Polyhedra and Euler Characteristics

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April 2025

Abstract

In this paper, we cover the traits of polyhedra and introduce topology. For polyhedra, we prove the existence of limited Platonic solids and introduce Euler's characteristic to explain why Euler's formula applies to polyhedra. Regarding topology, we will cover $\mathbb{RP}^2$ and cellular decomposition.

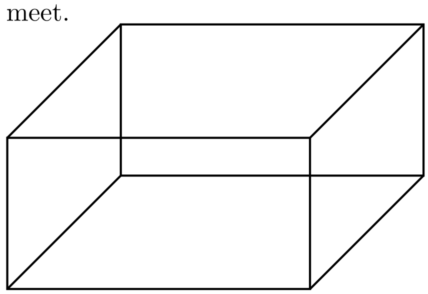
Contents

1 Introduction

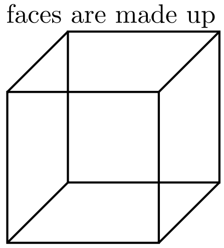
Polyhedra are fascinating mathematical constructs with topological characteristics of 3-dimensional shapes despite their simple structure of flat, 2-dimensional faces. These shapes abide by particular universal characteristics, such as Euler's Formula, for polyhedra and topological spheres, and the generalized Euler Characteristic, for abstract topological spaces. Furthermore, these rules carry over for evaluating $\mathbb{RP}^2$ and cellular decomposition. Such reliance on polyhedra characteristic reflects the complicated and interwoven relationship between math concepts in Geometry and Topology, where structures must traverse dimensions to be evaluated.

2 Definitions

Definition 2.1. A *polyhedron* is a convex 2-dimensional shape living in 3 dimensions consisting of vertices, edges, and faces. Edges are straight segments between two vertices and are formed where two faces meet. Faces are closed, flat surfaces bordered by edges. Vertices are the points at which where 3 edges meet.



Definition 2.2. A *platonic solid* or *regular polyhedron* is a polyhedron whose faces are made up of congruent regular polygons.



Definition 2.3. The *real projective space*, denoted by $\mathbb{RP}^2$, is a 2-dimensional space we obtain by adding the points at infinity in the direction of each line to the usual plane $\mathbb{R}^2$.

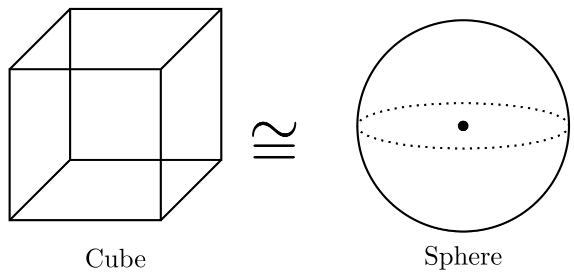
Definition 2.4. *Induction* refers to the proof technique of reducing complex structures into simpler, elementary forms with similar structure to prove the validity of a rule. If the rule holds true for the basic example, the proof enables the conclusion that the rule holds true for all cases in which it is applicable. To prove the rule's unconditional validity, the proof assumes that the same conclusion will be reached for all possible applications, given that the initial example's scalability is confirmed.

Definition 2.5. Two spaces A and B are *homeomorphic* if there exists a continuous function $f : A \to B$ with a continuous inverse $f^{-1} : B \to A$. Informally, we can describe it as, after A is distorted to create B if the distortion follows the following criteria:

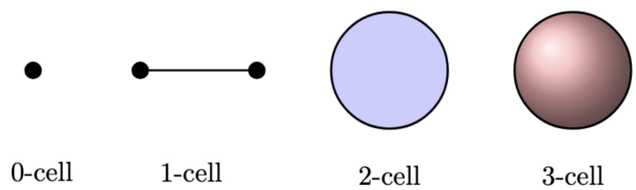
1. The object may not go through itself.

