Pick's Theorem: how to calculate the area of a polygon

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May 21, 2025

Abstract

Pick's theorem, first proved by Georg Alexander Pick in 1899, gives a formula for calculating the area of a simple polygon with lattice points as vertices. We give a proof of Pick's theorem using Euler's formula (graph theory) inspired by *Mathematics Galore!* and *Proofs from the Book.* We also describe several different ways to calculate the area of a triangle and a modification of Pick's theorem for polygons with holes.

1 Introduction

There are many formulas for calculating areas of shapes. Pick's theorem is a theorem used to find the area of a simple polygon, namely a polygon where the edges don't intersect and there are no holes, whose vertices are lattice points, using the number of its boundary and interior lattice points in a generalized formula. It was first discovered by Georg Alexander Pick in 1899, and has many known proofs. We provide one such proof using Euler's formula about the Euler characteristic of a planar graph.

Theorem 1.1. Consider a simple polygon with lattice points as vertices, i interior lattice points and b boundary lattice points, the area A of the polygon is $A = i + \frac{b}{2} - 1$.

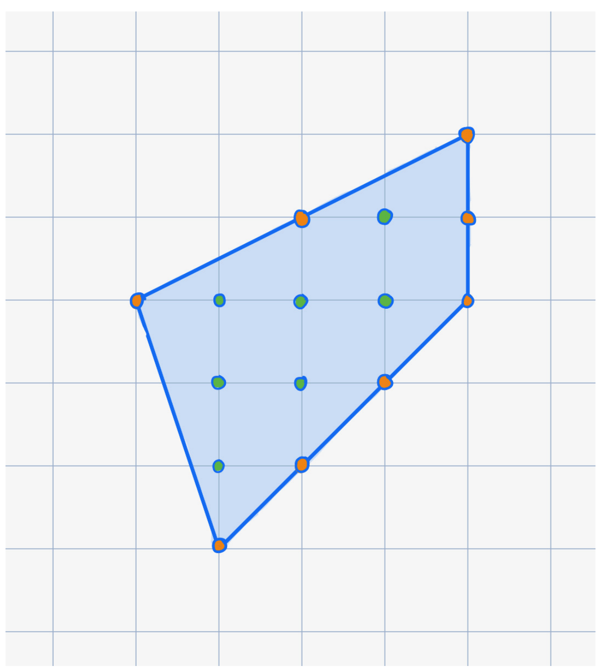


Figure 1: Depicted above is a polygon with 8 boundary lattice points shown in orange and 7 interior lattice points shown in green, so the area is $A = 7 + \frac{8}{2} - 1 = 10$.

This paper is organized as follows. Section 2 discusses the definitions for terminology and concepts used in the paper. Section 3 discusses various strategies for finding the area of a particular triangle.

Section 4 discusses how to find the area of a particular triangle using Pick's Theorem. Section 5 discusses the proof for Pick's theorem. Section 6 discusses further generalizations of Pick's Theorem.

2 Definitions

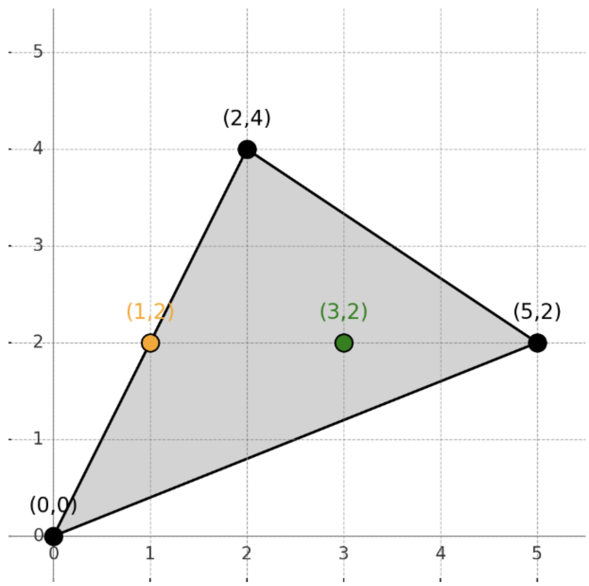
Definition 2.1. A polygon is a closed figure made up of line segments in a two-dimensional plane.

