sometimes the **edge chromatic number**) and is denoted by $\chi'(G)$.

Definition 3.11. [CZ12] An edge coloring that uses k colors is a **k-edge coloring**.

Theorem 3.12 (Vizing's Theorem [Har85])**.** *For any nonempty graph G, the chromatic index (sometimes called line-chromatic) number satisfies the inequalities*

$$\Delta(G) \le \chi'(G) \le \Delta(G) + 1.$$

By Vizing's Theorem, the chromatic index of every nonempty graph G is one of two values: $\Delta(G)$ or $\Delta(G) + 1$.

We illustrate this result using cycles. For every cycle C_n, where $n \ge 3$, each vertex has degree 2, so $\Delta(C_n) = 2$. Therefore,

$$\chi'(C_n) = 2 \quad \text{or} \quad \chi'(C_n) = 3.$$

First suppose that n is even. Label the edges of C_n in cyclic order and assign colors alternately, using colors 1 and 2. Since consecutive edges receive different colors and the cycle has even length, the final edge receives a different color from the first edge. Thus, this produces a valid edge coloring using two colors. Therefore,

$$\chi'(C_n) = 2 \quad \text{when } n \text{ is even.}$$

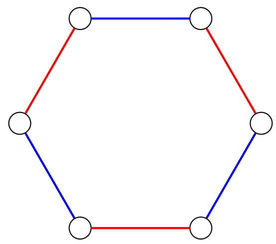

Figure 9: A 2-edge coloring of an even cycle

Now suppose that n is odd. Attempting the same alternating coloring results in a contradiction: after assigning colors alternately around the cycle, the final edge receives the same color as the first edge. Since these two edges are adjacent, this violates the definition of an edge coloring.

To make this argument precise, consider any edge coloring of C_n using two colors. Any two edges assigned the same color cannot be adjacent, so edges of the same color form a matching. In a graph with n vertices, a matching can contain at most $\lfloor n/2 \rfloor$ edges. When n is odd, this is $\frac{n-1}{2}$.

Thus, each color can be used on at most $\frac{n-1}{2}$ edges. Using two colors, we can color at most

$$\frac{n-1}{2} + \frac{n-1}{2} = n - 1$$

edges. However, the cycle C_n contains n edges, so it is impossible to color all edges with only two colors.

Therefore, at least three colors are required. Hence,

$$\chi'(C_n) = 3 \quad \text{when } n \text{ is odd.}$$

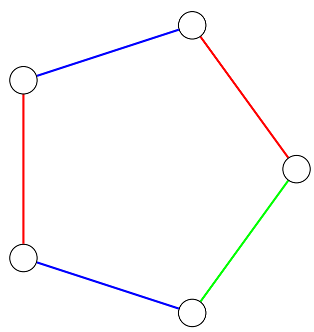

Figure 10: A 3-edge coloring of an odd cycle

Theorem 3.13. *[CZ12] Let G be a graph of odd order n and size m. If*

$$m > \frac{(n-1)\Delta(G)}{2},$$

then $\chi'(G) = 1 + \Delta(G)$.

Proof. Suppose, for contradiction, that $\chi'(G) = \Delta(G)$. Then there exists an edge coloring of G using exactly $\Delta(G)$ colors.

In any edge coloring, edges assigned the same color form a matching. Since G has odd order n, each matching contains at most

$$\frac{n-1}{2}$$

edges. Because there are $\Delta(G)$ colors, the total number of edges that can be colored is at most

$$\Delta(G) \cdot \frac{n-1}{2} = \frac{(n-1)\Delta(G)}{2}.$$

However, by assumption,

$$m > \frac{(n-1)\Delta(G)}{2},$$

which contradicts the existence of such an edge coloring. Therefore,

$$\chi'(G) \neq \Delta(G).$$

By Vizing's Theorem,

$$\chi'(G) \leq \Delta(G) + 1.$$

Hence,

$$\chi'(G) = \Delta(G) + 1.$$

□

3.3 Coloring Applications

Example 3.14. Eight mathematics majors at a small college are permitted to attend a meeting dealing with undergraduate research during final exam week provided they make up all the exams missed on the Monday after they return. The possible time periods for these exams on Monday are:

(1) 8:00–10:00 (2) 10:15–12:15 (3) 12:30–2:30
(4) 2:45–4:45 (5) 5:00–7:00 (6) 7:15–9:15

Use graph theory to determine the earliest time on Monday that all eight students can finish their exams if two exams cannot be given during the same time period whenever some student must take both exams.

