Number Theory - The Difference Game

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Abstract

This paper explores the Difference Game in Number Theory, as well as gives definitions for relevant terms and a brief overview of Proof by Contradiction and Proof by Induction. We will also explore concepts such as the Well Ordering Principle, division algorithm, and greatest common divisor, as all three are crucial in explaining the Difference Game.

1 Introduction

What is number theory?

Number theory is a branch of mathematics that involves the study of various types of numbers and their unique properties. These numbers include odd and even numbers, perfect squares, rational numbers, and many others. Mathematicians who specialize in this field, known as number theorists, aim to uncover deep and intrinsic relationships between numbers, often exploring how they interact, how they can be classified, and what patterns they exhibit. The study of number theory can be both abstract and surprisingly applicable, revealing elegant structures that underpin the foundation of mathematics itself.

This paper will draw on several key concepts and problems presented in *An Introduction to The Theory of Numbers.* by Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery, a foundational text in the field of number theory that we have been reading from over the course of the past 5 months. In particular, while we will touch on a variety of number theory principles, the central focus of this paper will be an exploration of the "Difference Game," a mathematical game that illustrates several core ideas in number theory through strategy, logical deduction, and numerical reasoning.

To provide a roadmap we will begin with the basic proof methods of proof by induction, contradiction, and the well-ordering principle, applying them to the difference game once we have explained the game rules, through our four lemmas.

2 Preliminaries

2.1 Definition of Divisibility

An integer b is *divisible* by an integer a if $\frac{b}{a}$ can be reduced and expressed as an integer, leaving no remainder. This is denoted as $b \mid a$ and read as "b divides a." This allows us to claim that a is a *divisor* of b.

Example 1. *Consider* $\frac{10}{5} = 2$. *Because there is no remainder we know that 5 is a divisor of 10 and that* $5 \mid 10$. *However the counterexample* $\frac{10}{4} = 2 + 2/4$ *where 2/4 is the remainder. As a result of obtaining a non-integer remainder we know that 4 is* ***not*** *a divisor of 10 and that* $4 \nmid 10$

2.2 Definition of GCD

A *common divisor* is a divisor that is shared by two or more integers. The *greatest common divisor* (GCD) is the largest divisor that is shared. The greatest common divisor of integers a and b is denoted by $(a, b) = g$.

A set of numbers are *coprime* when their greatest common divisor is 1, while a set of numbers are not coprime when their greatest common divisor is greater than 1.

Example 2. *Consider the greatest common divisor of 24 and 40. If we expand both values we get* $8*3$ *and* $8*5$*. Because both integers are multiples of 8, and 8 is the* ***greatest divisor*** *they share, then the greatest common divisor of 24 and 40 is 8, denoted by (24, 40) = 8.*

2.3 Definition of Prime and Composite Numbers

Prime numbers are positive integers that have no divisor other than itself and 1. Composite numbers are positive integers that are not prime.

Example 3. *Some example of prime numbers: 2, 3, 5, 7, 11, and 13. Some examples of composite numbers are 4, 6, 8, 10, 12, and 14.*

3 Proofs to Know

Two proofs that will be needed for this paper are Proof by Contradiction and Proof by Induction.

3.1 Proof by Contradiction

Proof by Contradiction is a logical method used by mathematicians to prove a statement is true by assuming the opposite and then showing how that assumption leads to a contradiction or impossibility.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Table 1: Proof by Contradiction shown with Truth Tables where if the statement "if p then q" is true we can also prove it takes on the same truth values for the contrapositive statement "if not q then not p" and opposite truth values for the negated statement, "if not p then not q."

Proof. Let's take an example. Suppose we are solving the equation $2x + 1 = 7$ and want to prove that $x \neq 2$. Instead of trying to prove this directly, we could assume the opposite: that $x = 2$. Following this line of thought, we would then substitute $x = 2$ into the equation.

$$\begin{aligned} 2x + 1 &= 7 \\ 2(2) + 1 &= 7 \\ 4 + 1 &= 7 \\ 5 &= 7 \\ 5 &\neq 7 \end{aligned}$$

5 does *not* equal 7, so we have run into a contradiction. We could now conclude that our original assumption ($x = 2$) is false. Therefore, $x \neq 2$. ☐

One of the earliest known proofs using contradiction can be attributed to the ancient Greek mathematician Euclid.