2. The object may not attach itself to itself.

3. The object may not be cut or separated from itself.



Definition 2.6. The *cellular decomposition* of a topological space X is a space homeomorphic to X consisting of several balls, called *cells*, glued together. An n-dimensional cell is called an *n-cell*.



A 0-cell is visualized as an isolated point, a 1-cell as a line segment with open endpoints, a 2-cell as a filled disk, and a 3-cell as a solid ball with filled volume.

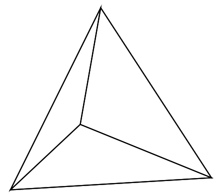
Definition 2.7. For a space X with a cellular decomposition, the *Euler characteristic* $\chi(X)$ of X is a sum of the number of n-cells, added or subtracted in an alternating pattern:

$$\chi(X) = (\# \text{ of 0-cells}) - (\# \text{ of 1-cells}) + (\# \text{ of 2-cells}) - \dots$$

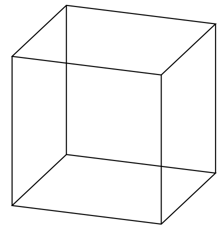
3 Polyhedra

3.1 Platonic Solids

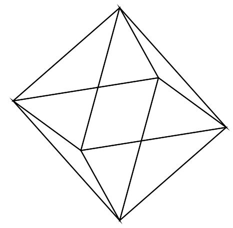
There are five Platonic solids, namely the tetrahedron, cube, octahedron, dodecahedron, and icosahedron.



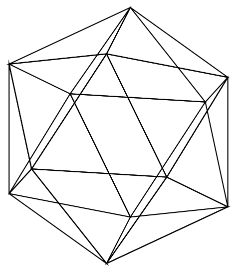
Tetrahedron



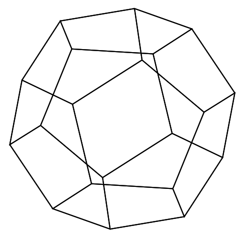
Cube



Octahedron



Icosahedron



Dodecahedron

Theorem 3.1. There exist only five platonic solids.

While it might initially seem like there could be infinite platonic solids - shapes with regular polygonal sides, seeing movies with massive force fields, only 5 actually exist.

Before proving this statement, we must display the following properties of 3-dimensional shapes. Consider the following statement where n is the number of faces joined by a vertex and where α is the measure of the each side's angles at said vertex.

$$n \cdot \alpha < 360^\circ$$

This means to display the maximum angle and faces a vertex can have before being forced to lie in a plane and therefore become unable to become part of a polyhedron. Consider the following figures.