The courses are:

{AC, DE, G, GT, LP, MA, S, T}.

Proof. We construct a graph G whose vertices represent the courses. Two vertices are adjacent if there exists at least one student enrolled in both courses.

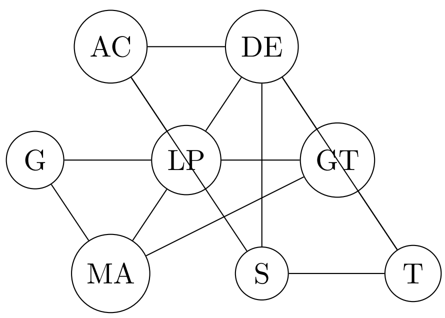

Figure 11: Conflict graph for the exam scheduling problem

A proper coloring of this graph corresponds to assigning time slots so that no two adjacent vertices (courses) share the same color.

Thus, the chromatic number $\chi(G)$ represents the minimum number of time periods required.

A proper coloring of the graph using four colors is possible, for example:

Color 1	AC, G, T
Color 2	DE, MA
Color 3	LP, S
Color 4	GT

This shows that four time periods are sufficient. The graph also contains a set of four courses that cannot all be scheduled at the same time, so fewer than four colors are not enough. Therefore,

$$\chi(G) = 4.$$

Since the fourth time period is 2:45–4:45 PM, the exams can be completed by 4:45 PM. □

Example 3.15. Alvin (A) has invited three married couples to his summer house for a week: Bob (B) and Carrie (C), David (D) and Edith (E), and Frank (F) and Gena (G).

Each guest will play a tennis match against every other guest except his or her spouse. In addition, Alvin will play a match against each of David, Edith, Frank, and Gena. If no one is to play two matches on the same day, determine a schedule of matches over the smallest number of days.

Proof. We construct a graph H whose vertices are the people:

$$V(H) = \{A, B, C, D, E, F, G\}.$$

Two vertices are adjacent if the corresponding people are to play a match.

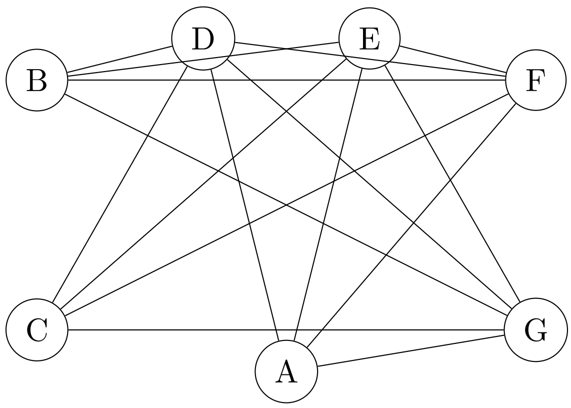

Figure 12: Graph H representing tennis matches

Each edge represents a match, and an edge coloring corresponds to assigning matches to days so that no vertex (person) is incident with two edges of the same color (day).

The maximum degree of H is $\Delta(H) = 5$. By Vizing's Theorem,

$$\chi'(H) = 5 \text{ or } 6.$$

Since H has $n = 7$ vertices and $m = 16$ edges, and

$$16 > \frac{(7-1) \cdot 5}{2} = 15,$$

it follows that $\chi'(H) = 6$.

Thus, at least 6 days are required.