2.1 Lattice point

Definition 2.2. A lattice point is a point with integer coordinates in the plane $\mathbb{R}^2$, which is the Cartesian plane.

Definition 2.3. A boundary point of a polygon is a lattice point that sits on an edge or vertex of the polygon.

Definition 2.4. An interior point of a polygon is a lattice point that is strictly within the boundaries of the polygon created by the edges.



In the graph shown above, the orange point is an example of a boundary point and the green point is an example of an interior point (both are examples of lattice points).

2.2 A Basis

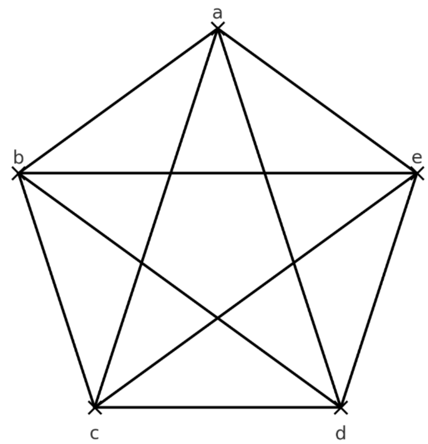
Definition 2.5. A basis for $\mathbb{Z}^2$ is a pair of vectors $(a, b), (c, d)$ so that every element of $\mathbb{Z}^2$ can be expressed as $k(a, b) + l(c, d)$ when $k, l \in \mathbb{Z}$.

2.3 Graphs

Definition 2.6. A graph $G = (V, E)$ is a set V and set E. The elements of V are called vertices, and the elements of E are called edges. An edge is a pair of vertices that can be ordered or unordered.

Definition 2.7. If the edges are ordered, the graph G is directed, and if unordered, the graph is undirected.

Definition 2.8. A planar graph is a graph that can be drawn with no edges intersecting at any point other than the vertices.



An example of a nonplanar graph is depicted above with vertices a, b, c, d, e and edges $(a, b), (b, c), (c, d),$ $(d, e), (e, a),$ $(a, c), (b, d),$ $(c, e), (d, a), (e, b)$.

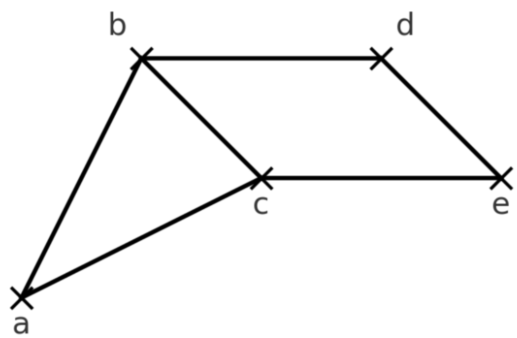


Figure 2: This drawing depicts an example graph $G_1 = (\{a, b, c, d, e\}, \{ab, ac, bc, bd, ce, de\}$

The example graph $G_1 = (\{a, b, c, d, e\}, \{ab, ac, bc, bd, ce, de\}$ is a planar graph because it can be drawn without intersecting edges.

2.4 Euler characteristic

The Euler characteristic χ is a quantity defined as $v - e + f$, where v is the number of vertices, e is the number of edges, and f is the number of faces. The number of faces of a planar graph corresponds to how many pieces the plane is divided into; therefore, the Euler characteristic applies to planar graphs and not non-planar graphs, which do not create distinct faces due to the ambiguity of intersecting edges.

