Transformations In Geometry

Mogos Ghile

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Abstract

In this paper, we will cover functions, transformations, and relate them to linear algebra. For functions and transformations, we will delve into the definitions and characteristics as well as applications. Regarding linear algebra, we will introduce matrices and vectors, demonstrating how we can utilize these tools in transformational geometry. Overall, we will focus on the importance of these topics in the study of geometry.

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1 Introduction

Transformational Geometry is a modern approach to geometry that focuses on studying figures by manipulating characteristics about them such as, position and size using functions. Although moving geometric figures around is an ancient approach to geometry, the focus more recently shifted to a more function-based approach. As a result of the study of polynomials in the early nineteenth century, algebraic transformations and groups emerged. It wasn't until early in the twentieth century that physicists realized the power of transformations, starting with Einstein's theory of relativity and then with quantum mechanics.

2 Understanding functions and maps

2.1 Sets

Before introducing functions, we will first go over sets. **Sets** are collections of distinct objects, in the case of this paper, numbers. $N = \{1, 2, 3\}$.

Important characteristics of sets include:

- The order in which we write elements doesn't matter. $\{0, 1\} = \{1, 0\}$.
- There are no repeated elements in a set.
- To prove that two sets are equal, it needs to be proven that they share the same elements.
- A set can have no elements; this is called an empty set.

Sets are fundamental concepts in Transformational Geometry as it is needed to define core concepts such as functions and transformation groups.

2.2 Functions

In Transformational Geometry, **functions** are rules that map every figure (input) in a geometric figure to a new location (output) often times changing qualities of the figure. Functions are defined by their name, followed by the input set (domain), and then the output set (codomain).

Two important conditions that a function ($F : \mathbb{R}^2 \to \mathbb{R}^2$) can satisfy are:

1. One-to-one (Injective): $f(x, y) = f(x'y')$ This means that each input has a unique output, in other words, no two different points in the domain can map to the same point in the codomain.
2. On to (Surjective): $\forall (x', y') \in \mathbb{R}^2, \exists (x, y) \in \mathbb{R}^2$ such that $f(x, y) = (x', y')$. This means that every point in the codomain came from a point in the original.

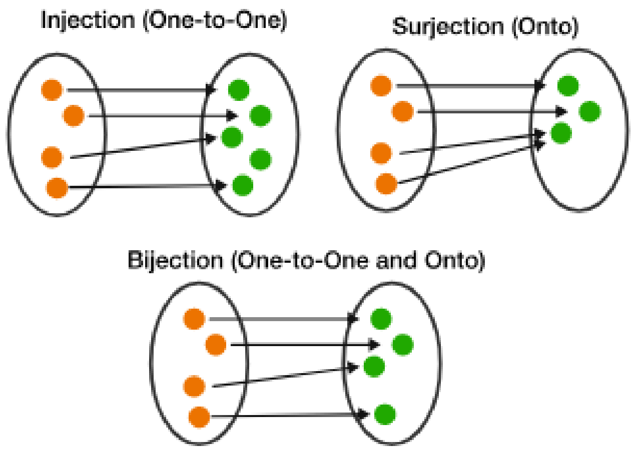

Figure 1: A demonstration of Injectivity, Surjectivity, and Bijectivity [1].

Let's examine a function that squares the domain, $f : \mathbb{R} \to \mathbb{R}$ which has both a domain and codomain of all real numbers. First, we can test if it is injective by analyzing the domain which contains both positive and negative real numbers. Due to the function squaring the domain, then real numbers with the same absolute value will map to the same codomain, therefore making this function not injective. For example, $f : 3 \to 9$ and $f : -3 \to 9$. Moreover, this function is not surjective because the codomain which claims to include both positive and negative real numbers cannot have elements with negative values. This is a result of squaring the domain which can only produce positive outputs.

3 Transformations

3.1 Algebraic Representation

Transformations are functions from $\mathbb{R}^2 \to \mathbb{R}^2$ that are bijective (both injective and surjective).

