SURFACES IN KNOT THEORY

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Abstract. This paper studies the significance of surfaces in knot theory, and focuses on surfaces bounded by knots, including their topological properties and Seifert surfaces. The paper is set up in three parts: an introduction to knots, followed by a definition of surfaces and their characteristics, and finally, properties of Seifert surfaces. Lastly we discuss genus, orientability, compressibility, and the concept of the Euler characteristic, which all act as invariants for our surfaces, and we introduce Seifert surfaces as orientable surfaces that are related to knots and links.

1. Introduction

Definition 1.1 (Knots)**.** A knot is a closed loop embedded into $\mathbb{R}^3$. It cannot cut, be glued to, or intersect itself.

Definition 1.2 (Ambient isotopy)**.** An ambient isotopy is a continuous deformation of the entire three dimensional space ($\mathbb{R}^3$) that carries one knot into another.

Two knots are equivalent to each other if one can be deformed into the other via ambient isotopy, or by shifting the strands of the knot around in relation to each other without cutting the knot or intersecting itself in any way. To study knots, we look at them as two dimensional projections, taking what is a three dimensional object and looking at it from a specific angle, translating what we see into $\mathbb{R}^2$.

1.1. **Examples of knots.** Figure 1 shows three knots, one that's an example of the trivial knot and two examples of prime knots, which will be defined soon.

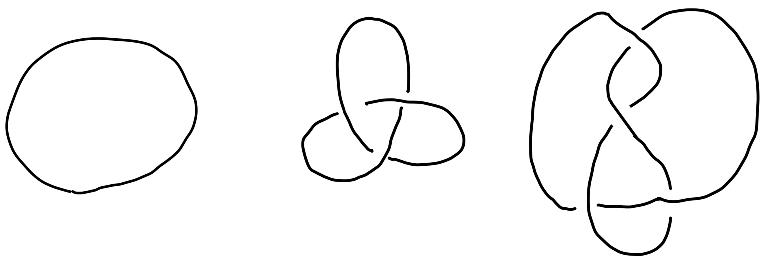

Figure 1. (From left to right) Unknot, Trefoil knot, Figure Eight knot.

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The unknot is a trivial knot and is the simplest knot as it is not knotted within itself. It can be twisted, tangled, looped, and deformed, but it can always be untangled into a simple loop via ambient isotopy. The trefoil and figure eight knots are non-trivial knots, which mean that they cannot be deformed by ambient isotopy to the unknot, and have intrinsic features that prevent them from doing so. The trefoil knot and figure eight knot are also examples of **prime knots**.

Definition 1.3 (Prime knots)**.** Prime knots are non-trivial knots that cannot be expressed as the **composition** of other non-trivial knots.

Definition 1.4 (Composition of knots)**.** We define a composition of knots as the result of the process of removing two small arcs from two separate knots, and joining them via the loose strands such that they form one composite knot.

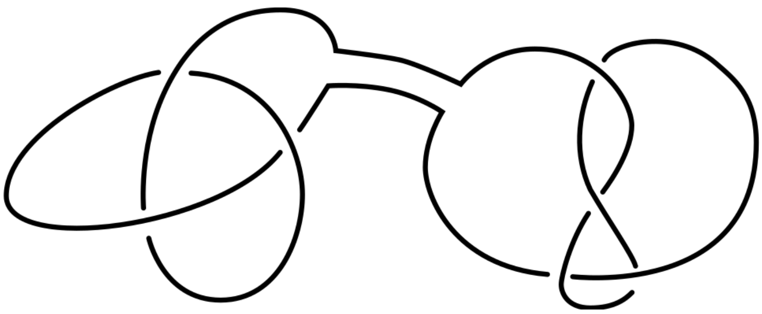

FIGURE 2. Composition of two knots, the trefoil (left component) and the figure eight (right component).

1.2. **Properties of Knots.** All knots have certain traits that one uses to distinguish different knots from one another. We call these characteristics **invariants**. A knot invariant is a property that does not change under deformation of the knot, as long as no cutting and gluing is used in the process. In general, these properties don't change under an ambient isotopy. One famous example of an ambient isotopy used to distinguish knots while conserving these invariants includes **Reidemeister moves**.