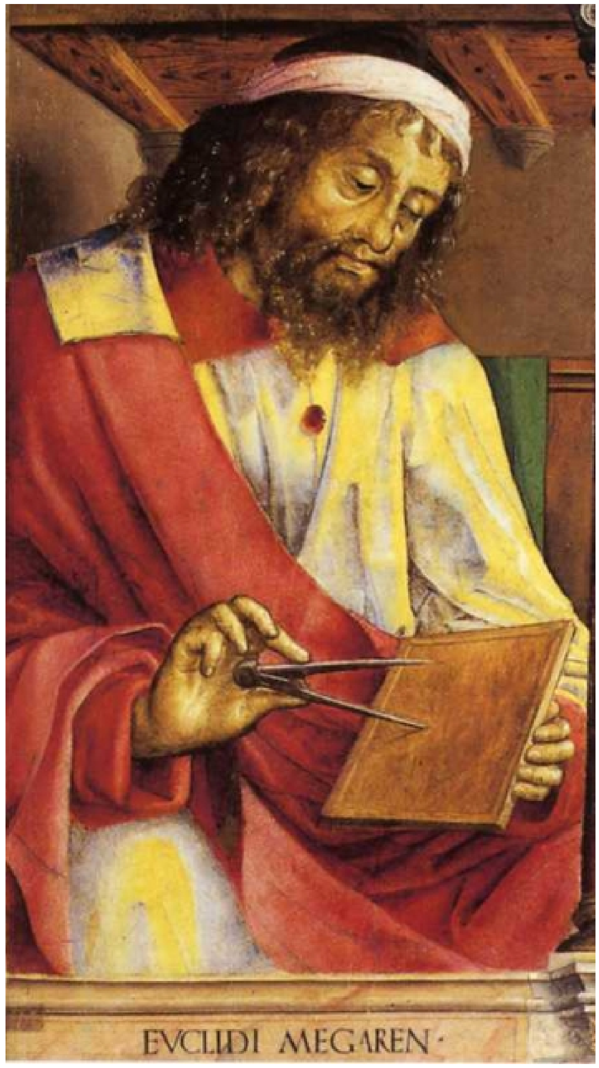


Figure 1: Joos van Gent and Pedro Berruguete, 'Euclid'. Mid-15th century

Let's take Euclid's Proof of the Infinitude of Primes as an example of Proof by Contradiction.

Proof. We assume, for the sake of contradiction, that there are only finitely many prime numbers. Let these primes be $p_1, p_2, ..., p_n$. Consider the expression

$$N = p_1, p_2, \cdots, p_n + 1$$

This number N is either prime or composite. If N is prime, then it is a prime not in our original list, contradicting the assumption that we had listed all primes. If N is composite, then it must be divisible by some prime. However, none of the primes $p_1, p_2, \cdots, p_n$ divide N, since each divides the product $p_1, p_2, \cdots, p_n$ which leaves a remainder of 1 when dividing N. Therefore, N must have a divisor that is prime, and that divisor is not in our list, again contradicting our assumption. As a result, there must be infinitely many primes. □