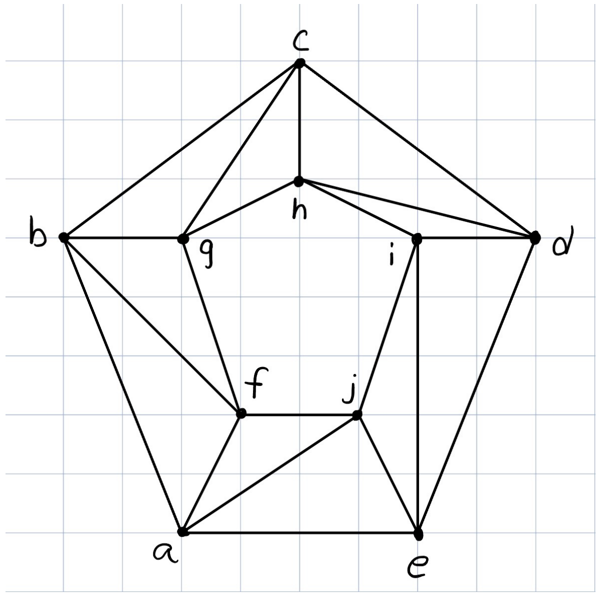


Figure 3: This image depicts the graph $G_2 = (\{a, b, c, d, e, f, g, h, i, j\}, \{ab, bc, cd, de, ae, fg, gh, hi, ij, fj, bg, ch, id, je, fa, gc, hd, ie, ja, fb\})$

G_2 contains 10 vertices, 20 edges, and 12 faces. Using the definition of the Euler characteristic of χ, $v - e + f = \chi$

$$10 - 20 + 12 = \chi \quad (1)$$
$$2 = \chi \quad (2)$$

Therefore, the Euler characteristic is 2 for G_2.

3 Finding the area of a particular triangle

Triangle T has vertices A at $(0, 0)$, B at $(5, 2)$, and C at $(2, 4)$. There are many ways to find the area of T. We give a few of them in Subsections 3.1, 3.2, 3.3, and 3.4.

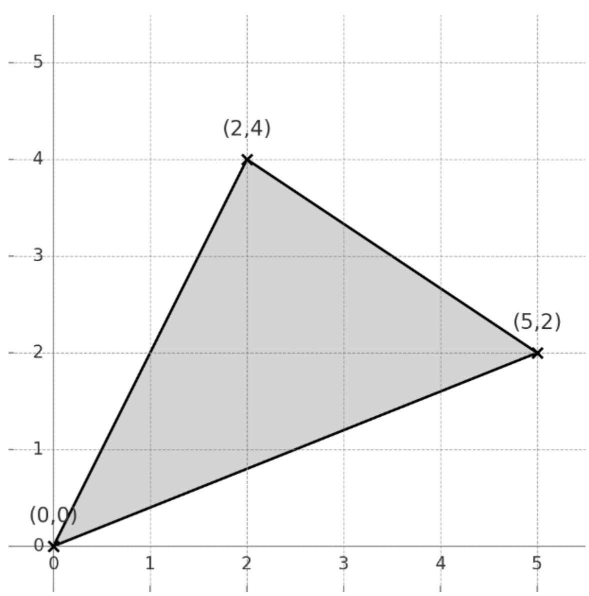


Figure 4: This image depicts a triangle with vertices $(0, 0), (5, 2), (2, 4)$.

3.1 Strategy 1: Enclose it in a box

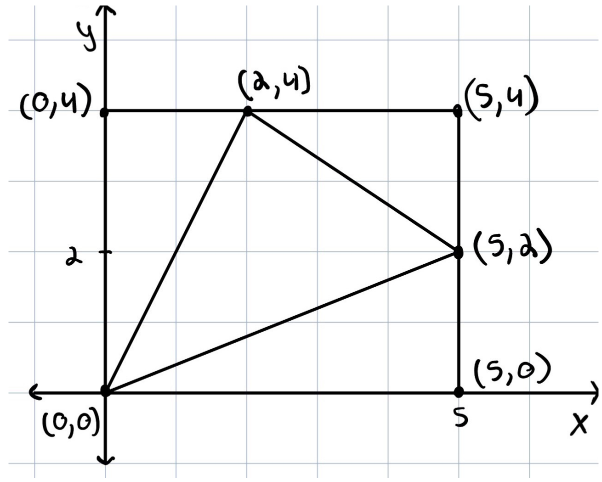


Figure 5: This image depicts the same triangle with vertices $(0,0),(5,2),(2,4)$ enclosed in a box with vertices $(0,0),(5,0),(5,4),(0,4)$.