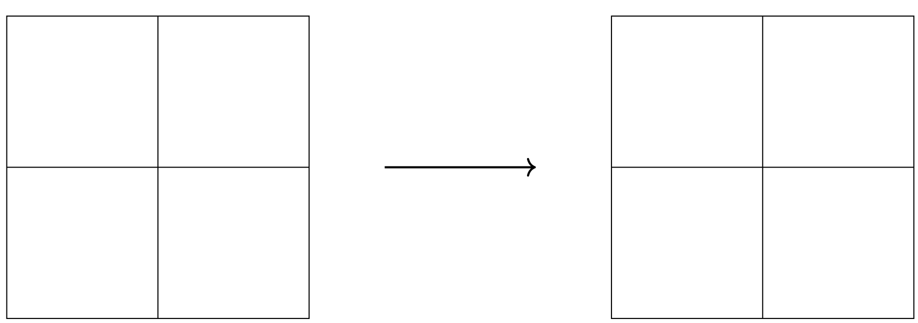


Figure 3.1.1

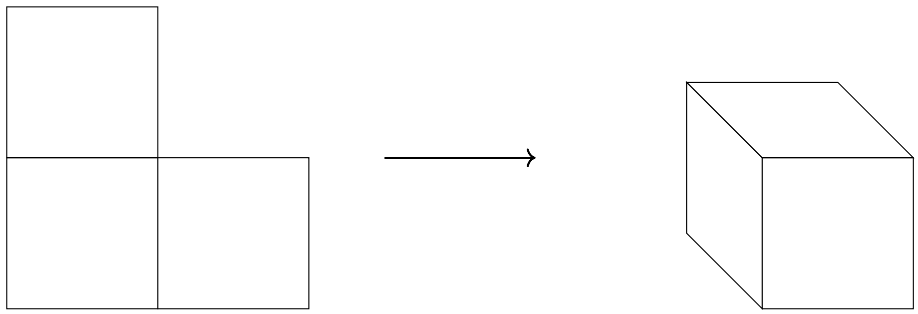


Figure 3.1.2

As seen above, the figure on the left cannot be folded to become a part of a 3-dimensional object and must remain rigid in 2 dimensions. This is consistent with the above law, $n \cdot \alpha < 360°$. $4 \cdot 90° = 360°$. 360 is not less than 360, and therefore the vertex cannot be 3-dimensional.

In contrast, Figure 2 can be folded to become a part of a 3-dimensional object and will fold to make a corner. Again, this is consistent with the law $n \cdot \alpha < 360°$. $3 \cdot 90° = 270°$. $270°$ is less than $360°$, so the vertex can be 3-dimensional.

Now that we have illustrated the formula, we will move on to proving the initial statement of there existing only 5 platonic solids by going through every possible option of n and α.

Proof. Again, n is the number of faces joined by a vertex and where α is the measure of the each side's angles at said vertex.

The most likely and smallest αs will be the following: $60°$ from an equilateral triangle, $90°$ from a square, $108°$ from a regular pentagon, and $120°$ from a regular hexagon. The following:

$$\alpha = 60°$$

$$\alpha = 90°$$

$$\alpha = 108°$$

$$\alpha = 120°$$

In addition, n cannot equal 1 or 2 for the following reasons. If n were 1, the shape would simply remain with one side and remain as a 2D object. If n was 2, the shape would simply glue itself to itself, once again collapsing to become a 2D object. Therefore, we must start at $n = 3$. We will go from the smallest α to the highest α that meets the conditions of the inequality $n \cdot \alpha < 360°$. Therefore, we will start with $\alpha = 60°$. $3 \cdot 60° = 180°$. $180°$ is less than $360°$, so the statement is fulfilled. Next, we move on to $\alpha = 90°$. $3 \cdot 90° = 270°$. $270°$ is less than $360°$, so the statement is fulfilled again. We will do the same as for $\alpha = 108°$. $3 \cdot 108° = 324°$. $324°$ is less than $360°$, so the statement is fulfilled. Lastly, we have a case where $\alpha = 120°$. $3 \cdot 120° = 360°$. $360°$ is not less than $360°$, and therefore the vertex cannot be 3-dimensional and no other shape with a greater angle can become part of a platonic solid when $n = 3$ and at all, as $n = 3$ is the smallest possible n. We should keep track of the cases that met the inequality.

$$3 \cdot 60° < 180°$$

$$3 \cdot 90° < 270°$$

$$3 \cdot 108° < 324°$$

These correspond to the tetrahedron which has 4 sides, the cube, which has 6, and dodecahedron, which has 12, respectively. Next, we will move on to the cases where $n = 4$. Again, we will start with $\alpha = 60°$. Then, we get $4 \cdot 60° = 240°$. $240°$ is less than $360°$, so this situation is possible. We try again for $\alpha = 90°$. Then, we get $4 \cdot 90° = 360°$. $360°$ is not less than $360°$, so this situation is impossible and any other situation where $\alpha > 90°$ is possible. Therefore, we add only one situation to our list of inequalities.