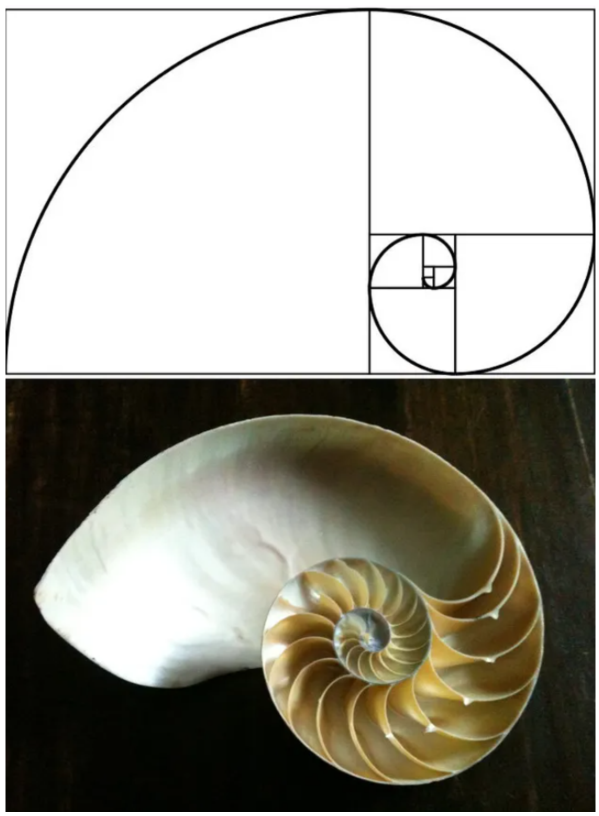
3.2 Proof by Induction

WEAK AND STRONG INDUCTION!

Proof by Induction works very similarly to a stack of dominoes. Imagine that we have stacked up a long train of dominoes, all of them equally spaced. If we push the first domino, this will trigger the next one to fall, so on and so forth, until all of the dominoes have fallen. In Proof by Induction, if we know a statement is true for term $\boldsymbol{n}$, we hope to show that it is true for **n+1**, the next term in the sequence.

In other words, we can prove a base case (i.e., Domino 1 falls) and the inductive step (i.e., if Domino n falls, then Domino n+1 falls), and this proves the statement for all n (i.e., all n dominoes fall).

Speaking of sequences, let's take the Fibonacci Sequence as our example. **The Fibonacci Sequence is a sequence where each number is the sum of the two preceding ones**. The first two terms of the Fibonacci Sequence (f_1 and f_2), are 1 and 1. $1+1$ is 2. This means the 2 is the third term of the Fibonacci Sequence. The fourth term of the sequence would be 3, as $1+2$ is 3. The fifth term of the sequence would be 5, as $2+3$ is 5. To move things forward, the Fibonacci sequence goes, $1, 1, 2, 3, 5, 8, 13...$ so on and so forth.



A Beautiful Example of the Fibonacci Sequence Found in Nature!

Proof. The official equation for the Fibonacci Sequence is this:

$$f_{n+1} = f_n + f_{n-1}$$

f_{n-1}, f_n, and f_{n+1} are three terms in the Fibonacci Sequence that appear in the order that was just listed.

Now, we are given an equation, and we want to know if it is true or false. Here it is:

$$f_1 + f_2 + ... + f_n = f_{n+2} - 1 \tag{1}$$

In other words, adding all the Fibonacci numbers up to f_n results in a total that is one less than the term located two places ahead in the sequence. To assess whether this equation is true, we are going to take a **base case. A base case is a small, initial step where we try to prove that a statement is true for the smallest possible value.** In our case, this will be:

$$n = 1 \tag{2}$$

We substitute the corresponding values into the equation:

$$f_1 + f_2 + ... + f_n = f_{n+2} - 1 \tag{3}$$

$$f_1 = f_3 - 11 = 2 - 1 \tag{4}$$

Since our equation checks out for the base case, we can assume it works for $n = k$, where k is a natural number. We show that the equation holds for $n = k$ by rewriting the equation we were given in terms of k. This gives us:

$$f_1 + f_2 + ... + f_k = f_{k+2} - 1 \tag{5}$$

Now we are going to take the inductive step. The **inductive step is taken to prove that the *next* number in the sequence *also* holds true to the equation.**

In our case, we know that the equation holds for f_k, but does it hold for f_{k+1}? We add f_{k+1} to both sides and get:

$$f_1 + f_2 + ... + f_{k+1} = f_{k+2} + f_{k+1} - 1 \tag{6}$$

f_{k+1} and f_{k+2} are just two consecutive terms, and in the Fibonacci sequence, we know that those two terms will just sum up to the next term in the sequence, f_{k+3}. So, the sum of all terms until f_{k+1} is one less than the value of f_{k+3}. This is precisely the original equation, simply one term ahead. We have proved that the original statement holds for $n = k$, where n can equal as little as 1 (our base case), and we proved it was true when $n = k + 1$ (inductive step). Because of this logic, we can conclude that the equation will hold for all successive terms where $k \in \mathbb{N}$. This shows that $f_1 + f_2 + ... + f_n = f_{n+2} - 1$ holds for any value of k, since k is a natural number. □

3.3 Proof by Well-Ordering Principle

In number theory and combinatorics, a foundational concept often employed in proofs and strategy development is the **Well-Ordering Principle (WOP)**. It states that **every non-empty subset of the natural numbers has a least element**. This seemingly simple property of the natural numbers underpins many inductive and recursive arguments, particularly in contexts where minimality or termination is central to analysis.