First we can find the area of the box by multiplying the length and width, which in this case would be 5×4, making the area of the box 20. From here there are 3 smaller right triangles surrounding the main triangle, so we can use the sides of the box and the formula that the area of a right triangle is the base multiplied by the height divided by 2 for each of these right triangles. The right triangle consisting of vertices $(0,0),(0,4),(2,4)$ has an area of 4, the right triangle consisting of vertices $(2,4),(5,4),(5,2)$ has an area of 3, and the right triangle consisting of vertices $(5,2),(5,0),(0,0)$ has an area of 5. Using the areas of the 3 right triangles, we can add them together and subtract the sum from the area of the box. $4+3+5=12$, and $20-12=8$, therefore, the area of the triangle is 8.

3.2 Strategy 2: Heron's formula

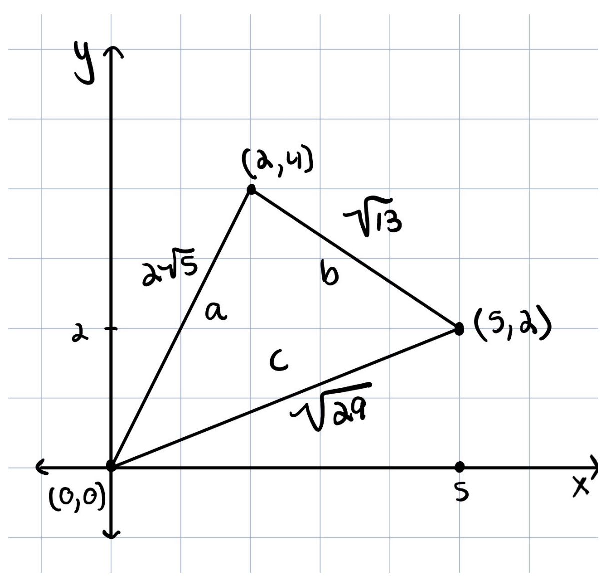


Figure 6: This image depicts the same triangle from Figures 1 and 2, but with side lengths included.

Definition 3.1. Given a triangle with side lengths a,b,c, the perimeter of the triangle is $a+b+c$ and the semiperimeter is $\frac{a+b+c}{2}$.

Theorem 3.1. A triangle with side lengths a, b, c and semiperimeter s has area $\sqrt{s(s-a)(s-b)(s-c)}$

The semiperimeter for this particular triangle is $\frac{2\sqrt{5}+\sqrt{13}+\sqrt{29}}{2}$, and the area is 8.

3.3 Strategy 3: Cut the triangle in half

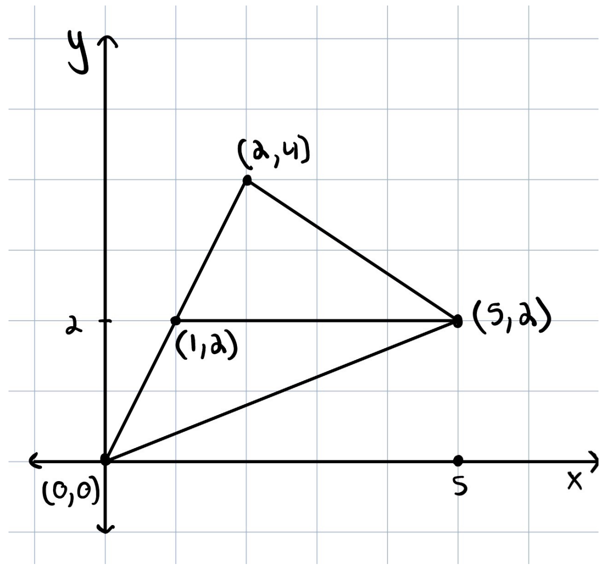


Figure 7: This image depicts the same triangle from Figures 1-3, but there is now a line segment connecting points $(1, 2)$ and $(5, 2)$, cutting the original triangle into two smaller ones. The smaller triangles have a horizontal base and vertical height, making their areas easier to calculate than the original triangle.