$$4 \cdot 60° < 360°$$

This situation corresponds to the octahedron, which has 8 sides. Next, we will move on to the cases where $n = 5$. Again, we will start with $\alpha = 60°$. Then, we get $5 \cdot 60° = 300°$. $300°$ is less than $360°$, so this situation is possible. We try again for $\alpha = 90°$. Then, we get $5 \cdot 90° = 450°$. $450°$ is greater than $360°$, so this situation is impossible and any other situation where $\alpha > 90°$ is possible. Therefore, again, we can add only one situation to our list of inequalities.

$$5 \cdot 60° < 360°$$

This situation creates the vertex of an icosahedron, which has 20 sides. Finally, we will have $n = 6$. Here, we try again for $\alpha = 60°$. Then, we get $6 \cdot 60° = 360°$. $360°$ is equal to and not less than $360°$, so this situation is impossible and any other situation where $\alpha > 60°$ is possible. In addition, no other situation where $n > 6$ is possible either, as $\alpha > 60°$ makes α as small as it can be. Therefore, we cannot add any situations to our list of inequalities and our list is complete with the following.

Calculation	**Sum of Angles**	**Polyhedron**
$3 \cdot 60°$	$180° < 360°$	Tetrahedron
$3 \cdot 90°$	$270° < 360°$	Cube
$3 \cdot 108°$	$324° < 360°$	Dodecahedron
$4 \cdot 60°$	$240° < 360°$	Octahedron
$5 \cdot 60°$	$300° < 360°$	Icosahedron

The illustrations of these solids can be seen below in section 3.2. □

3.2 Euler's Formula

Euler's Formula for polyhedra states that for a polyhedron with V vertices, E edges, and F faces:

$$V - E + F = 2$$

Euler's Formula is fundamentally a topological property. The rule is maintained even when a polyhedron is deformed or flattened into a 2-dimensional shape (known as a planar graph), as long as the polyhedron's properties of vertices, edges, and faces remain intact. A polyhedron can be represented on a flat planar graph by replacing faces with edges and vertices. The polyhedron's vertices are represented as nodes, and the edges of the polyhedron become edges of the graph. This method enables the formation of a skeleton graph.

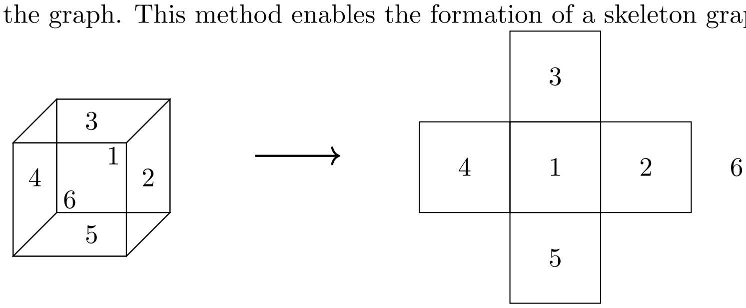
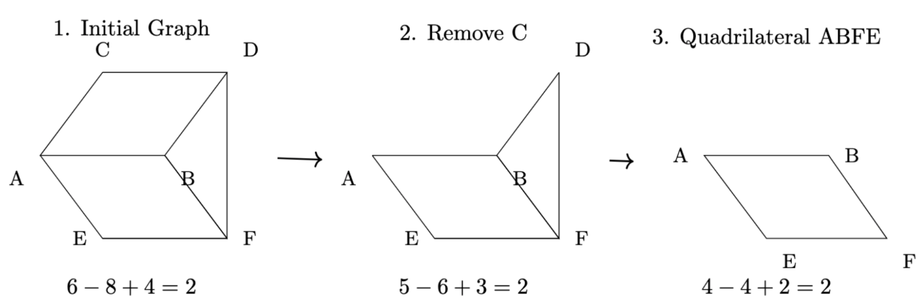


Figure 3.2.2: Cube and its 2D net representation.

Note that on the planar graph, the sixth face exists in the region outside of the graph.

Theorem 3.2. Euler's Rule can be proved through induction.



