

Analyzing the Logistic Map as a Dynamical System

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Abstract

Consider the iterated equation $x_n = 1.1x_{n-1}$. We can write this as a dynamical system, $f(x) = 1.1x$. We examine the dynamical system $L_\lambda(x) = \lambda x(1-x)$ and how it behaves with the orbits of specific points. We look at attracting/repelling points, an abstraction of the function to a sequence space, and the dynamics of the function's complex variant.

1 Dynamical Systems

1.1 What is a Dynamical System?

In this paper, we will be looking at discrete dynamical systems. We use the word discrete to refer to functions or dynamical systems which are defined for input values starting at zero and progressing through the positive integers. In general, a *dynamical system* is a system that changes through time. In a discrete dynamical system, time is discrete, so it too usually starting at zero and progressing through the positive integers.

Example 1.1.1. Let's say I put some money in the bank, which gives me 10% interest every year. So, if I put $x_0 = \$100$ in the bank, then:

After zero years, I will have $x_0 = 100$ dollars.

After one year, I will have $x_1 = x_0 \cdot 1.1 = 110$ dollars.

After two years, I will have $x_2 = x_1 \cdot 1.1 = 121$ dollars.

We can model this using a function, which takes an amount of money as an input and returns the amount of money with interest added. In this case, we have $F(x) = 1.1x$, meaning that if I put x dollars into the bank I will have $1.1x$ dollars a year later. We can rewrite our early statements as

$$x_0 = \$100.$$

$$x_1 = F(x_0) = 1.1x_0 = \$110.$$

$$x_2 = F(x_1) = 1.1x_1 = \$121.$$

Often we want to study the behavior of a dynamical system over long periods of time. In this case, we can see that if we put any positive real number of dollars into the bank, as more and more years pass, the amount of money we have will grow without bound; that is, it will diverge to positive infinity.

Example 1.1.2. We can also look at more abstract dynamical systems. Define the *logistic map* as $L_\lambda(x) = \lambda x(1-x)$, which is used in modeling population growth. We can then *iterate* this function (apply it repeatedly to a given starting value) and see what happens with different starting points.

Note how this function has a *parameter*, λ , which we can change the value of. For this function, we will only look at $\lambda > 0$; one of the goals of this paper is to analyze the behavior the logistic function for all such values of λ .

Definition 1.1.3. In general, a *discrete dynamical system* on a set S (generally \mathbb{R}) is specified by a function $F : S \rightarrow S$ to itself, where for a given x_0 , we recursively define $x_{n+1} = F(x_n)$, where n represents our discrete time variable. The goal is to analyze the behavior of different inputs as we iterate the given function. Some functions one might analyze are

- $Q_c(x) = x^2 + c$,
- $S(x) = \sin(x)$, and
- $E_\lambda(x) = \lambda e^x$.

We will focus primarily on the logistic map in this paper.

1.2 Orbits and Iteration

Let's introduce some notation to work with these systems.

Definition 1.2.1. We define the *n*th iterate of F as

$$F^n(x) := \underbrace{(F \circ F \circ \dots \circ F)}_{n \text{ F's}}(x) = \underbrace{F(F(F(\dots F(x))))}_{n \text{ F's}},$$

that is, applying the function F^n to an input is equivalent to applying F to that input n times. We then define the *orbit* of a point x_0 under F to be

$$x_0, x_1, x_2, \dots, x_n, \dots \text{ where } x_i = F^i(x_0).$$

You may recognize this notation from Example 1.1.1, where we put \$100 in the bank with 10% interest; using our new terminology we can now state that the orbit of 100 under $F(x) = 1.1x$ begins 100, 110, 121..., and diverges to positive infinity.

Example 1.2.2. Let's look at a specific orbit under our logistic map $L_\lambda(x) = \lambda x(1 - x)$. We choose $\lambda = 1$ and set $x_0 = \frac{1}{2}$. To find the orbit of x_0 , we calculate

$$\begin{aligned} x_0 &= \frac{1}{2} \\ x_1 &= L_1(x_0) = 1 \cdot \left(\frac{1}{2}\right) \cdot \left(1 - \frac{1}{2}\right) = \frac{1}{4} \\ x_2 &= L_1^2(x_0) = 1 \cdot \left(\frac{1}{4}\right) \cdot \left(1 - \frac{1}{4}\right) = \frac{3}{16} \\ x_3 &= L_1^3(x_0) = 1 \cdot \left(\frac{3}{16}\right) \cdot \left(1 - \frac{3}{16}\right) = \frac{39}{256}. \end{aligned}$$

So the orbit of $\frac{1}{2}$ under F_1 begins $\frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{39}{256}, \dots$, and it seems that it will slowly approach zero. This in fact is true, and we will make this idea more rigorous later.

1.3 Fixed and Periodic Points

There are some specific types of orbits that we care about.

Definition 1.3.1. A *fixed point* of a function F is a point x for which $F(x) = x$; it is unchanged by the function. The orbit of a fixed point x will simply be x, x, x, \dots since $F(x) = x$ implies $F^n(x) = x$ for all x .

Example 1.3.2. The interest function from earlier, $F(x) = 1.1x$, has a fixed point at $x = 0$ since $F(0) = 1.1 \cdot 0 = 0$. In context, this means that if you put zero dollars into the bank, you will still have zero dollars after any number of years.

Example 1.3.3. Let's find all the fixed points of the logistic function. To do this, we solve the equation $L_\lambda(x) = \lambda x(1 - x) = x$. Note that 0 is a solution of this equation; to find other solutions, we can divide by x to find that

$$x = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

Therefore L_λ has two fixed points; one at 0 and one at $p = \frac{\lambda - 1}{\lambda}$ (recall that this is always defined since $\lambda > 0$). Note that these are distinct, unless $\lambda = 1$, in which case $p = 0$ and there is only one fixed point.