Because the two small triangles make up the entire large (original) triangle, to find the area of the original we can find the areas of the two smaller ones, and add their values together. The small triangle at the top has a base of length 4, and a height of 2. So, using the formula for finding the area of a triangle, we can calculate that the area is 4. Next, we can find the area of the small bottom triangle. It has a base of 4 and a height of 2 as well, so we can calculate that the area of this triangle is also 4. Now, we can add up the areas of the two smaller triangles to get 8, the total area of the original triangle.

3.4 Strategy 4: Determinant formula

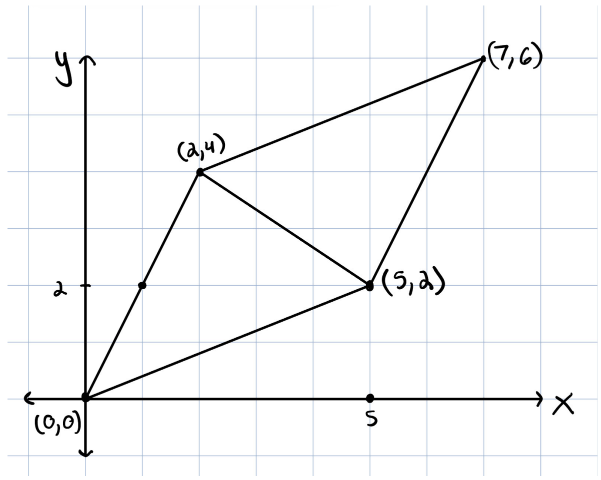


Figure 8: This image depicts the triangle from Figure 1 but with an additional point at $(7, 6)$, forming a parallelogram with vertices $(0, 0), (5, 2), (2, 4), (7, 6)$ and a diagonal from $(2, 4)$ to $(5, 2)$.

Definition 3.2. Consider 2×2 matrix

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

The determinant of M is $p * s - q * r$.

Example 3.1. The determinant of

$$M = \begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix}.$$

The determinant of M is 16.

To calculate the area of the triangle using a determinant, we write vertices $(5, 2)$ and $(2, 4)$ as column vectors in a 2×2 matrix. The area of the triangle is half the absolute value of the determinant value, so in this case, the area of the triangle would be 8.

4 Pick's theorem: example

4.1 Triangle

In this subsection, we will cover how to find the area of the same triangle T with vertices A at $(0, 0)$, B at $(5, 2)$, and C at $(2, 4)$, using Pick's Theorem, our paper's topic.

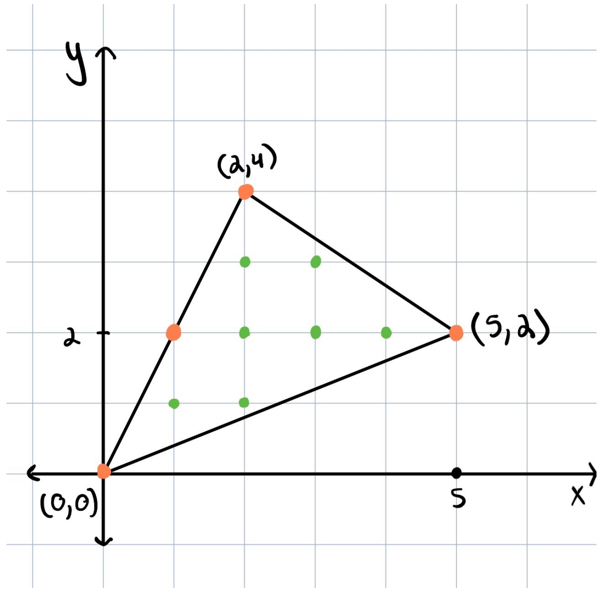


Figure 9: This image depicts our triangle, with boundary lattice points b in orange, and interior lattice points i in green.