A classic example of transformations is translations. **Translations** are a transformation that slides every point of a figure the same distance in a specified direction. For instance, $T(x, y) = (x + a, y + b) = (c, d)$. In this case, we have a function T that translates the x-coordinate a units and the y-coordinate b units producing the new point (c, d).

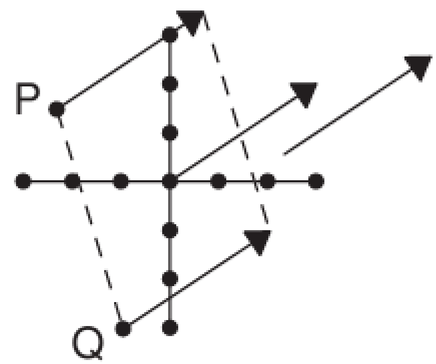

Figure 2: A translation on the figure with points P and G [2].

There is also a transformation that does not change the input points, which is called an **identity** transformation. Consider $I(x, y) = (x, y)$ and if this is indeed a transformation. This satisfies injectivity since each input has a unique output. Additionally, this satisfies surjectivity since every point in the codomain came from a point in the domain. The importance of the identity transformation is to be a base from which nothing happens that is referred to when doing inverse transformations. Since applying a transformation and then its inverse on a point will result in no changes, it relates to an identity transformation.

Furthermore, with every transformation f there is an **inverse transformation** f^-1 that undoes the changes. Continuing the example with translation T that transforms the point (x, y) to $(x + a, y + b)$, the inverse transformation would undo this by translating in the opposite direction. $T^-1(x + a, y + b) = (x + a - a, y + b - b) = (x, y)$.

3.2 Applications

To apply what we've learned, let's examine the equation of a unit circle. The unit circle has specific qualities such as being centered around the origin, and having a radius of 1. These qualities simplify the general equation of a circle, $R^2 = (x - a)^2 + (y - b)^2$ to the specified equation of a unit circle, $1 = x^2 + y^2$. But if we were interested in transforming the unit circle into a more arbitrary circle, how would we do that?

1. First we would need to change the radius from being exactly one to an arbitrary length R. To do this would require a dilation, D by a factor of R.

2. Second we would need to move the center of the unit circle to an arbitrary point (a, b). To do this would require a translation T that translates the point (x, y) to $(x+a, y+b)$.

The order in which we would apply these two transformations is significant in the same way the expression $5 + 2 * 3$ results in 11 only if the operations were done in the correct order. In this case, the dilation, D would be done first and then the translation, T would be done last. These two transformations can be **composed** which is an algebraic operation where the output of one function becomes the input of another. $T \circ D(x, y) = T(D(x, y))$. If we were now to apply the transformation on the point (x, y) we would get $T(D(x, y)) = T(Rx, Ry) = (Rx + a, Ry + b)$.