Definition 1.5 (Reidemeister moves)**.** Reidemeister moves are deformations of a local subsection of a knot that can be used to rearrange the knot into a new two dimensional projection of the same three dimensional shape.

By definition, since Reidemeister moves are ambient isotopies, if two knots differ by only these moves and can be rearranged using these moves to have the same representation, they are the same knot. For example, the two knots in Figure 3 are the same knot, which can be shown via a sequence of Reidemeister moves to get from one to the other.

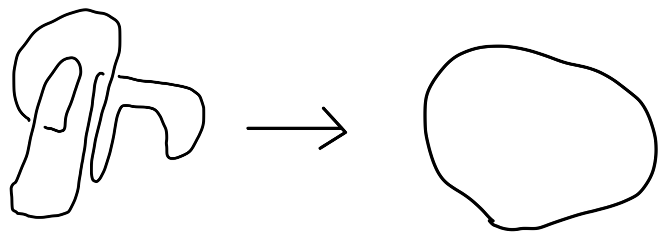

FIGURE 3. Two representations of the unknot.

There are three types of Reidemeister moves, each unique in their function, yet contributing to the same goal. First we have Type I Reidemeister moves, identified by their "twisting" or "untwisting" motion which either adds or removes a crossing between strands as shown in Figure 4.

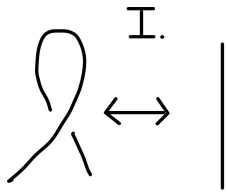

FIGURE 4. Type I Reidemeister Move.

Next we have Type II Reidemeister moves, which can be described as pulling one strand over or under another strand as shown in Figure 5.

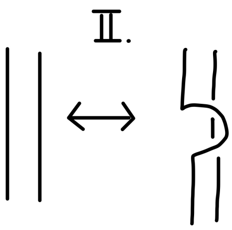

FIGURE 5. Type II Reidemeister Move.

Lastly we have Type III Reidemeister moves, described as sliding a strand over or under a crossing formed by two other strands as seen in Figure 6.

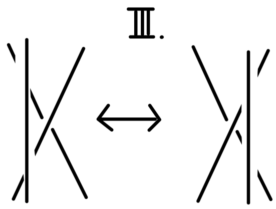

Figure 6. Type III Reidemeister Move.

Reidemeister moves are important because they give us a complete set of local transformations. We can say that two knot diagrams represent the same knot if and only if one can be obtained from the other by a finite sequence of Reidemeister moves. This means that to prove two knots are the same, it suffices to find such a sequence.

2. Surfaces without Boundary

Definition 2.1 (Surfaces)**.** A surface is a **two-dimensional manifold**, or a space where each point has a neighborhood that looks like an open disc in two-dimensional Euclidean space. This means that locally, it looks flat. At any point on a surface, there exists some disc-like neighborhood lying on the surface surrounding the point.

We give a few examples of surfaces and non-surfaces:

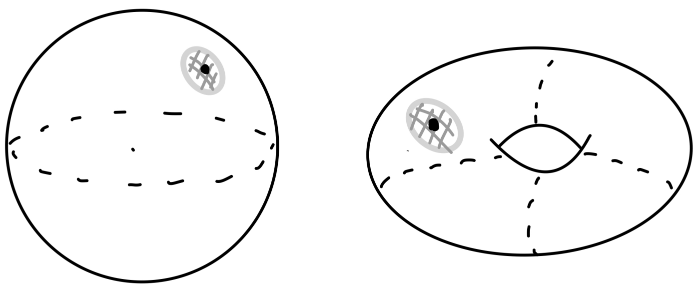

Figure 7. Examples of a surface.

The surfaces in Figure 7 both satisfy the property that at any point on the surface, a disc-like neighborhood can be drawn around it. The surface on the left is called a sphere, while the surface on the right is a torus, which takes the shape of a donut — specifically, only the outer shell, like the glaze on a donut rather than the solid object itself.

In Figure 8, the surface on the left is extended by a line and a point, such that any point on this line cannot have a disc made around it. The surface on the right also contains a single point, similarly not surrounded by a disc. Therefore, these two examples are not surfaces.