To find the area of the triangle using Pick's Theorem, we will use the formula $A = i + \frac{b}{2} - 1$, where b is the number of boundary points and i is the number of interior points. To find the value for b for triangle T we can count the number of orange points, which is 4. To find the value for i for triangle T we can count the number of green points, which is 7. We can plug in these values to get the equation $A = 7 + \frac{4}{2} - 1$, which simplifies to $A = 8$. So, we have used Pick's Theorem to calculate that the area of the triangle is 8.

4.2 Quadrilateral

Pick's theorem can also be applied to quadrilaterals to find their area. For example, we can calculate the area of the parallelogram PQRS with vertices $(0, 0), (2, 4), (7, 6), (5, 2)$.

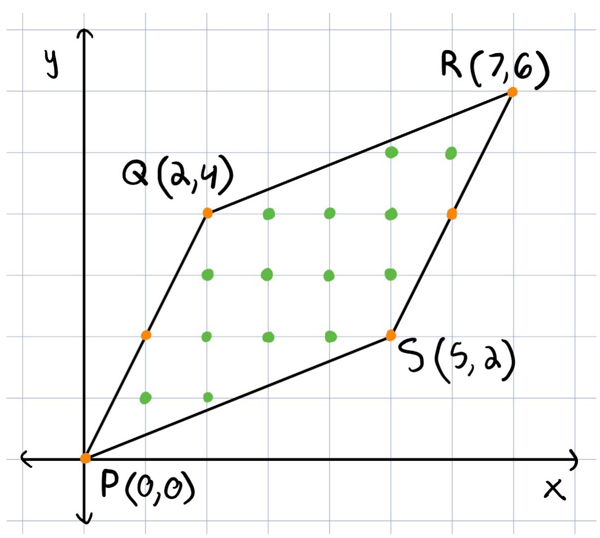


Figure 10: This image depicts parallelogram PQRS with vertices $(0, 0), (2, 4), (7, 6), (5, 2)$, the interior lattice points labeled in green and the boundary lattice points labeled in orange.

In the figure above, there are 14 interior lattice points and 6 boundary lattice points, therefore $i = 14$ and $b = 6$. Using Pick's Theorem $A = i + \frac{b}{2} - 1$:

$$A = 14 + \frac{6}{2} - 1 \tag{3}$$
$$A = 16 \tag{4}$$

Therefore, using Pick's theorem to calculate the area of parallelogram PQRS, we found that the area is 16.

5 Pick's theorem: proof

Theorem 5.1. Let Δ be a triangle that contains no boundary or interior lattice points except for the three vertices. Then Δ has area $1/2$.

Lemma 5.1. If the parallelogram with vertices $(0,0),(a,b),(c,d),(a+c,b+d)$ has no boundary or interior lattice points except for the vertices, then $\{(a,b),(c,d)\}$ is a basis for $\mathbb{Z}^2$.

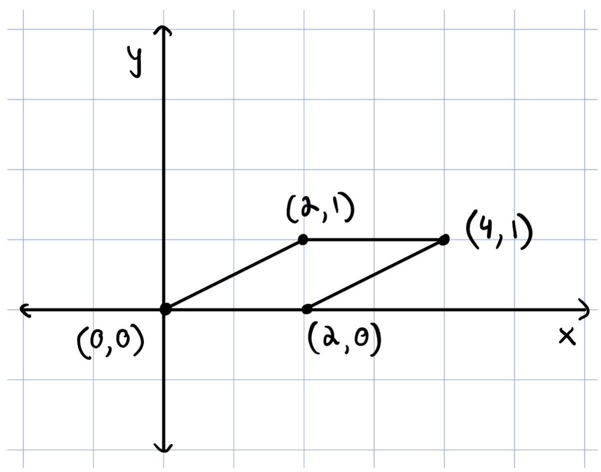


Figure 11: In the graph of $\mathbb{Z}^2$ depicted above, the parallelogram $P'Q'R'S'$ with points $P' = (0,0)$, $Q' = (2,0)$, $R' = (2,1)$, and $S' = (4,1)$ contains no boundary or interior lattice points except the vertices.