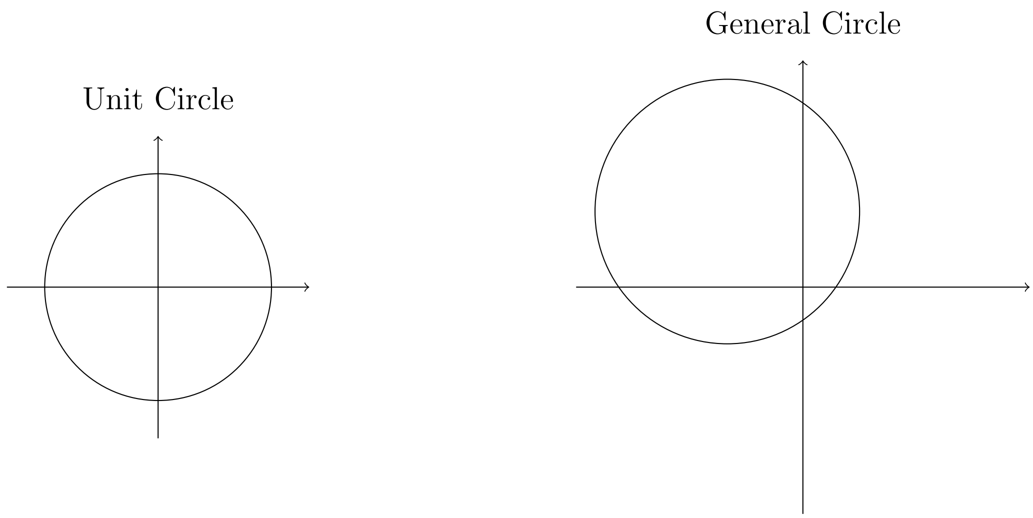

Figure 3: Unit Circle and General Circle

4 Connecting to Linear Algebra

4.1 Transformation Groups

A **group** is defined as containing a set and an operation and has to satisfy four conditions. The example with the set being $\mathbb{R}$ and the operation of addition $(\mathbb{R}, +)$ will be demonstrated for each condition.

- (Closure) If any two elements in the set are combined using the operation, the result stays inside the set. $2 + 3 = 5$.
- (Identity) There exists a special element, e that leaves every other element unchanged when combined with it. $7 + \mathbf{0} = 7$.
- (Inverses) There exists some method to arrive back at e from any element in the set. $8 + (-8) = \mathbf{0}$.
- (Associativity) The order in which elements are combined is consistent. $(9 + 5) + 3 = 9 + (5 + 3)$.

A **transformation group** is defined as a group with the set being a collection of transformations and the operation being the composition of them.

4.2 Vector Space

A **Vector** is defined as a quantity with direction and magnitude, often depicted as an arrow. $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. Vectors can be added or subtracted with one another. $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \pm \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \end{pmatrix}$.

In addition, there is a **Vector Space** V which is essentially a group with the set of all vectors, S and the operation being addition. $(S, +)$ On top of this, in the Vector Space, there is **scalar multiplication** where vector $\vec{a}$ can be multiplied with a scalar (real number) λ. $\lambda\vec{a} = \lambda \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \end{pmatrix}$.

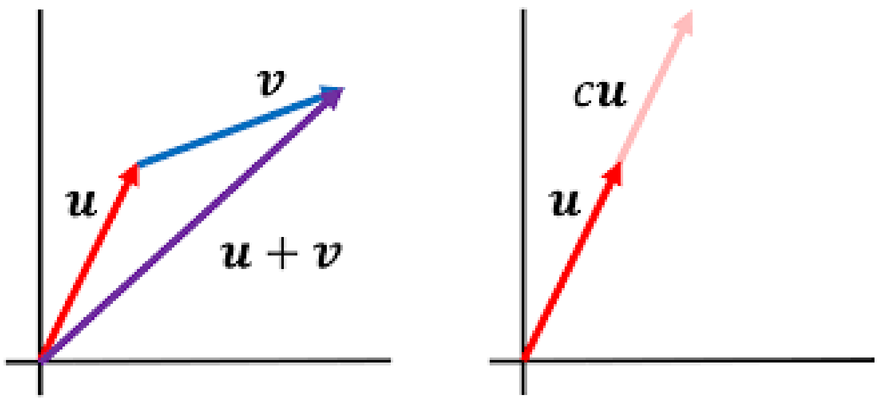

Figure 4: On the left, vector addition; on the right, scalar multiplication [3].

4.3 Linear Independence

For the set of vectors S, we can get the **linear combination** which is the sum of the vectors and each vector is multiplied by some scalar. $\lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} ... + \lambda_n \begin{pmatrix} x_n \\ y_n \end{pmatrix}$. When the only solution to the linear combination equaling 0 is when the scalars all equal 0, then the set is **Linearly Independent**.