We will invoke the Well-Ordering Principle in our exploration of the **Difference Game**, a combinatorial process governed by iterated subtraction. The game's structure lends itself naturally to analysis via WOP, as it involves sequences and sets of numbers generated under fixed rules, where understanding the existence or behavior of minimal elements is key to deducing strategic outcomes or proving termination.

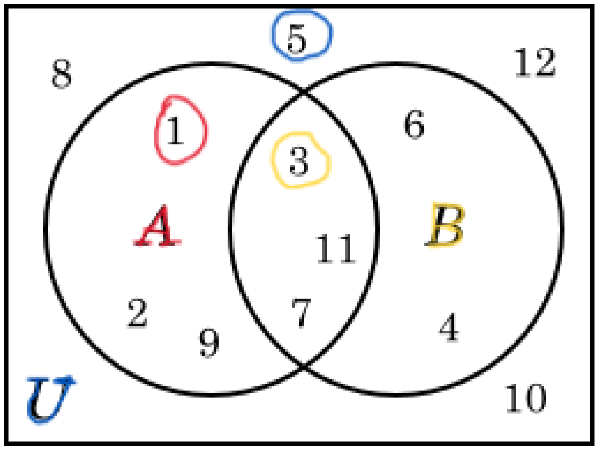


Figure 2: All sets have a least element. An annotated image color codes the least number to the set it belongs to

Proof. We aim to prove that every non-empty subset $S \subseteq \mathbb{N}$ has a least element.

Assume for contradiction that $S \subseteq \mathbb{N}$ is non-empty and has no least element.

$$\text{Assume } S \subseteq \mathbb{N},\ S \neq \emptyset, \text{ and } S \text{ has no least element.}$$

We will show this leads to a contradiction using mathematical induction.

Let us define the property $P(n)$: "$n \notin S$".

$$P(n): \quad n \notin S$$

We will prove $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case: $n = 0$

$$P(0): \quad 0 \notin S$$

If $0 \in S$, then 0 would be the least element of S, which contradicts our assumption.

Inductive Step: Assume $P(k)$ holds for all $k < n$, i.e., no number less than n is in S.

$$\text{Assume } \forall k < n,\ k \notin S$$

Suppose $n \in S$. Then n is smaller than any other element in S, since all smaller numbers are not in S.

$$\Rightarrow n \text{ is the least element of } S$$

But this contradicts the assumption that S has no least element. Therefore,

$$n \notin S$$

By the principle of mathematical induction, we conclude that no $n \in \mathbb{N}$ is in S:

$$\forall n \in \mathbb{N},\ n \notin S$$

This implies that S is empty, contradicting our assumption that $S \neq \emptyset$.

$$\Rightarrow \text{Contradiction}$$

Hence, every non-empty subset of $\mathbb{N}$ must contain a least element.

$$\boxed{\text{Q.E.D.}}$$

□

4 The Difference Game (Rules of the Game)

Since this paper has a focus on The Difference Game, we will turn to that now.

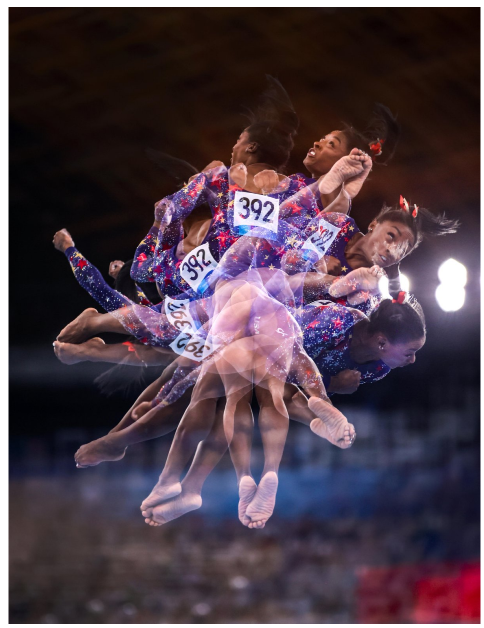
The Difference Game may begin once two positive integers have been written on a chalkboard and there are two players ready to play, who must alternate turns.

