Sentential Logic

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Abstract

This paper offers a foundational overview of sentential logic, sometimes known as propositional logic. This paper will cover many topics, beginning with the basic connectives, notation, and definitions that are the building blocks of logical expressions.

Next, it will introduce truth tables as a tool for evaluating compound logical statements. The paper then explores tautologies, which are statements that are always true. The paper will conclude with a discussion of logical laws, including the identity and domination laws, which aid in simplifying logical expressions. Through examples and formal definitions, this paper aims to build a clear understanding of how sentential logic operates and why it is central to mathematical reasoning.

1 Introduction

Sentential logic, also known as propositional logic, is the study of how statements (or sentences) can be combined and manipulated using logical connectives.

The roots of sentential logic trace back to around 225 B.C.E. with the work of the ancient Greek philosopher Chrysippus, who laid the foundations of formal logic. Although much of this knowledge was lost during the Middle Ages, it was later rediscovered and further developed by medieval scholars such as Peter Abelard in the 12th century.

In sentential logic, complex expressions, called composites, are built from simpler statements known as prime or atomic sentences. These are connected using logical connectives, which allow us to represent logical relationships.

Definition 1.1 (Sentential Logic). *Sentential logic* is a system of logic in which prime (atomic) sentences with defined truth values are combined using logical connectives to form compound statements.

Definition 1.2 (Prime (Atomic) Sentence). A *prime sentence* is a statement that cannot be broken down into simpler logical components and contains no logical connectives. Each prime sentence has a truth value, either true (T) or false (F).

For example, the statement "Dogs are blue" could be a prime sentence with the truth value false (F), and the statement "Dogs have four legs" could be a sentence with a true (T) value.

Definition 1.3 (Composite Sentence). A *composite sentence* is a logical expression formed by connecting two or more prime sentences using logical connectives.

Definition 1.4 (Logical Connectives). *Logical connectives* are symbols used to join or modify sentences in logic. The basic connectives include the following:

- **Negation** ($\neg$): "not"

  When you negate a term, the truth value will be the opposite, for example, inputting true will output false and vice versa. This is also different from the other logical connectives because it only has one input.

- **Conjunction** ($\wedge$): "and"

  Both sides of the statement must be true in order for the whole to be true. Otherwise, it is false.

- **Disjunction** ($\vee$): "or" (inclusive)

  Only one side of the statement must be true in order for the whole to be true. If both are false, then it is false.

- **Conditional** ($\rightarrow$): "if... then..."

  This statement is only false if the first side is true and the second is false, because a true statement cannot imply a false one.

- **Biconditional** ($\leftrightarrow$): "if and only if"

  This statement is true when both sides are equal, for example, true and true or false and false will result in true.

These are the main logical connectives that are vital to understanding to perform any sort of task within sentential logic. Let's try multiple examples for each logical connective.

Example 1.5. Let A be a true statement and B be a false statement. Then:

- $\neg B$ is true
  This is because B is false and the negation of false is true.

- $A \wedge B$ is false
  This is because A and B are not both true.
- $A \vee B$ is true
  This is because at least one of A or B is true.
- $A \to B$ is false
  This is because a true statement cannot imply a false one.
- $A \leftrightarrow B$ is false
  This is because A and B do not have the same truth value.

2 Truth Tables

In sentential logic, truth tables are essential tools for systematically determining the truth values of logical expressions. They provide a clear and structured way to evaluate the behavior of how the prime statements interact with the logical connectives, allowing the exploration of how different combinations of truth values for individual atomic sentences affect the overall truth value of more complex expressions.

Definition 2.1 (Truth Tables). A *truth table* is a mathematical tool used to determine the truth values of a logical expression based on the truth values of its components. Each row in the table represents a unique combination of truth values (true or false) for the variables. The number of rows in a truth table is 2^n, where n is the number of input variables. The resulting truth value of the entire expression is computed according to the logical connectives involved.

Truth tables are essential in analyzing and understanding logical expressions. They allow the systematic exploration of the behavior of logical connectives and are the foundation in sentential logic. By using truth tables, we can determine the validity of logical statements and much more.

2.1 Basic Truth Tables

In the following table, the truth values of the logical expression $A \vee B$, where $\vee$ (read as "or") is the logical connective. This operator returns true if at least one of the propositions A or B is true, and false only when both are false. As you can see below, we write out all the possible truth values for both A and B.

A	B	$A \vee B$
True	True	True
True	False	True
False	True	True
False	False	False

If the values of the overall logical statement do not vary and are always true no matter what, then the expression is called a **tautology**.

Definition 2.2 (Tautology). A tautology is a logical expression that is always true, regardless of the truth values of its atomic components.