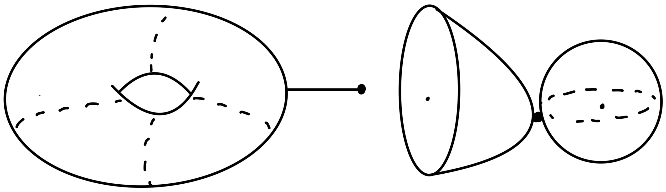

FIGURE 8. Examples that are not surfaces.

2.1. **Triangulation.** Triangulation is the process of separating a surface into a finite number of triangles such that neighboring triangles share edges and vertices, without overlapping. As shown below in Figure 9, each triangulation is different, or each surface is divided into a different number of triangles. Triangulation also allows us to compute invariants such as the Euler characteristic.

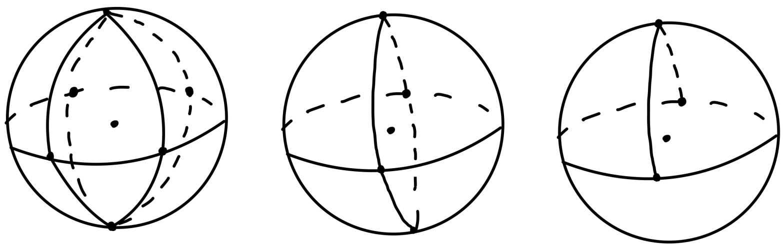

FIGURE 9. Triangulation of the same sphere.

Definition 2.2 (Euler Characteristic)**.** A surface's Euler characteristic is defined by the following equation:

$$\chi = V - E + F \tag{2.1}$$

Where V is the number of vertices, E is the number of edges, and F is the number of faces obtained when a surface is triangulated. The Euler characteristic is an invariant of a surface, where any representation of the same surface will have the same value for its Euler characteristic. Even if the surface is triangulated differently, the Euler characteristic remains constant for the same surface.

We can determine that the above statement is true by looking at what happens when we add a vertex on an edge and in the middle of a face. If we add a vertex on an edge, it splits that edge into two line segments, thus changing our formula to:

$$\chi = (V + 1) - (E + 1) + F = \boxed{V - E + F}$$

No change in overall formula for the Euler characteristic. If we instead add a vertex within a face and add three line segments or edges that connect the new vertex to the three on the outside of the triangular face (making three separate triangular faces in the process) our formula changes to:

$$\chi = (V + 1) - (E + 3) + (F + 2) = \boxed{V - E + F}$$

No change in overall formula, and thus Euler characteristic remains the same for the same shaped surface regardless of triangulation process.

2.2. **Connected Sum.** The connected sum of two surfaces S and T, denoted by $S\#T$, is formed by removing a small open disc from each surface and gluing the two surfaces together along the resulting circular boundary components.

The connected sum is the sum of the Euler characteristics of the two composing surfaces subtracted by two. This is because when we remove a disc from each surface and glue along the boundaries, this disc can be considered to be a triangle in the triangulation of each surface. We take away three edges and three vertices, and two fewer faces. When calculated, we get a correctional factor of $3 - 3 - 2 = -2$. Thus, the Euler characteristic of the connected sum is found using the following:

$$\chi(S\#T) = \chi(S) + \chi(T) - 2 \tag{2.2}$$

Where S and T are surfaces.

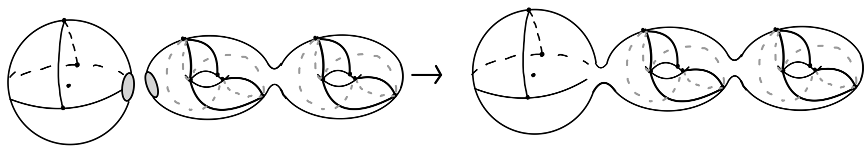

FIGURE 10. The connected sum of two surfaces.

Definition 2.3 (Homeomorphism)**.** Two surfaces are homeomorphic if there exists a continuous bijection between them with a continuous inverse — intuitively, one can be continuously deformed into the other without tearing or gluing. Equivalently, two surfaces are homeomorphic if they have the same topological type: the same genus, the same number of boundary components, and the same orientability.