To kick off the game, the first player takes their turn, and in doing so they must try to write a new positive number on the board. This ”new number” is the difference of *two* numbers that are already written up there.

The next player proceeds, but now their number can be created by finding the difference of any two of the now *three* numbers on the chalkboard, with the exception of utilizing the intial two numbers that the first player found the difference of. The only way to lose the game is if you are the first player who is unable to form a new number. For our purposes we are assuming no human error, meaning that the game continues until it is genuinely impossible to produce another unique difference and number.

Hypothetically, suppose that a game begins with the numbers 55 and 25 and that the players are Emmy Noether, a female German mathematician who proved proved two theorems that were foundational for both general relativity and elementary particle physics in 1918, and Simone Biles, the most decorated Olympic gymnast of all time and the owner of the restaurant, *The Taste of Gold.* Their game might proceed as follows:

Emmy Noether



Simone Biles

VS.

Turn	Player	Move and Explanation
1	Noether	$30 = 55 - 25$
2	Biles	$5 = 30 - 25$
3	Noether	$50 = 55 - 5$
4	Biles	$20 = 25 - 5$
5	Noether	$35 = 55 - 20$
6	Biles	$15 = 30 - 15$
7	Noether	$10 = 25 - 15$
8	Biles	$45 = 55 - 10$
9	Noether	$40 = 50 - 10$
10	Biles	No valid move left for Noether. **Biles wins!**

Table 2: A sample game of the Difference Game between Emmy Noether and Simone Biles

Observe that all numbers created during the sample game are multiples of 5 and less than 55.

5 Proving the Difference Game

Simone Biles may be celebrating her victory by doing the "Biles"- a double back layout with a half twist -she has defeated one of the most famous mathematicians of all time in a game of intellect!! Or has she? But little does Biles know that the winner of the Difference Game is actually determined by the two numbers she and Noether started with, as well as the player who chooses to go first. These two factors combined sealed Noether's fate to lose, while the strategies of the gymnast and mathematician while playing the game actually don't matter at all!

5.1 Lemma 1

Our first lemma is that every number generated is a multiple of the GCD of the starting numbers. Our goal is to prove that the set S is empty. Set S includes all indices (all i) such that the ith element is not a multiple of the GCD. We want to prove that this is impossible, and that S is empty. We can do this by making the claim that l is the least element in this set S and then we show that this lth least element doesn't exist. If there is no least element in Set S, then Set S must be empty (Well-Ordering Principle).

Proof. If l is the least element in S, we know that our GCD does not divide x_l. In order to produce x_l, we must let there be two numbers to begin the Difference Game. These two numbers are x_m and x_n, such that they appear before x_l. According to the rules of the Difference Game, x_l will be the difference of these two previous numbers, x_m and x_n. However, since we have claimed that l is the least element of Set S, that means that all terms that come before x_l are all divisible by the GCD. So, the GCD *does* divide x_m and x_n. In fact, the GCD divides *all* m and n less than l. This means we can factor the GCD out of the differences generated, telling us that x_l would also be divisible by the GCD. This means that x_l is not in Set S, but since x_l was the least element of the set, Set S must now be empty as all sets must have a least element (Well-Ordering Principle). □

5.2 Lemma 2

Proof. The next thing we must prove is that every number generated is a multiple of the smallest number generated. Set S (different from our previous Set S) now includes all of the numbers generated in our Difference Game, and d is the smallest/least element of S. We started our game with numbers m and n, (different from previous m and n) and we know that the GCD of m and n is g. We also know that x is a difference generated in our difference game, the same way that g is also a difference generated. Let's assume that d does *not* divide X.

$$d \nmid x$$

This tells us that we can write x as a multiple of d plus r (remainder). r must be greater than or equal to 1 and less than d, because if r were 0 or d, then we would just have a case where d *does* divide x.