Proof. So, as we tile the plane with the parallelogram, all corners of copies of the parallelogram would collectively reach every lattice point, making $\{(a,b),(c,d)\}$ a basis for $\mathbb{Z}^2$. This reasoning works for parallelograms other than $P'Q'R'S'$ with no boundary or interior lattice points except the vertices. ☐

Lemma 5.2. If (a,b) and (c,d) form a basis for $\mathbb{Z}^2$, then

$$\left| \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right| = 1.$$

Proof. The matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ must have an integer determinant because all the entries are integers. Since it is always possible to convert from one basis to the other basis within $\mathbb{Z}^2$, the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ must be invertible to allow for the multiplication of the basis vectors with the inverse to set one basis to another basis as a linear transformation. When considering the inverse matrix

$$\frac{1}{\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

it also must have an integer determinant because it has all integer entries, therefore the determinant of the inverse matrix must be ± 1, since any other value for the determinant would result in a fraction, which is not possible with all integer entries. Therefore, $\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right|$ must be 1 and cannot be -1 since we are taking the absolute value of the determinant.

□

Proof of Theorem 5.1. The triangle Δ can be translated so that one of its vertices is at the origin. This translated triangle Δ' has the same area as Δ. Let the vertices of Δ' be $(0,0), (a,b), (c,d)$. The area of Δ' is

$$\frac{1}{2}\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right|$$

By Lemmas 5.1 and 5.2,

$$\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = 1.$$

So the area of Δ' is $1/2$, which implies the area of Δ is also $1/2$. □

Definition 5.1. A triangle Δ with no boundary or interior points except for the three vertices is called a special triangle.

A polygon with lattice points as vertices can be subdivided into special triangles. Here is an example.

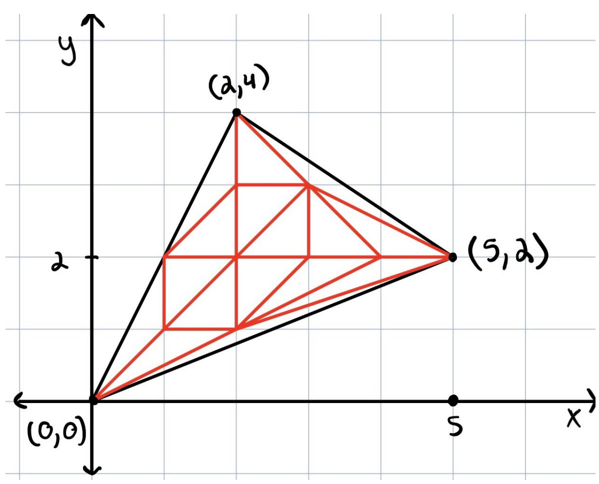


Figure 12: This image depicts the same triangle T subdivided into 16 special triangles. This construction can be regarded as a graph G_T.

A is the area of the graph G_T, b is the number of boundary lattice points for the graph G_T, and i is the number of interior lattice points for the graph G_T.

Each special triangle can be considered a face of the graph. Since each special triangle Δ has an area of $\frac{1}{2}$, $2A$ is equivalent to the number of faces within the boundary of the graph. The total number of faces on the graph G_T will be $2A+1$ because the outer plane also counts as a face.

Now we count the edges of G_T. Why is $6A = 2e - b$? This is because both sides count the number of edges of special triangles. To get the left hand side, since each special triangle has area $1/2$, the number of special triangles is $2A$. Since each triangle has 3 sides, we get $6A$ edges of special triangles. to get the right hand side, the number of edges of special triangles is $2e - b$ because each edge participates in 2 triangles except for the edges on the boundary.