We also care about *periodic points*; a point x is periodic with *period* n (also called an *n-cycle*) if $F^n(x) = x$. If n is the least such n , we call n the *prime period* of x (since every 2-cycle is also technically a 4-cycle, etc.). The orbit of a periodic point with prime period n will be $x_0, x_1, \dots, x_n, x_0, x_1, \dots$, and will loop forever. Note that every fixed point is periodic with prime period 1.

Example 1.3.4. Let's see if L_λ has any 2-cycles. This requires solving

$$L_\lambda^2(x) = L_\lambda(\lambda x(1-x)) = \lambda(\lambda x(1-x))(1-\lambda x(1-x)) = x.$$

This simplifies to

$$-\lambda^3 x^4 + 2\lambda^3 x^3 - (\lambda^3 + \lambda^2)x^2 + (\lambda^2 - 1)x = 0.$$

Note that, since our two fixed points 0 and $p = \frac{\lambda-1}{\lambda}$ are also 2-cycles (they both satisfy $L_\lambda^2(x) = x$), they are roots of this polynomial; we can thus divide this polynomial by $x(x - \frac{\lambda-1}{\lambda})$ to obtain the quadratic

$$(-\lambda^3)x^2 + (\lambda^3 + \lambda^2)x + (-\lambda^2 - \lambda) = 0.$$

We can then apply the quadratic formula to find the solution

$$q_\pm = \frac{\lambda + 1 \pm \sqrt{(\lambda - 3)(\lambda + 1)}}{2\lambda}.$$

One can confirm that $L(q_+) = q_-$ and $L(q_-) = q_+$. Note that, since $(\lambda + 1)$ is positive for all $\lambda > 0$, the inside of the square root is nonnegative if and only if $\lambda \geq 3$, meaning that this 2-cycle only exists for these values of λ . When $\lambda = 3$, we have $q_+ = q_- = p = \frac{2}{3}$; this is an example of a *period-doubling bifurcation*, as we will see later.

1.4 Attracting and Repelling Points

Two other notable types of points we care about are *attracting* and *repelling* points. A fixed point is called *attracting* when the orbits of nearby points converge toward the fixed point. Similarly, a fixed point is called *repelling* when the orbits of nearby points diverge away from the fixed point.

Example 1.4.1. Take L_λ with $\lambda = 2$. This puts the fixed points of L_λ at 0 and $\frac{1}{2}$. First, let us look at the fixed point at $x = 0$. We can choose values close to 0 and compute their orbits to determine if 0 is an attracting or repelling fixed point. Looking at the orbits of -0.1 and 0.1 , we get

$$\begin{array}{l} -0.1, -0.22, -0.54, -1.65, \dots \\ 0.1, 0.18, 0.29, 0.42, \dots \end{array}$$

This shows that point close to the fixed point at 0 diverge away upon iteration, meaning 0 is a repelling fixed point. Doing the same with $\frac{1}{2}$, we see that the orbits of 0.4 and 0.6 are

$$\begin{array}{l} 0.4, 0.48, 0.499, 0.49999, \dots \\ 0.6, 0.48, 0.499, 0.49999, \dots \end{array}$$

which shows that the fixed point at $\frac{1}{2}$ is attracting.

A fixed point can be determined to be attracting or repelling using its derivative. Given x_0 as a fixed point for a function F :

- if $|F'(x_0)| < 1$ then x_0 is attracting;
- if $|F'(x_0)| > 1$ then x_0 is repelling;
- and if $|F'(x_0)| = 1$ then x_0 is neutral.

To determine whether a cycle is repelling or attracting, we find the derivative of F^n instead of F at any point on the cycle.

Example 1.4.2. Using the chain rule, we may analyze the points of the 2-cycle of L_λ to determine whether they are attracting or repelling. We have $L'_\lambda(x) = \lambda - 2\lambda x$, so

$$\begin{aligned} (L_\lambda^2)'(q_+) &= L'_\lambda(q_+)L'_\lambda(q_-) \\ &= (\lambda - 2\lambda q_+)(\lambda - 2\lambda q_-) \\ &= (\lambda - (\lambda + 1 + \sqrt{(\lambda - 3)(\lambda + 1)}))(\lambda - (\lambda + 1 - \sqrt{(\lambda - 3)(\lambda + 1)})) \\ &= (-1 - \sqrt{(\lambda - 3)(\lambda + 1)})(-1 + \sqrt{(\lambda - 3)(\lambda + 1)}) \\ &= 1 - (\lambda - 3)(\lambda + 1). \end{aligned}$$

Keeping in mind that this 2-cycle only exists for $\lambda > 3$, we know that this quantity is always less than 1; it becomes less than -1 when $(\lambda - 3)(\lambda + 1) > 2$, which is equivalent to $\lambda > 1 + \sqrt{6} \approx 3.4$. So this cycle is attracting for $3 < \lambda < 1 + \sqrt{6}$, neutral for $\lambda = 1 + \sqrt{6}$, and repelling for $\lambda > 1 + \sqrt{6}$.

There are two theorems relating to attracting and repelling fixed points detailing the convergence and divergence of neighboring points, appropriately named the Attracting Fixed Point Theorem and the Repelling Fixed Point Theorem.

Attracting Fixed Point Theorem. *Suppose x_0 is an attracting fixed point for a function F . There must be an interval I that contains x_0 in its interior such that for all $x \in I$ it is that $F^n(x) \in I$ for all n and $F^n(x) \rightarrow x_0$ as $n \rightarrow \infty$.*

Repelling Fixed Point Theorem. *Suppose x_0 is a repelling fixed point for a function F . There must be an interval I that contains x_0 in its interior such that for all $x \in I$ where $x \neq x_0$ there exists an $n > 0$ such that $F^n(x) \notin I$.*

The full proof relies on the Mean Value Theorem; it can be found in Devaney's book.

1.5 Bifurcations

The literal meaning of the word bifurcation is a division in two, splitting apart, which is a strong way to describe the behavior of a function as it undergoes a bifurcation. For a function F_c with parameter c , F_c is said to undergo a *bifurcation* at a parameter value c_0 when there is a change in the number of fixed points of F_c between c values less than c_0 and c values greater than c_0 .