$$x = dq + r$$
$$1 \leq r < d$$

We can rewrite this as:

$$r = x - dq$$

We now list all the numbers so far in our difference game:

$$x, d, g, m, n \tag{7}$$

We can produce $x - d$ as a difference, and then we can take that number and subtract d from it and do this repetitively again and again. This results in:

$$x - d \tag{8}$$
$$x - 2d \tag{9}$$
$$x - 3d \tag{10}$$
$$... \tag{11}$$
$$x - dq \tag{12}$$

This leads us to the conclusion that we can definitely produce $x - dq$ as a valid difference in our game. In addition, $x - dq$ is really just r, so r *exists* as a difference in the difference game, and we know that r is less than d. This is a contradiction to our original assumption because this means that d is not the least element of Set S. □

5.3 Lemma 3

Lemma 5.1. *Let S be the set of all positive integers generated by repeatedly taking absolute differences of two starting numbers $m, n \in \mathbb{N}$, with $m, n > 0$. Let d be the smallest element of S. Then:*

$$d = \gcd(m, n)$$

Proof. Let $g = \gcd(m, n)$. Our goal is to prove:

$$d = g$$

To do this, we will show:

$$d \geq g \quad \text{and} \quad d \leq g$$

Step 1: Show that $g \mid d$, i.e., $d \geq g$.

From Lemma 1, we know that every number in S is a multiple of g. In particular, the smallest number $d \in S$ must also be a multiple of g. So there exists a natural number q such that:

$$d = qg \quad \text{for some } q \in \mathbb{N}$$

Since $q \geq 1$, we conclude:

$$d \geq g$$

Step 2: Show that $d \mid m$ and $d \mid n$, i.e., $d \leq g$.

The numbers m and n are elements of S by definition. Since d is the smallest positive element of S, and S is closed under subtraction, every number in S must be divisible by d. In particular:

$$d \mid m \quad \text{and} \quad d \mid n$$

So d is a common divisor of m and n. But $g = \gcd(m, n)$ is the greatest such divisor, hence:

$$d \leq g$$

Conclusion.
Combining both inequalities:

$$d \geq g \quad \text{and} \quad d \leq g$$

we conclude:

$$d = g$$

Therefore, the smallest number in S is equal to the greatest common divisor of the starting numbers m and n. □

5.4 Lemma 4

Lemma 5.2. *Let $m, n \in \mathbb{N}$ with $m \geq n > 0$, and let $g = \gcd(m, n)$. In the Difference Game starting with m and n, all numbers generated belong to the set:*

$$S = \{kg \in \mathbb{N} \mid g \leq kg \leq m\}$$

That is, all numbers in the process are positive multiples of g, and no generated number exceeds m.

Proof.
From Lemma 1, we know that all numbers generated in the Difference Game are multiples of $g = \gcd(m, n)$. So every element $x \in S$ satisfies:

$$x = kg \quad \text{for some } k \in \mathbb{N}$$

Next, we show that the values generated cannot exceed m. This is because the game proceeds by taking absolute differences of previously generated values. Since we start with m and n, and at each step we take differences such as $|a - b|$, every new number is less than or equal to the largest number seen so far. Thus:

$$\forall x \in S, \quad x \leq m$$

Now, we show that *every* multiple of g between g and m is eventually generated.

We start with m, and by Lemma 3, we know that $g \in S$ since $g = \gcd(m, n)$. Since subtraction of multiples of g preserves multiples of g, we can generate a sequence:

$$m, m - g, m - 2g, \ldots$$

As long as $m - kg > 0$, we generate all values:

$$m, m - g, m - 2g, \ldots, g$$

This sequence includes all positive multiples of g less than or equal to m. Thus, we have shown:

$$S = \{kg \in \mathbb{N} \mid g \leq kg \leq m\}$$

□

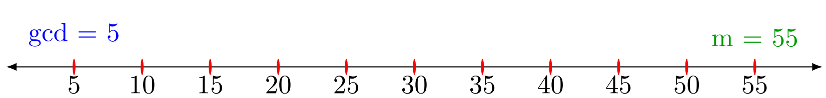


Figure 3: Multiples of $\gcd(55, 25) = 5$ on the interval $[5, 55]$.

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