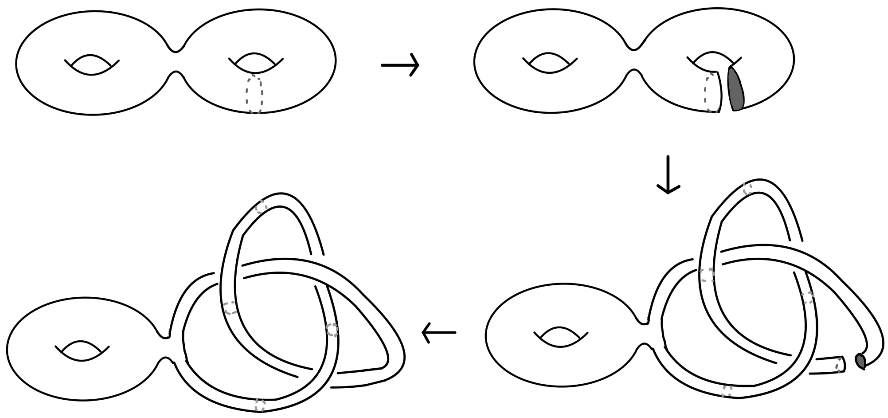

FIGURE 11. Two surfaces that are homeomorphic.

Definition 2.4 (Compressibility)**.** A surface embedded in $\mathbb{R}^3$ is considered compressible if there exists a simple, closed curve on that surface that bounds a disc in $\mathbb{R}^3$, but not a disc on the surface itself.

Informally, a surface is compressible if there is a closed loop on it that could be 'pushed inward' to bound a disc in the ambient space, without that disc intersecting the surface itself. When the surface is a Seifert surface for a knot K, we additionally require the disc to lie in $\mathbb{R}^3 \setminus K$, i.e., not intersecting the knot.

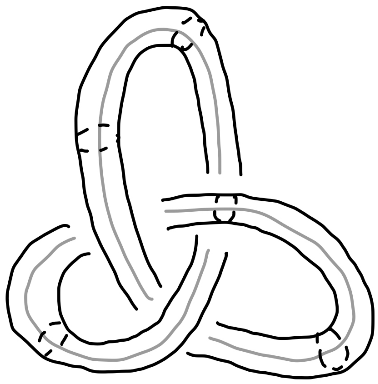

FIGURE 12. A non-compressible surface.

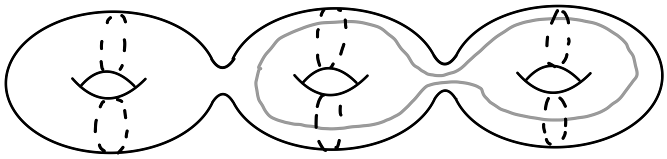

FIGURE 13. A compressible surface.

This surface is compressible because there exists a closed curve on the surface that bounds a disc such that it doesn't cut or intersect the knot. This disc can be drawn on the first hole on the left of the surface.

2.3. **Genus.**

Definition 2.5 (Genus)**.** Intuitively, the Genus of a surface is determined by the number of holes in that surface, i.e. a sphere has Genus 0, and a torus, which in standard form looks like a donut, has genus 1, seen in Figure 14 on the next page.

The Genus of a surface is a topological invariant, making it a useful tool to classify surfaces.

Theorem 2.6 (Using genus to calculate Euler characteristic)**.**

$$\chi = 2 - 2g \tag{2.3}$$

This is how to calculate a surface's Euler characteristic using its genus instead of calculating it through triangulation.

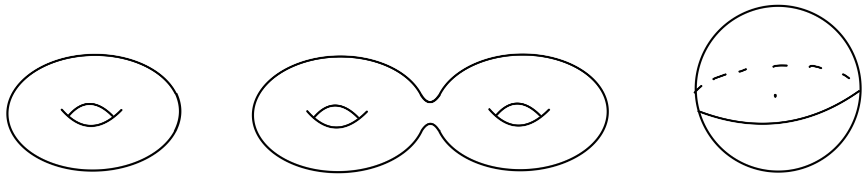

FIGURE 14. Surfaces with different genera.

Proof. We proceed by induction on genus g.