Because $6A = 2e - b$, we can solve for e to get the equation $e = \frac{6A+b}{2}$.

To express the number of vertices in G_T, we can add the number of interior lattice points i and boundary lattice points b, since each boundary and interior lattice point is a vertex of at least one special triangle Δ within the graph G_T.

To summarize,

$$v = i + b$$
$$e = \frac{6A+b}{2}$$
$$f = 2A + 1.$$

Using Euler's characteristic $v - e + f = 2$ and the expressions $v = i + b$, $f = 2A + 1$, and $e = \frac{6A+b}{2}$, we can find Pick's theorem.

$$(i+b) - \frac{6A+b}{2} + (2A+1) = 2 \tag{5}$$
$$-\frac{6A+b}{2} + 2A = 1 - i - b \tag{6}$$
$$-3A - \frac{b}{2} + 2A = 1 - i - b \tag{7}$$
$$-A = 1 - i - \frac{b}{2} \tag{8}$$
$$A = i + \frac{b}{2} - 1 \tag{9}$$

Note that this argument doesn't just work for the triangle T, it works for any general polygon with lattice point vertices! So Pick's theorem is proved.

6 Future directions

Pick's theorem might fail if the polygon doesn't have lattice point vertices. The area of a square with vertices at $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$ is 1, but this square has $i = 1$ and $b = 0$, so Pick's theorem gives an area of $i + b/2 - 1 = 0$, which is wrong. To calculate the area of a simple polygon whose vertices are not necessarily lattice points, we can use the shoelace formula.

There is a generalization of Pick's theorem to a polygon with at least one hole.

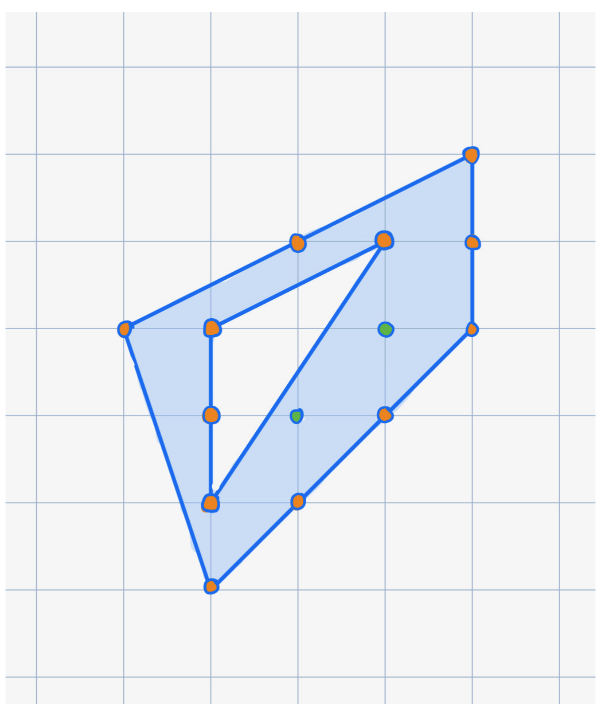


Figure 13: Depicted above is a polygon with a hole. The number of interior lattice points i=2, boundary lattice points b=12, and holes h=1. So, $A = i + \frac{b}{2} + h - 1 = 2 + 6 = 8$.

Theorem 6.1. Consider a polygon with lattice points as vertices, with i interior lattice points, b boundary lattice points, and h holes, then the area A of the polygon is $A = i + \frac{b}{2} + h - 1$.

7 Acknowledgments

Carolena and Hannah would like to thank their mentor Katherine Tung, PRIMES Circle head mentors Mary Stelow and Marisa Gaetz, and other PRIMES Circle staff and MIT for facilitating their participation in the PRIMES Circle program.

Carolena would like to acknowledge the support of her family and friends in her journey throughout PRIMES Circle.

Hannah would like to thank her parents and family for their support of her participation in the PRIMES Circle program.

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