The most basic kind of bifurcation is known as a *saddle-node bifurcation*. A function F_c undergoes a *saddle-node bifurcation* at the parameter value c_0 if there is an open interval I and an $\epsilon > 0$ such that:

1. For $c_0 - \epsilon < c < c_0$, F_c has no fixed points in I .
2. For $c = c_0$, F_c has one fixed point in I which is neutral.
3. For $c_0 < c < c_0 + \epsilon$, F_c has two fixed points in I , which one attracting and one repelling.

This definition of a saddle-node bifurcation is fairly intuitive, a function undergoes a saddle-node bifurcation when a fixed point suddenly appears and splits in two. The order of the inequalities is not important, as they can be flipped and still result in a saddle node bifurcation. The following example demonstrates this.

Example 1.5.1. Take $Q_c(x) = x^2 + c$ at $c = \frac{1}{4}$. For $c > \frac{1}{4}$ there are no fixed points, for $c = \frac{1}{4}$ there is one neutral fixed point, and for $-\frac{3}{4} < c < \frac{1}{4}$ there is a pair of fixed points with one attracting and one repelling. To be absolutely rigorous, we may choose $\epsilon = 1$ and $I = \mathbb{R}$ to show that $c = \frac{1}{4}$ is indeed a saddle-node bifurcation.

Another important bifurcation is the *period-doubling bifurcation*. A function F_c is said to undergo a period-doubling bifurcation for a parameter value c_0 if there is an open interval I and $\epsilon > 0$ such that:

1. For each c in $[c_0 - \epsilon, c_0 + \epsilon]$ there is a unique fixed point p_c in I .

2. For $c_0 - \epsilon < c \leq c_0$, F_c has no 2-cycles in I and p_c is attracting.
3. For $c_0 < c < c_0 + \epsilon$, there is a unique attracting 2-cycle q_c^1, q_c^2 in I with $F_c(q_c^1) = q_c^2$, and p_c is repelling.
4. As $c \rightarrow c_0$, $q_c^i \rightarrow p_{c_0}$.

Example 1.5.2. Take L_λ with $\lambda = 3$. For values of λ near 3 there is a unique fixed point around $x = \frac{2}{3}$. For parameter values less than 3 this fixed point is attracting and there are no 2-cycles nearby. For parameter values greater than 3 there is always a unique attracting 2-cycle nearby and the fixed point becomes repelling. As parameter values tend toward 3 the unique fixed points approach $\frac{2}{3}$. This shows the logistic map undergoes a period-doubling bifurcation at $\lambda = 3$.

2 Chaos

In this section, we will examine how the logistic map exhibits seemingly “chaotic” properties, and give a rigorous definition of a chaotic dynamical system. To use this definition, however, we need the tools from a branch of dynamical systems theory called *symbolic dynamics*, which we will look at after taking a brief excursion in orbit diagrams.

2.1 Orbit Diagrams

A useful tool in the analysis of dynamical systems are *orbit diagrams*. An *orbit diagram* is a diagram used to describe the behavior of all orbits of a dynamical system. In an orbit diagram, the locations of fixed points are plotted along the y axis, and the value of a parameter is plotted along the x axis.

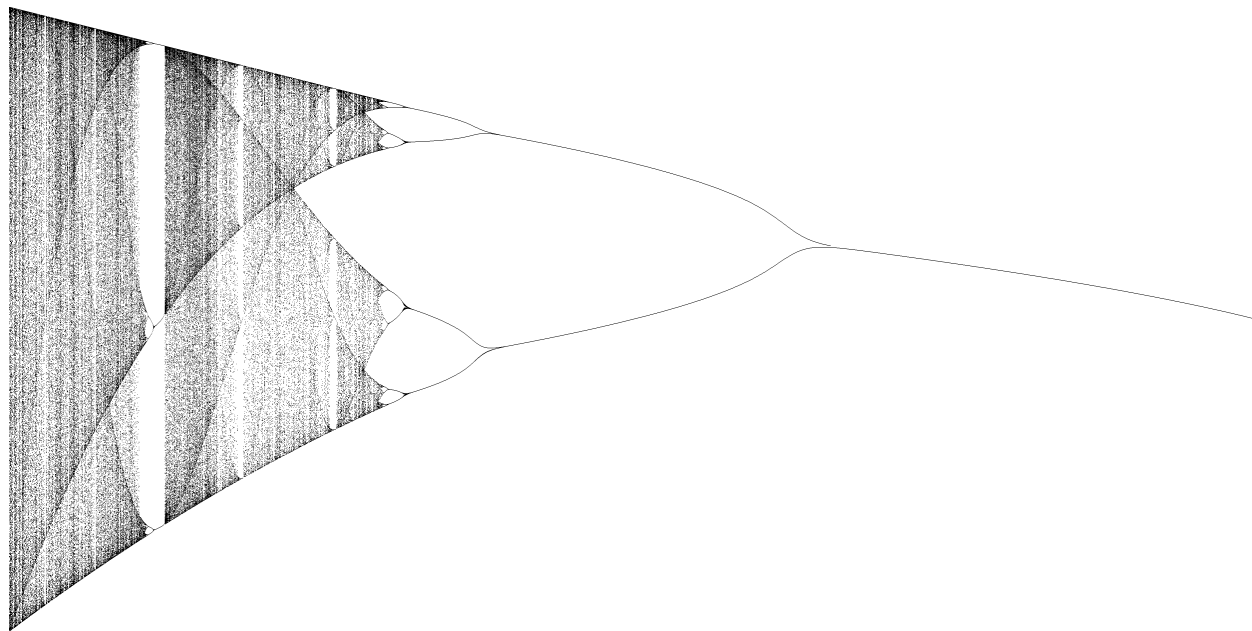


Figure 1: The orbit diagram of $Q_c(x) = x^2 + c$

In Figure 1 we observe divisions in the locations of the fixed points of Q_c which correspond to period doubling bifurcations in Q_c . The black lines represent the x values of the fixed points, which bifurcate repeatedly as the parameter value decreases.