(1) Base case: the torus has genus 1, and $\chi = 2 - 2(1) = 0$. ✓
(2) Next, let g be the genus of a surface. A surface with genus $g+1$ can be obtained by cutting and adding a cylinder such that this adds another hole to the surface.
(3) Given that $\chi(1) = 0$, the Euler characteristic of the surface with genus $g+1$ is $\chi(g+1) = \chi(g) + \chi(1) - 2 = \chi(g) - 2$ by using the connected sum theorem.
(4) By the inductive hypothesis $\chi(g) = 2 - 2g$, so $\chi(g+1) = \chi(g) - 2 = (2-2g) - 2 = -2g = 2 - 2(g+1)$, completing the induction.

□

Theorem 2.7 (Euler characteristic with boundaries)**.**

$$\chi = 2 - 2g - b \tag{2.4}$$

Equation (2.3) describes χ for surfaces without boundary. The formula above extends this to surfaces with b boundary components.

Proof. A surface with one boundary component can be obtained from a closed surface by removing an open disc, which corresponds to removing one face from a triangulation while leaving vertices and edges unchanged. By $\chi = V - E + F$, this decreases χ by 1. Applying this argument inductively, removing b such discs decreases χ by b, giving $\chi = (2-2g) - b = 2 - 2g - b$ □

3. Surfaces with Boundary

Having established the properties of closed surfaces, we now turn to surfaces that have a boundary — a one-dimensional edge curve — such as a disc or cylinder. The key new property for these surfaces is orientability.

Definition 3.1 (Orientability)**.** A surface is considered orientable if it has a choice for a consistent normal vector. It can be described as having two distinct sides such that if one side was painted black, the other side must be painted a different color to cover it. A non-orientable surface has only one continuous side, such that if one side was painted black, the entire surface would be black.

Definition 3.2 (Möbius Strip)**.** A Möbius strip is a non-orientable surface that does not have two distinct faces. Rather, it has one continuous side.

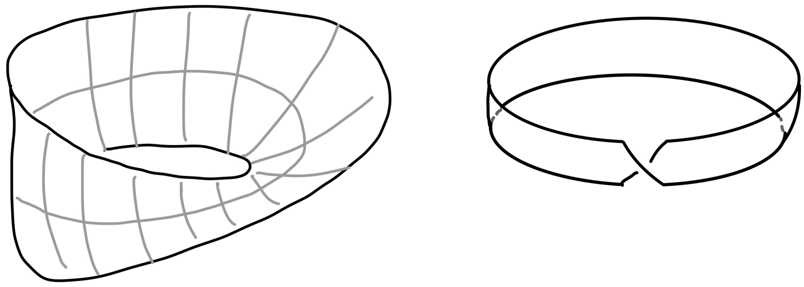

FIGURE 15. Two representations of a non-orientable Möbius strip.

3.1. **Seifert Surfaces.**

Definition 3.3 (Seifert surfaces)**.** A Seifert surface of a knot K is a fully connected and orientable surface whose boundary is exactly the knot K it bounds to.

Every knot has at least one Seifert surface. Different Seifert surfaces can have different genera. However, the **minimum genus** is especially important as it is a knot invariant. One natural way to construct a Seifert surface is via the Seifert algorithm: assign an orientation to the knot, smooth each crossing consistently with the orientation, and fill in the resulting circles as discs connected by twisted bands. This guarantees the existence of at least one Seifert surface for every knot, though other Seifert surfaces for the same knot may not arise from this algorithm.

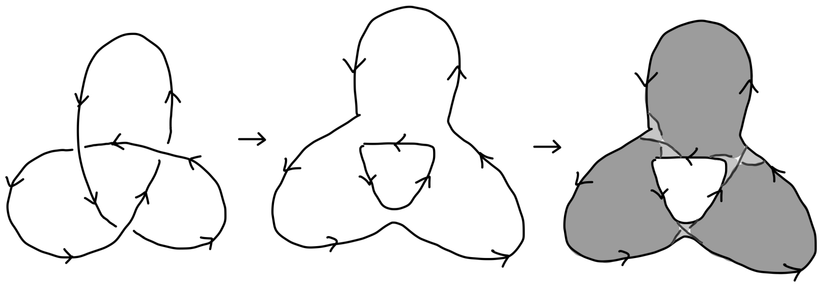

FIGURE 16. Seifert surface from the trefoil.