For example, take the expression $A \vee \neg A$. This statement reads as "A or not A." It will always be true; either A is true or $\neg A$ is true, which means that one of them must hold in every case.

A	$\neg A$	$A \vee \neg A$
True	False	True
False	True	True

Therefore, we know that $A \vee \neg A$ is a tautology.

Now that we know the basics of truth tables, we can use them to explore deeper relationships between logical statements. This brings us to our next definition: **logical equivalence**.

Definition 2.3 (Logical Equivalence). Two logical expressions are said to be *logically equivalent* if they have the same truth value in every possible scenario. That is, their columns in a truth table are identical for all combinations of truth values of their atomic components.

For example, given that *Dogs are blue* is the prime sentence B and *Dogs have four legs* is the prime sentence L, since the expressions *Dogs are blue* (B) $\vee$ *Dogs have four legs* (L) and *Dogs have four legs* (L) $\vee$ *Dogs are blue* (B) have the same truth table, then they are logically equivalent and can be written as

$$B \vee L \equiv L \vee B$$

.

Now that we have the basics of sentential logic and truth tables down, let uss move on.

3 Logical Laws

Now that we understand how to evaluate logical expressions using truth tables, we can introduce a set of rules known as *logical laws*. These laws help us simplify expressions and prove logical equivalences more efficiently.

Before stating the laws, we define two special logical constants:

Definition 3.1 (Logical Constants).

- $\top$ (read as verum) represents a statement that is always true.
- $\bot$ (read as falsus) represents a statement that is always false.

3.1 Identity Laws

The following statements are considered the Identity Laws.

$$A \lor \bot \equiv A$$
$$A \land \top \equiv A$$

This statement $A \lor \bot \equiv A$ is true because the logical OR ($\lor$) requires only one of the values to be true. Since $\bot$ (false) is always false, the truth value depends entirely on A.

Similarly, the statement $A \land \top \equiv A$ is true because the logical AND ($\land$) requires both values to be true. Since $\top$ (true) is always true, the total truth value depends solely on A.

The identity laws state that combining a logical statement with $\top$ or $\bot$ using a conjunction or disjunction leaves the input truth value as the output as well. It acts like other identities in mathematics, such as multiplying by 1, adding 0, and multiplying a matrix by the identity matrix.

Below is a truth table proving and verifying the identity law $A \lor \bot \equiv A$:

A	$\bot$	$A \lor \bot$
True	False	True
False	False	False

As we can see, the resulting column for $A \lor \bot$ matches the original values of A, confirming that the expression is logically equivalent to A.

Next, we will move onto the common Domination Laws.

3.2 Domination Laws

The domination laws (also sometimes called null or annihilation laws) state that combining a logical statement with $\top$ or $\bot$ using a disjunction or conjunction also results in a constant truth value.

$$A \lor \top \equiv \top$$
$$A \land \bot \equiv \bot$$

The statement $A \lor \top \equiv \top$ is true because in a logical OR ($\lor$), if one value is always true ($\top$), the whole expression is always true regardless of A. This law acts like multiplying by 0 in basic math, where the output is 0 no matter what the input is.

Similarly, the statement $A \wedge \bot \equiv \bot$ is true because in a logical AND ($\wedge$), if one value is always false ($\bot$), the whole expression is always false regardless of A.

Like done for the Identity Laws, a Domination Law will also be proven using a truth table. Considering the Domination Law $A \wedge \bot \equiv \bot$:

A	$\bot$	$A \wedge \bot$
True	False	False
False	False	False

Here, it is shown that the expression $A \wedge \bot$ is always false, regardless of the truth value of A, demonstrating the Domination Law.

3.3 Summary of Basic Laws

As shown in the chart below, there are a few summarized common laws that one should be familiar with. They are all used to help figure out the truth value of a statement or simplify these statements. We already have gone over the Identity and Domination Laws.

Law	**Form**
Identity	$A \vee \bot \equiv A, \quad A \wedge \top \equiv A$
Domination	$A \vee \top \equiv \top, \quad A \wedge \bot \equiv \bot$
Double Negation	$\neg(\neg A) \equiv A$
De Morgan's Laws	$\neg(A \wedge B) \equiv \neg A \vee \neg B, \quad \neg(A \vee B) \equiv \neg A \wedge \neg B$
Distributive	$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

The Double Negation Law states that if you negate a statement and then negate it again, the result is logically equivalent to the original statement. In other words, reversing the truth value twice brings you back to the original truth.

De Morgan's Laws explain how negation affects compound statements, the negation of a conjunction becomes the disjunction of the negations, and the negation of a disjunction becomes the conjunction of the negations.

The Distributive Law shows how logical connectives work together, allowing expressions to be rewritten by distributing one connective to others.

4 References

References

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