2.2 The Sequence Space

In this section, we will begin introducing the tools of symbolic dynamics.

Definition 2.2.1. We define the *sequence space* as the set

$$\Sigma = \{(s_0s_1s_2, \dots) \mid s_i = 0 \text{ or } 1\},$$

that is, Σ is the set of all sequences constructed of ones and zeroes. For example, the sequence $(000\dots) = (\bar{0})$ of all zeroes is in Σ , as well as the sequence $(111\dots) = (\bar{1})$. We can then turn this set into a mathematical object called a *metric space* by defining a *metric* on it, which is essentially a distance function: it takes two inputs in Σ and returns the "distance" between them. We define this metric as

$$d[\mathbf{s}, \mathbf{t}] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i},$$

where $\mathbf{s} = (s_0s_1s_2\dots)$ and $\mathbf{t} = (t_0t_1t_2\dots)$ are elements of Σ . The reader may want to experiment with some examples; note that the farthest distance between any two sequences is 2, and more generally, given any two sequences \mathbf{s} and \mathbf{t} with $s_i = t_i$ for $0 \leq i \leq n$, we have $d[\mathbf{s}, \mathbf{t}] \leq \frac{1}{2^n}$.

Definition 2.2.2. We now define the *shift map* $\sigma : \Sigma \rightarrow \Sigma$ as

$$\sigma((s_0s_1s_2\dots)) = (s_1s_2s_3\dots),$$

that is, σ drops the first digit of its input sequence. While this may seem simple, we will see that this function actually has very complicated behavior – one might even say chaotic. What does that mean? Let's define it.

2.3 The Definition of Chaos

We will first provide a few definitions to assist in defining chaos.

Definition 2.3.1. Given a set X , a function $d : X \times X \rightarrow \mathbb{R}$ is a *metric* on X if for all $x, y \in X$, it is that:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$, and
3. for all $z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A set X with a metric d is considered a *metric space*.

Definition 2.3.2. Given a metric space X and a subset $Y \subseteq X$, we say that Y is *dense* in X if, for every $x \in X$ and every $\epsilon > 0$, there exists some $y \in Y$ such that the distance between x and y is less than ϵ .

Example 2.3.3. The rational numbers \mathbb{Q} are dense in the real numbers \mathbb{R} because, for any real number and any $\epsilon > 0$, there is always some rational number within ϵ distance of the chosen real number (that is, you can approximate any real number to an arbitrary precision with a rational number).

There are three conditions that we use to define chaos. We say a function $F : X \rightarrow X$ is *chaotic* if:

1. The periodic points of F are *dense* in X ,
2. F is *transitive*,
3. and F is *sensitive to initial conditions*.

Condition 1 of chaos thus states that, for any point in X , there is always a periodic point of F arbitrarily close to it; in other words, no matter where you look in X , there is always a point whose orbit is predictable close by.

Example 2.3.4. The periodic points of the shift map are dense in Σ . To prove this, choose an arbitrary $\mathbf{s} = (s_0s_1s_2\dots) \in \Sigma$ and let $\epsilon > 0$. Choose some n such that $\epsilon > \frac{1}{2^n}$. Then consider the point $\mathbf{s}_n = (\overline{s_0s_1s_2\dots s_n}) = (s_0s_1s_2\dots s_ns_0s_1s_2\dots)$. This point is periodic with period $n+1$, since applying σ to it $n+1$ times (which is equivalent to removing the first $n+1$ digits) leaves the sequence unchanged. This point is also within $\frac{1}{2^n}$ (and thus within ϵ) of \mathbf{s} , since it matches digits 0 through n . Since we have shown there exists a periodic point arbitrarily close to any point in Σ , we have proven that the periodic points of σ are dense in Σ .

Now let's look at the second condition.

Definition 2.3.5. A function $F : X \rightarrow X$ is *transitive* if, given arbitrary $x, y \in X$ and $\epsilon > 0$, there is some $z \in X$ within ϵ of x and some n for which $F^n(z)$ is within ϵ of y .

In words, for any two points in X , there is always some orbit of F that passes arbitrarily close to them. Note that if a given function has a dense orbit – that is, a point whose orbit goes arbitrarily close to every point in its domain – then it is automatically transitive.

Example 2.3.6. The shift map has a dense orbit (and is thus transitive). Consider the point

$$\hat{\mathbf{s}} = (0 \ 1 \quad \underbrace{00 \ 01 \ 10 \ 11}_{\text{all possible 2-digit blocks}} \quad \underbrace{000 \ 001 \ \dots \ 111}_{\text{all possible 3-digit blocks}} \quad \underbrace{0000 \ 0001 \ \dots \ 1111 \ \dots}_{\text{all 4-digit blocks}} \dots).$$

We claim that the orbit of $\hat{\mathbf{s}}$ is dense in Σ . To prove this, we choose arbitrary $\mathbf{s} \in \Sigma$ and $\epsilon > 0$. Again choose n such that $\epsilon > \frac{1}{2^n}$. Because $\hat{\mathbf{s}}$ contains every $n+1$ -digit block, we know that some point on the orbit of $\hat{\mathbf{s}}$ will match the first $n+1$ digits of \mathbf{s} , making \mathbf{s} within ϵ of the orbit of $\hat{\mathbf{s}}$. Therefore the orbit of $\hat{\mathbf{s}}$ is dense in Σ .

Finally, we turn to the third condition, perhaps the most intuitive for causing chaos; sensitivity to initial conditions.

Definition 2.3.7. A function $F : X \rightarrow X$ is *sensitive to initial conditions* if there exists a “sensitivity constant” β for which, for every $\epsilon > 0$ and every $x \in X$, there exists some $y \in X$ within ϵ of x and some k for which $F^k(x)$ and $F^k(y)$ are at least β apart.

Note that a function which is sensitive to initial conditions must evade iterated numerical approximation; the tiniest discrepancy between desired input and the floating-point approximation will balloon into a gap as wide as β after enough iterations.