The resulting simple curves are called **Seifert circles**. The **boundary components** of a surface are the disjoint curves that form a boundary. For a Seifert surface of a knot, there is exactly one boundary component corresponding to the knot it bounds. Let s denote the number of Seifert circles, c denote the number of crossings, and b denote the number of boundary components. From here, we can obtain these two equations:

$$\chi = s - c \tag{3.1}$$

This is a representation of the Euler characteristic using the number of Seifert circles and crossings a surface has.

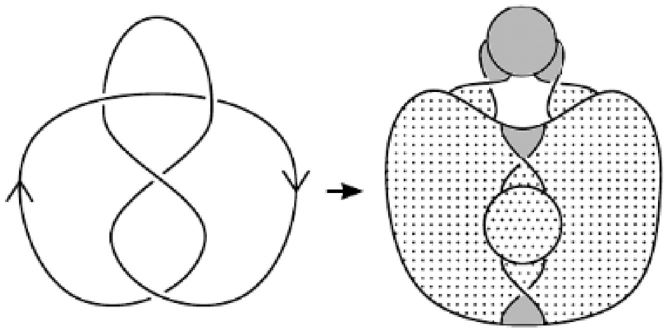

FIGURE 17. Seifert surface from the figure eight knot.

$$g = (c - s + 1)/2 \tag{3.2}$$

This calculates the genus of the Seifert surface from the number of crossings and Seifert circles, and assumes $b = 1$ (i.e., the surface bounds a knot with a single boundary component).

These equations come from the Seifert algorithm, and can be used in any situation that regards a Seifert surface.

We now examine how genus behaves under the connected sum operation, extending our earlier result from Euler characteristics to genus directly. If we define $J\#K$ as the composition of two knots J and K, then we can also say that:

Theorem 3.4 (Additivity of genus under connected sum)**.**

$$g(J\#K) = g(J) + g(K).$$

Genus thus behaves additively under connected sum, as illustrated in Figure 18.

Proof. Let g_1 and g_2 denote the genera of surfaces J and K, respectively. Using what we know from connected sum, we can say that $\chi(J\#K) = \chi(J) + \chi(K) - 2$. Similarly, we know that we can derive a surface's Euler characteristic using the genus of a surface, with the formula $\chi(g) = 2 - 2g$ as $\chi(g_{total}) = 2 - 2(g_{total})$. Therefore, $\chi(J) = 2 - 2g_1$ and $\chi(K) = 2 - 2g_2$. Thus $\chi(J\#K) = (2 - 2g_1) + (2 - 2g_2) - 2 = 2 - 2g_1 - 2g_2 = 2 - 2(g_1 + g_2)$. This means $\chi(J\#K) = 2 - 2(g_1 + g_2)$.

We know that $\chi(J\#K) = 2 - 2(g_1 + g_2)$; therefore we can say, $2 - 2(g_{total}) = 2 - 2(g_1 + g_2)$. Simplifying this results in $-2(g_{total}) = -2(g_1 + g_2)$ which is $g_{total} = g_1 + g_2$. Thus, the composition of a genus under the connected sum of two surfaces is the sum of the individual surfaces' genera.

□

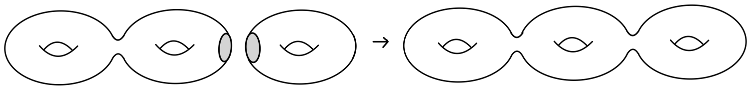

FIGURE 18. Composition of two surfaces with different genera.

Conclusion

Throughout this paper, we have developed the foundational tools necessary in order to study surfaces in knot theory. Starting from knots and their diagrams, we introduced surfaces as two-dimensional manifolds and studied their invariants, namely the Euler characteristic, genus, orientability, and compressibility. We then focused on Seifert surfaces. This topic connected our topological concepts that we covered directly to knots: every knot bounds a Seifert surface, and the minimum genus of such a surface is itself a knot invariant. The additivity of genus under connected sum further illustrates how these invariants interact. These ideas open the door to deeper questions, such as how to compute the genus of a given knot, what other invariants can be extracted from Seifert surfaces, and how these properties pertain to modern fields of study, such as computational biology, quantum computing, and more.

References

[1] Colin C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, American Mathematical Society, 2004.

[2] Sakuma, Makoto. *A survey of the impact of Thurston's work on Knot Theory.* arXiv:2002.00564, 2020.