Proof. We start with any connected planar graph. The first graph consists of 6 vertices, labeled A through F. Each vertex is connected by an edge, totaling 8. The shape contains 3 faces within the graph and 1 external to the structure, adding up to 4 total faces. Algebraically, adding vertices, subtracting edges, and adding faces would equate to a value of 2. Step 2 of the proof involves the reduction of a planar graph by removing select vertices and the corresponding edges and faces they form. Breaking down the initial graph by removing vertex C results in quadrilateral $ABDFE$, where 1 vertice, 2 edges, and 1 face are removed. Euler's characteristic ($V - E + F = 2$) is preserved, given that the loss of vertex C corresponds to the loss of edges AC and CD, and the loss of face $ACDB$. Euler's Rule is maintained. Further breaking down the figure into Quadrilateral $ABFE$ removes vertice D and corresponding edges BD and DF and face BDF. Figure 3 showcases the tenacity of polyhdrons despite reduction, as the figure's 4 vertices, 4 edges, and 2 faces total to a value of 2 in $V - E + F$. □

3.3 Exceptions to Euler's Rule

Euler's formula applies solely to convex polyhedra and planar graphs equivalent (in topology) to spheres. Shapes with holes, such as donut-shaped tori, use:

$$V - E + F = 0.$$

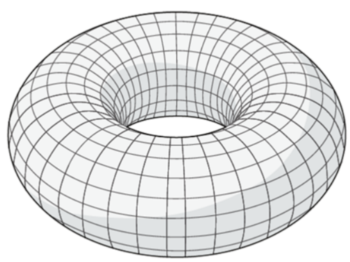


Figure 3.2.3

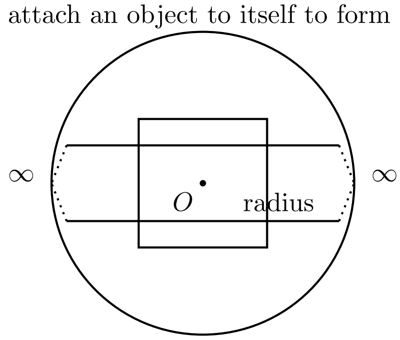
The torus is an exception to the standard Euler's formula because of its hole, which restricts the shape from topologically morphing into a sphere. This hole changes how the vertices, edges, and faces of the polyhedron connect. Every loop on the torus cannot be glued onto a single point to become universally connected, as can a sphere.

4 Topology

4.1 Real projective plane, $\mathbb{RP}^2$

4.1.1 Construction of $\mathbb{RP}^2$

Imagine a point A and a distinct point B with parallel lines a and b running through their respective points. Normally, these lines would never intersect, being parallel. However, in the plane of $\mathbb{RP}^2$, they do. When one looks down a long hallway with parallel walls, the walls seem to converge, but they never touch. Similarly, in our process of creating $\mathbb{RP}^2$, we state that a and b converge at distance ∞, which we denote as a point C. To continue, we create an infinite number distinct versions of A,B, a, b, and C oriented in different directions until the points C create a circle with radius ∞. In relation to homeomorphism, one can attach an object to itself to form a new object.



We will use this action of attaching an object to itself to complete our new plane, $\mathbb{RP}^2$. We do not want two parallel lines converging twice, as that could imply that they are the same line. Therefore, we attach each point C at infinity to its corresponding opposite point C. Finally, we have created $\mathbb{RP}^2$.