Example 2.3.8. The shift map is sensitive to initial conditions. Choose $\beta = 1$, and choose any $\mathbf{s}, \mathbf{t} \in \Sigma$ with $\mathbf{s} \neq \mathbf{t}$. Then there must be some digit, say the k th digit, at which the two sequences differ (otherwise they would be equal). Then consider $F^k(\mathbf{s})$ and $F^k(\mathbf{t})$. They differ at the first digit, therefore the distance between them is at least 1, our chosen β ; thus σ is sensitive to initial conditions.

We have thus proven:

Theorem 1. The shift map $\sigma : \Sigma \rightarrow \Sigma$ is chaotic.

2.4 Conjugacies

We begin this section with three definitions.

Definition 2.4.1. Let X and Y be two metric spaces with respective distance functions d_X and d_Y . A function $F : X \rightarrow Y$ is *continuous* if, for every $\epsilon > 0$, there exists some $\delta > 0$ such that, for every $x_1, x_2 \in X$ the statement $d_X(x_1, x_2) < \delta$ implies that $d_Y(F(x_1), F(x_2)) < \epsilon$.

Definition 2.4.2. A *homeomorphism* is a continuous, bijective (one-to-one and onto) function whose inverse is also continuous.

Definition 2.4.3. Given two functions $F : X \rightarrow X$ and $G : Y \rightarrow Y$, a homeomorphism $h : X \rightarrow Y$ is a *conjugacy* if $h \circ F = G \circ h$.

Given a conjugacy h between F and G , we find something interesting happens when we expand the expression $h \circ F^n$:

$$\begin{aligned}
h \circ F^n &= (h \circ F) \circ F^{n-1} \\
&= (G \circ h) \circ F^{n-1} \\
&= G \circ (h \circ F) \circ F^{n-2} \\
&= (G^2 \circ h) \circ F^{n-2} \\
&\quad \vdots \\
&= (G^{n-1} \circ h) \circ F \\
&= G^n \circ h
\end{aligned}$$

This shows the primary use of conjugacies, which is that conjugacies preserve orbits; applying h to $F^n(x)$ does the same thing as taking G^n of $h(x)$. Because orbits are preserved, this means that F and G have essentially equivalent behavior as dynamical systems; specifically, if F is conjugate to a chaotic dynamical system, F must also be chaotic.

To this end, we will construct a conjugacy between L_λ and σ . For the rest of this section, we will work only with $\lambda > 4$. Let Ω denote the set of points whose orbits are bounded under L_λ . We can make some initial observations about Ω ; first, note that any orbit which goes below zero will diverge to negative infinity, because for $x < 0$, $\lambda > 4 \implies \lambda(1-x) > 4(1-x) > 4 \implies \lambda x(1-x) < 4x < x$ and so the orbit is strictly decreasing. Also note that $L_\lambda(x) > x(1-x) \geq 0$ is true if and only if $x \in [0, 1]$; therefore $\Omega \subseteq [0, 1]$, and we can state

$$\Omega = \{x \in [0, 1] \mid L_\lambda^n(x) \in [0, 1] \text{ for all } n\}.$$

Now consider the values in $[0, 1]$ that are mapped outside of $[0, 1]$ by L_λ . Since all values in $[0, 1]$ get mapped to positive values by L_λ , we can look at all points x for which $L_\lambda(x) > 1$. It follows that

$$\lambda x(1-x) > 1 \implies x - x^2 > \frac{1}{\lambda} \implies x^2 - x + \frac{1}{\lambda} < 0 \implies x \in \left(\frac{1 - \sqrt{1 - \frac{4}{\lambda}}}{2}, \frac{1 + \sqrt{1 - \frac{4}{\lambda}}}{2} \right)$$

where the last implication follows from the quadratic formula. Because the orbit of any x in the above interval leaves $[0, 1]$ and thus diverges, we have

$$\Omega \subseteq [0, 1] \setminus \left(\frac{1 - \sqrt{1 - \frac{4}{\lambda}}}{2}, \frac{1 + \sqrt{1 - \frac{4}{\lambda}}}{2} \right) = \left[0, \frac{1 - \sqrt{1 - \frac{4}{\lambda}}}{2} \right] \cup \left[\frac{1 + \sqrt{1 - \frac{4}{\lambda}}}{2}, 1 \right]$$

Let

$$I_0 := \left[0, \frac{1 - \sqrt{1 - \frac{4}{\lambda}}}{2} \right] \text{ and } I_1 = \left[\frac{1 + \sqrt{1 - \frac{4}{\lambda}}}{2}, 1 \right],$$

such that $\Omega \subseteq I_0 \cup I_1$. Note that I_0 and I_1 are disjoint. For all $x \in \Omega$, we know that $L_\lambda(x)^n \in \Omega$ (because by definition of Ω , the orbit of x does not diverge, and so every point on the orbit of x must not diverge as well). Therefore $L_\lambda^n(x)$ is contained in either I_0 or I_1 . We can use this to define a map from Ω to the sequence space that we will prove to be a conjugacy.

Definition 2.4.4. Define the *itinerary map* $S : \Omega \rightarrow \Sigma$ as

$$S(x) = (s_0 s_1 s_2 \dots \mid s_n = a \iff L_\lambda^n(x) \in I_a).$$

Example 2.4.5. To understand what this map does, let's look at some examples. For any λ , we have $S(0) = (0000\dots)$ because $L_\lambda^n(0) = 0 \in I_0$ for all n (recall that zero is a fixed point). Similarly, the itinerary of the other fixed point is $S(p) = S(\frac{\lambda-1}{\lambda}) = (1111\dots)$. The itinerary of 1 is $(1000\dots)$ because $1 \in I_1$ but $L_\lambda(1) = L_\lambda^n(1) = 0 \in I_0$.