For instance, if we define S to only contain $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$ then if we were to set the linear combination to 0 we would get $\lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. In this case, only when $\lambda_1, \lambda_2 = 0$ the linear combination equals to 0. This makes the set of these vectors S, Linearly Independent. $0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For another case, if we define S to only contain $\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right\}$. The linear combination would then be $\lambda_1\begin{pmatrix}1\\2\end{pmatrix}+\lambda_2\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$. In this situation, when $\lambda_1=-2$ and $\lambda_2=1$ then the linear combination equals 0. This makes the set of these vectors S, **Linearly Dependent**. $-2\begin{pmatrix}1\\2\end{pmatrix}+1\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

The importance of linear independence/dependence will be demonstrated when defining the basis of a vector space.

4.4 Spanning Set & Basis

For every set of vectors S we can also determine if it **spans** the Vector Space V. This means that for every vector $\vec{v}$ that is an element of V can be written as a linear combination of S. $v=\lambda_1\begin{pmatrix}x_1\\y_1\end{pmatrix}+\lambda_2\begin{pmatrix}x_2\\y_2\end{pmatrix}...+\lambda_n\begin{pmatrix}x_n\\y_n\end{pmatrix}$. The notation for spanning is as follows: $\text{span}\{v_1,v_2,\dots,v_n\}=V$.

Referring back to the previous example, the set of vectors S that only contains $\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}-3\\1\end{pmatrix}\right\}$ is considered spanning for the Vector Space V defined as $(\mathbb{R}^2,+)$. $\vec{v}=\lambda_1\begin{pmatrix}1\\2\end{pmatrix}+\lambda_2\begin{pmatrix}-3\\1\end{pmatrix}$.

Consider if we tried to find the vector $\begin{pmatrix}-1\\5\end{pmatrix}$ from the set of vectors S, defined earlier. In the case of $\lambda_1=2,\lambda_2=1$ we would get this: $2\begin{pmatrix}1\\2\end{pmatrix}+1\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}-1\\5\end{pmatrix}$. If we were to change the values of the scalars we would be able to get any vector in the vector space V. The actual method to determine if a set of vectors spans a vector space is outside of the scope of this paper.

The **basis** of the vector space is the set of vectors S that is both linearly independent and spanning. Continuing on with the example, the set of vectors S that only contains $\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}-3\\1\end{pmatrix}\right\}$ is considered the basis of the Vector Space V, as it is both spanning and linearly independent. Additionally, the **canonical basis** is the simplest set of vectors S that is the basis of the Vector Space V. For the Vector Space V defined as $(\mathbb{R}^2,+)$ the canonical basis would be $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$. Since it is spanning, any vector can be written in terms of the canonical basis. $\begin{pmatrix}3\\5\end{pmatrix}=3\begin{pmatrix}1\\0\end{pmatrix}+5\begin{pmatrix}0\\1\end{pmatrix}$.

4.5 Matrices

In transformational geometry, **matrices**, which are rectangular arrays of numbers, can be used to represent a transformation. For transformations in $\mathbb{R}^2$, a 2×2 matrix is typically used. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we were to apply a transformation represented by a matrix, to a vector it would look like this. $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$. The values a, b, c, d are based on the mappings of the canonical basis.

For example, if we were to have the matrix be the exact same values of the canonical basis then the transformation would make no changes on the vector. $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 0 \\ 0 + y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. This is an identity transformation which we examined earlier.

Consider, if we were to have the matrix be the the same values of the canonical basis but dilated by 3. $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 0 \\ 0 + 3y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$. This results in a dilation, as the vector is multiplied by a factor of 3.

References

[1] Brilliant.org, "Bijection, Injection, And Surjection" *Brilliant*, https://brilliant.org/wiki/bijection-injection-and-surjection/, accessed May 12 2026.

[2] Sibley, Thomas, *Thinking Geometrically: A Survey of Geometries*, Mathematical Association of America, 2015, p. 203.

[3] Bernstein, Matthew N., "Vector Spaces," https://mbernste.github.io/posts/vector_spaces/, accessed May 22, 2026.