4.1.2 Computing the Euler characteristic of $\mathbb{RP}^2$

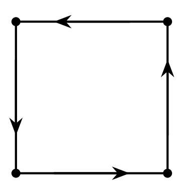
To find the Euler character, we can use cellular decomposition. To restate it, where T is the object and $\chi(T)$ is the Euler characteristic of the object,

$$\chi(T) = (\#\ \textit{of 0-cells}) - (\#\ \textit{of 1-cells}) + (\#\ \textit{of 2-cells}) - ... \pm (\#\ \textit{of n-cells})$$

We will not be using anything above 2 cells to decompose $\mathbb{RP}^2$, so we will disregard the content afterwards. That leaves us with the following formula.

$$\chi(\mathbb{RP}^2) = (\#\ \textit{of 0-cells}) - (\#\ \textit{of 1-cells}) + (\#\ \textit{of 2-cells})$$

As we stated in the creation of our $\mathbb{RP}^2$ plane, we had a circle with radius infinity at one point. Because Euler's characteristics are consistent across all homeomorphic objects, we can reduce that circle to a limited square, called the fundamental polygon.



To find the Euler character, we must fold the fundamental polygon so that the arrows are matched. When we fold them with the correct alignments, we get two 0-cells (vertices), two 1-cells (edges), and one 2-cell (the face). Therefore, from our prior equation, we get the following using our previous equation.

$$\chi(\mathbb{RP}^2) = (2) - (2) + (1) = 1$$
$$\chi(\mathbb{RP}^2) = 1$$

4.2 Cellular Decomposition

Theorem 4.1. An n-dimensional sphere can be constructed by gluing together cells of dimension $\leq$ n.

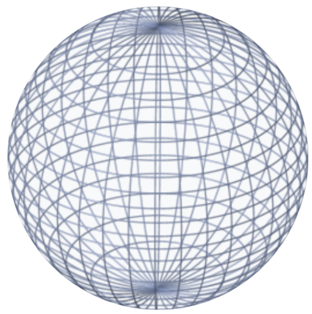


Figure 4.2.1: 2-sphere. Picture from Wikimedia Commons

Proof. A 2-sphere can be constructed with 0-cells, 1-cells, and 2-cells. Using a 0-cell as the base point, glue one 1-cell to form a continuous loop that connects the opposite sides of the base point. This forms a hollow ring, around which the structure will be built. Note that the 0-cell is necessary in this construction, given that the 1-cell may not glue onto itself. Construct the upper and lower hemispheres of the sphere using two separate 2-cells, one for each hemisphere. Glue the outer ring, or the circumference, of each 2-cell onto the corresponding circumference of the prior structure to form a sphere. The value of the Euler Characteristic $\chi(S)$ for a 2-sphere S equals to 2, as defined by the number of 0-cells (1) subtracted by the number of 1-cells (1) and summed with the number of 2-cells (2).

$$\chi(S) = (\#\text{ of 0-cells}) - (\#\text{ of 1-cells}) + (\#\text{ of 2-cells}) = 1 - 1 + 2 = 2.$$

□

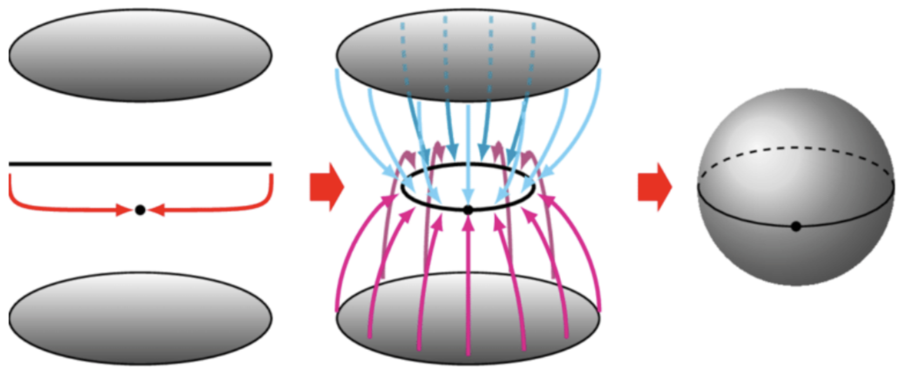


Figure 4.2.2: Visual representation of proof. Picture from Cohomology of Differential Forms and Feynman diagrams

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Wikipedia, *Mug and Torus morph*

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