We can also look at cycles: recall that $q_\pm = \frac{\lambda+1 \pm \sqrt{(\lambda-3)(\lambda+1)}}{2\lambda}$ is a 2-cycle of L_λ . We find that $S(q_+) = (101010\dots)$ and $S(q_-) = (010101\dots)$.

Note that

$$\begin{aligned} S(L_\lambda(x)) &= (s_0 s_1 s_2 \dots | s_n = a \iff L_\lambda^n(L_\lambda(x)) \in I_a) \\ &= (s_1 s_2 s_3 \dots | s_n = a \iff L_\lambda^n(x) \in I_a) \\ &= \sigma(S(x)). \end{aligned}$$

Therefore, we have $S \circ L_\lambda = \sigma \circ S$, and so proving S to be a conjugacy is equivalent to proving S to be a homeomorphism. We now prove that the itinerary map is a homeomorphism (and thus a conjugacy) between L_λ and σ for all values of λ greater than $2 + \sqrt{5} \approx 4.236$. We choose this value because, for all $\lambda > 2 + \sqrt{5}$, for any $x \in I_0 \cup I_1$, we have $|L'_\lambda(x)| > 1$; the theorem is true for all $\lambda > 4$ but the proof is beyond the scope of this paper.

Theorem 2. The itinerary map $S : \Omega \rightarrow \Sigma$ is a conjugacy between L_λ and σ for $\lambda > 2 + \sqrt{5} \approx 4.236$.

There are four parts to proving S a homeomorphism; S must be injective (one-to-one), surjective (onto), continuous, and its inverse must be continuous.

Injectivity. We prove by contradiction. Assume $x, y \in \Omega$ with $x \neq y$, letting $y > x$ without loss of generality, and suppose $S(x) = S(y)$, that is, $L_\lambda^n(x)$ and $L_\lambda^n(y)$ are both in I_0 or both in I_1 for all n . Consider the interval $[x, y]$. By the Mean Value Theorem, there is some $c \in [x, y]$ such that

$$\frac{L_\lambda(y) - L_\lambda(x)}{y - x} = L'_\lambda(c).$$

Since $[x, y] \subset I_0 \cup I_1$, we know that $|L'_\lambda(c)| > 1$, and so there exists some $\mu > 1$ with $|L'_\lambda(c)| > \mu > 1$. So $|L_\lambda(y) - L_\lambda(x)| \geq \mu|y - x|$. Now, if we replace x and y with $L_\lambda(x)$ and $L_\lambda(y)$ in the above argument, the logic still holds because $[L_\lambda(x), L_\lambda(y)] \subset I_0 \cup I_1$, so we can conclude that $|L_\lambda^2(y) - L_\lambda^2(x)| \geq \mu|L_\lambda(y) - L_\lambda(x)|$. Combining the two arguments tells us that $|L_\lambda^2(y) - L_\lambda^2(x)| > \mu^2|y - x|$. We can continue iterating this argument to see that

$$|L_\lambda^n(y) - L_\lambda^n(x)| > \mu^n|y - x|.$$

Since $\mu > 1$, $\mu^n \rightarrow \infty$, but $L_\lambda^n(y)$ and $L_\lambda^n(x)$ are both contained in I_0 or I_1 which have finite length; the only possibility is that $|y - x| = 0$, that is, $y = x$, but this is a contradiction since we assumed $x \neq y$. Therefore S is injective by contradiction.

Surjectivity. Given an interval I , let $L_\lambda^{-n}(I)$ denote the n th preimage of I , that is,

$$L_\lambda^{-n}(I) = \{x \in \mathbb{R} \mid L_\lambda^n(x) \in I\}.$$

Note that, for a closed interval in $[0, 1]$, its preimage consists of two closed intervals, one in I_0 and one in I_1 . This can be shown by explicit calculation: for $0 \leq x \leq y \leq 1$ we have

$$L_\lambda^{-1}([x, y]) = \left[\frac{1 - \sqrt{1 - \frac{4x}{\lambda}}}{2}, \frac{1 - \sqrt{1 - \frac{4y}{\lambda}}}{2} \right] \cup \left[\frac{1 + \sqrt{1 - \frac{4y}{\lambda}}}{2}, \frac{1 + \sqrt{1 - \frac{4x}{\lambda}}}{2} \right]$$

where the left interval is contained in I_0 and the right in I_1 ; these intervals have nonzero length if $x \neq y$. Now define the interval

$$I_{s_0 s_1 s_2 \dots s_n} = \{x \in [0, 1] \mid L_\lambda^i(x) \in I_{s_i} \text{ for } 0 \leq i \leq n\},$$

in other words, $I_{s_0 s_1 s_2 \dots s_n}$ is the set of points whose itinerary begins with $s_0 s_1 s_2 \dots s_n$. For example, I_{01} is the interval containing all points x in I_0 for which $L_\lambda(x) \in I_1$. By our definition of L_λ^{-n} , we can also write

$$I_{s_0 s_1 s_2 \dots s_n} = I_{s_0} \cap L_\lambda^{-1}(I_{s_1}) \cap \dots \cap L_\lambda^{-n+1}(I_{s_{n-1}}) \cap L_\lambda^{-n}(I_{s_n}).$$

This can be manipulated in two ways:

$$\begin{aligned} I_{s_0 s_1 s_2 \dots s_n} &= I_{s_0} \cap L_\lambda^{-1}(I_{s_1} \cap L_\lambda^{-1}(I_{s_2}) \cap \dots \cap L_\lambda^{-n+1}(I_{s_n})) \\ &= I_{s_0} \cap L_\lambda^{-1}(I_{s_1 s_2 \dots s_n}) \end{aligned}$$

and

$$I_{s_0 s_1 s_2 \dots s_n} = I_{s_0 s_1 s_2 \dots s_{n-1}} \cap L_\lambda^{-n}(I_{s_n})$$

The first expression tells us that, by induction, $I_{s_0 s_1 s_2 \dots s_n}$ is always a nonempty closed interval: the base cases I_0 and I_1 are nonempty closed intervals, and assuming $I_{s_1 s_2 \dots s_n}$ is nonempty and closed, $L_\lambda^{-1}(I_{s_1 s_2 \dots s_n})$ comprises two nonempty closed intervals, one in I_0 and one in I_1 , and so its intersection with I_{s_0} will be a single nonempty closed interval. The second expression tells us that $I_{s_0 s_1 \dots s_n}$ is contained in $I_{s_0 s_1 \dots s_{n-1}}$.