A valid schedule is:

Day 1: $D - F,\ E - G$
Day 2: $D - G,\ E - F$
Day 3: $A - D,\ B - E,\ C - F$
Day 4: $A - E,\ B - D,\ C - G$
Day 5: $A - F,\ B - G,\ C - D$
Day 6: $A - G,\ B - F,\ C - E$

Therefore, the matches can be completed in six days. □

4 Ramsey Numbers

Ramsey's theorem studies when complete disorder is impossible: if a structure is large enough and its parts are colored or arranged arbitrarily, some form of organized pattern is guaranteed to appear.

4.1 Ramsey's Theorem

Theorem 4.1 (Ramsey's Theorem [CZ12])**.** *For any $k \geq 2$ positive integers $n_1, n_2, \ldots, n_k$, there exists a positive integer n such that if each edge of K_n is colored with one of the colors $1, 2, \ldots, k$, then for some integer i with $1 \leq i \leq k$, there exists a complete subgraph whose edges are all colored i and which has n_i vertices.*

This theorem can be interpreted as a statement about structure emerging from large systems: no matter how the edges of a sufficiently large complete graph are colored, one is guaranteed to find a large monochromatic complete subgraph.

We now focus on the case where two colors are used, typically red and blue. A red-blue coloring of a graph assigns each edge one of these two colors.

Let F be a graph. A subgraph is called a **red** F if it is isomorphic to F and all of its edges are colored red. Similarly, a **blue** F is a subgraph isomorphic to F whose edges are all colored blue.

Let F_1 and F_2 be nonempty graphs, and consider a red-blue coloring of the complete graph K_n. Depending on the coloring, it may or may not contain a red copy of F_1 or a blue copy of F_2.

Ramsey's Theorem guarantees that for sufficiently large n, every red-blue coloring of K_n must contain at least one of these monochromatic subgraphs.

For graphs F_1 and F_2, the **Ramsey number** $r(F_1, F_2)$ is defined as the smallest positive integer n such that every red-blue coloring of K_n contains either a red F_1 or a blue F_2.

To verify that $r(F_1, F_2) = n$, one must show two things:

1. every red-blue coloring of K_n contains a red F_1 or a blue F_2, and
2. there exists a red-blue coloring of K_{n-1} that contains neither a red F_1 nor a blue F_2.

Example 4.2. $r(K_3, K_3) = 6$.

Proof. We first show that $r(K_3, K_3) \leq 6$. Consider any red-blue coloring of the edges of K_6. Let v_1 be a vertex of K_6. Since v_1 is incident with five edges, at least three of these edges must have the same color by the Pigeonhole Principle. Without loss of generality, assume that v_1v_2, v_1v_3, and v_1v_4 are red.

Now consider the triangle formed by v_2, v_3, v_4. If any of the edges among these vertices is red, then together with v_1 we obtain a red triangle K_3. Otherwise, all three edges among v_2, v_3, v_4 are blue, forming a blue K_3. Hence a monochromatic triangle always exists, so

$$r(K_3, K_3) \leq 6.$$

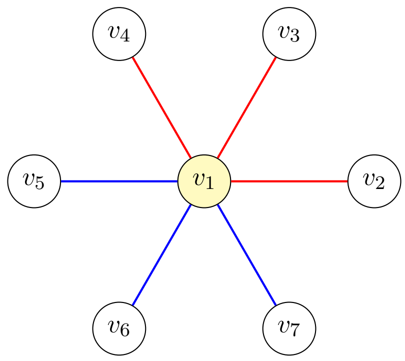

Figure 13: Three edges of the same color incident to v_1 in K_6

We now show that $r(K_3, K_3) \geq 6$. It suffices to construct a red-blue coloring of K_5 that contains no monochromatic triangle.

Let $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Color the edges of the 5-cycle

$$(v_1, v_2, v_3, v_4, v_5, v_1)$$

red, and color all remaining edges blue.