We're almost there. Let $\mathbf{s} = (s_0 s_1 s_2 \dots) \in \Sigma$ and consider the set

$$\bigcap_{n=0}^{\infty} I_{s_0 s_1 \dots s_n}.$$

This is an intersection of nonempty, nested, closed intervals, therefore it must also be a nonempty closed interval (in this case, since S is one-to-one, it converges to a single point). The itinerary of this point x must match \mathbf{s} to n digits for all n , therefore $S(x) = \mathbf{s}$. Therefore, for all $\mathbf{s} \in \Sigma$, there exists an x for which $S(x) = \mathbf{s}$, and we conclude that S is surjective.

Continuity. To prove S is continuous, we must show that for every $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, y \in \Omega$ with $|y - x| < \delta$, we have $d[S(x), S(y)] < \epsilon$. As with prior proofs, choose n such that $\frac{1}{2^n} < \epsilon$, meaning that $S(x)$ and $S(y)$ must have their first $n + 1$ digits match. Let $S(x) = (s_0 s_1 s_2 \dots)$, and consider all possible intervals $I_{t_0 t_1 t_2 \dots t_n}$. These intervals are disjoint, and the set Ω is contained in these intervals. For any x , we can therefore choose δ such that the interval of length 2δ centered on x is disjoint with all of the $I_{t_0 t_1 t_2 \dots t_n}$ with the exception of $I_{s_0 s_1 s_2 \dots s_n}$. Then y must also be in $I_{s_0 s_1 s_2 \dots s_n}$ such that $d[S(x), S(y)] < \frac{1}{2^n} < \epsilon$ as desired. Therefore S is continuous.

Continuity of Inverse. This proof is very similar to the last: we must show that for every $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, y \in \Omega$ with $d[S(x), S(y)] < \delta$, we have $|y - x| < \epsilon$. Consider the interval of length 2ϵ centered on x , and choose the smallest such n such that some $I_{s_0 s_1 s_2 \dots s_n}$ contains x and is contained in this interval. Then, letting $\delta = \frac{1}{2^n}$, the relevant implication follows, and so S^{-1} is continuous.

Therefore, S is a conjugacy between L_λ and σ , and since σ is chaotic and conjugacies preserve dynamical properties, we have the following corollary:

Corollary 1. The logistic map, $L_\lambda(x) = \lambda x(1 - x)$, is chaotic for $\lambda > 2 + \sqrt{5} \approx 4.236$. (Again, we remark that this holds for all $\lambda > 4$, but the proof is beyond the scope of this paper.)

3 Complex Dynamics

3.1 Prerequisites

Before we explore the complex dynamics of the logistic map we must establish a few concepts with complex numbers. First, we define a *complex number* to be any number of the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. For a complex number $z = x + yi$, we may call x the real part of z and y the imaginary part of z .

We now define a few operations on complex numbers. The *modulus* of a complex number $z = x + yi$, denoted $|z|$, is found by evaluating $|z| = \sqrt{x^2 + y^2}$, and represents the magnitude of the vector pointing to z originating from the origin. We note that a similar value, the *argument* of z , is the polar angle created by the vector pointing to z and the x -axis.

Multiplication is defined using the distributive property, giving $(a + bi) \cdot (x + yi) = ax - by + (bx + ay)i$. Complex numbers may be depicted in the cartesian plane, with a complex number $x + yi$ being placed at the point (x, y) . Complex numbers may be represented in an alternative way in polar form using Euler's formula, $e^{i\theta} = \cos \theta + i \sin \theta$, and the argument and modulus of a number. We may write $z = re^{i\theta}$ where r is the modulus of z and θ is the argument.

We define the *complex conjugate* of a number $z = x + yi$ to be $\bar{z} = x - yi$. Square roots of complex numbers may be found by representing them in polar form and then halving the argument as well as taking the square root of the modulus.

The simplest functions to describe the dynamics of on the complex plane are linear functions of the form $L_\alpha = \alpha z$, where $\alpha \in \mathbb{C} \setminus \{0\}$. We may write $\alpha = \rho e^{i\psi}$ and split the behavior of the functions into three cases:

1. $\rho < 1$: if $\rho < 1$ then $\rho^n \rightarrow 0$ as $n \rightarrow \infty$, meaning all orbits of L_α tend to 0. We call 0 an attracting fixed point in this case.
2. $\rho > 1$: if $\rho > 1$ then $\rho^n \rightarrow \infty$ as $n \rightarrow \infty$, so all nonzero orbits diverge to infinity. We call 0 a repelling fixed point in this case.
3. $\rho = 1$: if $\rho = 1$ then if ψ is rational all orbits are periodic, as iteration is equivalent to rotation around the origin by a rational multiple of π , making the orbit periodic. If ψ is irrational, then all orbits are dense on a circle centered around 0.

3.2 Julia Sets

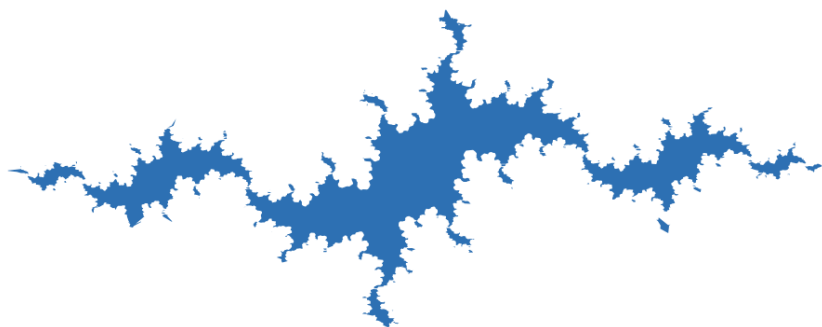


Figure 2: The Julia set of the logistic map with $\lambda = 3.368 + 0.21i$

We now turn to the behavior of the logistic map in the complex numbers. From now on, let $L_\lambda(z) = \lambda z(1 - z)$ be the complex logistic map where $\lambda, z \in \mathbb{C}$. We define the orbit of z under L_λ to be *bounded* if there exists K such that $|L_\lambda^n(z)| < K$ for all n . If this is not the case we say the orbit of z is *unbounded*. We now construct the *Julia set*.