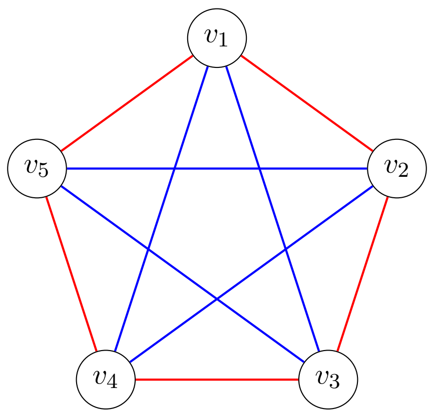

Figure 14: A red-blue coloring of K_5 with no monochromatic triangle

In this coloring, the red edges form a 5-cycle, which contains no triangle. The blue edges also do not contain a triangle. Therefore, no monochromatic K_3 exists in K_5, so

$$r(K_3, K_3) \geq 6.$$

Combining both results gives

$$r(K_3, K_3) = 6.$$

□

4.2 Applications of Ramsey Numbers: Biological Networks and Protein Interactions

Ramsey theory captures the idea that order must eventually appear in large enough systems, even when relationships are formed without any deliberate structure. In other words, once a network becomes sufficiently large, some form of regular pattern or uniform subgroup is unavoidable. This principle can be seen clearly in biological systems, particularly in protein–protein interaction networks.

Protein–protein interaction networks model how proteins inside a cell communicate and work together. In these networks, each vertex represents a protein, and each edge represents a physical or functional interaction between two proteins.

To make the model concrete, we consider several well-studied proteins involved in human cell regulation and signaling:

- *TP53* (tumor suppressor p53)
- *MDM2* (negative regulator of p53)
- *RB1* (retinoblastoma protein, cell-cycle control)
- *CDK2* (cyclin-dependent kinase, cell-cycle progression)
- *EGFR* (epidermal growth factor receptor, signaling receptor)
- *GRB2* (adaptor protein in signaling pathways)
- *SOS1* (activates Ras signaling)

Interactions in such models are often simplified into two categories: strong interactions, which correspond to stable binding within functional protein complexes, and weak interactions, which represent transient or context-dependent signaling relationships.

For example, TP53 and MDM2 form a strong regulatory feedback loop that controls cellular stress responses, while RB1 and CDK2 interact strongly within the cell-cycle regulation system. In contrast, EGFR, GRB2, and SOS1 form a signaling chain where interactions are more temporary and depend on cellular conditions.

This structure is illustrated in the following network.

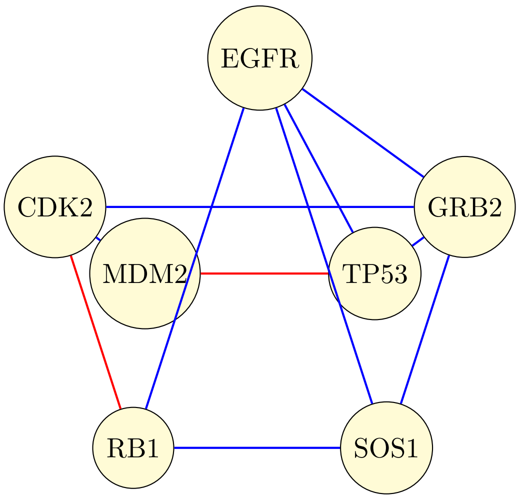

Figure 15: Protein–protein interaction network. Red edges represent strong functional interactions such as TP53–MDM2 and RB1–CDK2, while blue edges represent weaker signaling interactions such as EGFR–GRB2–SOS1.

Although the network appears complex and irregular at first glance, Ramsey-type reasoning implies that complete disorder cannot persist at large scale. Once the system is sufficiently large, subsets of proteins must exist in which interactions become uniform in type. These forced structures correspond biologically to functional modules such as stable protein complexes or coordinated signaling pathways.

From this viewpoint, Ramsey theory provides a mathematical explanation for why biological systems naturally organize into clusters and pathways, even without any global design or external coordination.

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