Definition 3.2.1. The *filled Julia set* of L_λ is the set of all points in the complex plane whose orbits are bounded. The *Julia set* of L_λ is the boundary of the filled Julia set.

We denote the filled Julia set by K_λ and the Julia set by J_λ .

There are a few interesting observations we may take down about the Julia set of the complex logistic map.

Observation 1. When $\lambda = 0$ the filled Julia set K_λ is the entire complex plane.

Observation 2. As $|\lambda| \rightarrow 0$ the size of the filled Julia set increases rapidly, and as $|\lambda| \rightarrow \infty$ the Julia set shrinks rapidly.

Observation 3. For certain values of λ , the filled Julia set appears to be in disconnected islands.

An example of a Julia set is shown above in Figure 2.

We can now touch upon the properties of the Julia sets generated by certain λ values. We provide without rigorous proof that when $|\lambda - 1| > 3$ it is that $J_\lambda = K_\lambda$ and that they are both Cantor sets. We will also supply that, if $|z| \geq |\lambda - 1| > 3$, then $|L_\lambda^n(z)| \rightarrow \infty$ as $n \rightarrow \infty$.

One observation that can be made that will carry into the next section is that for λ such that $|\lambda - 1| > 3$ it is that the orbit of $\frac{1}{2}$ to infinity under L_λ .

3.3 The Fundamental Dichotomy

We will now further discuss the connectedness of K_λ . As noticed earlier in observation 3, the filled Julia set of certain values of λ is disconnected. There is a pattern for when these sets are connected or disconnected.

The Fundamental Dichotomy. *Let $L_\lambda(z) = \lambda z(1 - z)$. Then either*

1. *The orbit of the critical point $\frac{1}{2}$ escapes to infinity and K_λ consists of infinitely many disjoint components, or*
2. *The orbit of $\frac{1}{2}$ remains bounded and K_λ is connected.*

It is important to note that this states that it will only ever be that K_λ is connected or K_λ is infinitely many disjoint parts. There are no filled Julia sets of the complex logistic map with two, three, twenty, or any finite number other than one components. We now continue on to defining an important set in the study of Julia sets.

3.4 The Logistic Mandelbrot Set

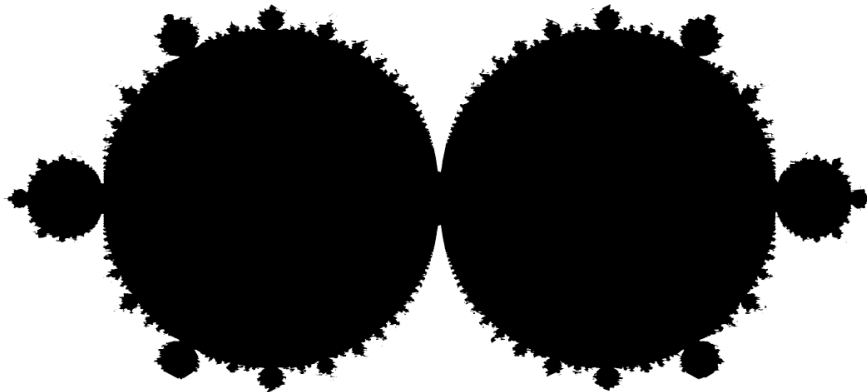


Figure 3: The Mandelbrot set of the logistic map

We define \mathcal{M} to be the *logistic Mandelbrot set*, depicted above in Figure 3 which consists of all the λ values where K_λ is connected. Equivalently,

$$\mathcal{M} = \{\lambda \in \mathbb{C} \mid |L_\lambda^n(\frac{1}{2})| \not\rightarrow \infty\}.$$

It is important to note that \mathcal{M} is contained in the complex plane of parameter values of λ , and not in the plane of values of z where the Julia sets are.

The more commonly discussed Mandelbrot set is that of the quadratic map, $Q_c(z) = z^2 + c$, which bears resemblance to our newly created logistic Mandelbrot Set. Note that the Mandelbrot set for the logistic map seems to be a cut off, shifted, and mirrored copy of the original Mandelbrot set.

An important aside to discuss with the Mandelbrot set is the computation of it, as the set builder definition does not provide much information to how it looks. For this we have an algorithm we can use:

Algorithm for the Mandelbrot Set. *Choose a maximum number of iterations N . Construct a grid of complex numbers, and for each point λ in the grid compute the first N iterations of the orbit of $\frac{1}{2}$ under L_λ . If $|L_\lambda^i(\frac{1}{2}) - 1| < 3$ for all $i \leq N$, color the square black. Else, color it based off of the largest i such that $|L_\lambda^i(\frac{1}{2}) - 1| < 3$.*

Experimenting with different λ values in the Mandelbrot set, we see behavior such as a period 4 bifurcation at $\lambda = \pm i$. We also see two large symmetrical bulbs with attracting fixed points. These large bulbs have smaller self-similar replicas attached to their ends on the real line. These bulbs all contain attracting cycles of period 2^n , where $n = 0$ for the largest bulb, $n = 1$ for the second largest bulb, and so on increasing as λ moves further away from the large bulbs.

Looking at the set more geometrically, we can see that it is constructed entirely of repeating tangent circles, unlike the Mandelbrot set for Q_c with a large cardioid.

4 Bibliography

Devaney, Robert. *A First Course in Chaotic Dynamical Systems*. Taylor and Francis, 2020.