# PERIODICITY OF $S$-PICK-UP-BRICKS

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Abstract. This paper introduces impartial game theory through the Sprague-Grundy Theorem. Beginning by explaining the types of positions, we build up to results about their sums and equivalence. We conclude our general overview of game theory by using the MEX Principle to prove the Sprague-Grundy Theorem. In the final sections, we use the MEX principle to prove that every single game of $S$-Pick-Up-Bricks (normally referred to as a subtraction game) has a corresponding periodic sequence, and find the period of that sequence for some sets $S$.

## 1. Introduction

Game theory is fundamentally a field focused on finding strategies in multiplayer games. It has developed rapidly over the past century with the discovery of the Nash Equilibrium and the Minimax theorem. A common practical application is economics, specifically, predicting the stock market and the behavior of major firms. Another application is political theory, in which game theory is used to determine the best possible formation of coalitions and tactics for negotiation.

This paper is concerned with combinatorial game theory, the study of what are essentially luckless games. Combinatorial games include Chess and Checkers, but exclude Poker, Rock-Paper-Scissors, and Backgammon. Upon consideration, the assertion that one of the players in a combinatorial game has a winning or tying strategy is intuitive; however, this makes the assertion no less profound: even in a game as complex as chess – which has more possible moves than molecules in the observable universe, has been subjected to extensive computational analysis, and has cemented itself in popular culture as a legitimate intellectual pursuit – all $10^{120}$ positions can be relegated to a simple statement of who has a winning or tying strategy. A game for which it is known which player has a winning or tying strategy is referred to as "ultra-weakly solved," a game for which this player's winning strategy from the starting position is known is referred to as "weakly solved," and a game for which the optimal strategy is known from all positions is referred to as "strongly solved." Contrary to what this nomenclature implies, ultra-weak solutions are often considered the most elegant, as weak or strong solutions have to consider an immense number of positions and therefore usually rely upon brute-force computation.

In combinatorial game theory, a game is referred to as "normal-play" if the last player to move wins. A normal-play game is considered "impartial" if, for every constituent position, the set of possible moves does not vary by player. This paper begins by exhibiting important results in the theory of impartial games, and concludes by applying these past results to explain original results in the strategic classification of positions in $S$-Pick-Up-Bricks, a simple game that can be used to analyze all other combinatorial games.

**Outline.** The rest of the paper is organized as follows: In Section 2, we explain the fundamental game theory necessary to understand Section 3, which describes core results of the theory of impartial games, including Nim, the MEX Principle, and the Sprague-Grundy Theorem. Both of the prior sections rely upon definitions and results from [1]. Section 4 applies this theory to the game of $S$-Pick-Up-Bricks, giving results about the periodicity of nimber sequences for certain sets $S$. Section 5 poses unanswered questions and conjectures about said periodicity. Theorem 4.4 was originally posed as a problem in [1] (the 20th problem of the 3rd chapter), but its solution and the rest of chapters four and five (beyond the basic definitions necessary to understand the statement of Theorem 4.4) are works of our own.

## 2. Preliminaries

This section mostly follows Chapter 1 of [1], though its two final definitions draw slightly from Chapter 2.

**Definition 2.1** (Combinatorial Games)**.** A *combinatorial game* is a game played between two players, Auden, and Daniil. Such a game must have

(1) A set of positions, or states of the game.
(2) A move rule, which determines, for each position, to which positions Auden and Daniil can move.
(3) A win rule, which states which positions are terminal (allow no subsequent moves). Each terminal position also must have an outcome: either Auden wins, Danill wins, or there is a tie.

**Game 2.2** (Pick-Up-Bricks)**.** *Pick-Up-Bricks*, a simple combinatorial game, consists of a pile of bricks of a chosen size. Each player takes turns taking either one or two bricks from the pile. The last player to take a brick wins.

**Definition 2.3** (Game Trees)**.** A *game tree* is a diagram showing, for a position and a player, to which trees or terminal positions that player can move.

**Example 2.4** (Pick-Up-Bricks example)**.** Here is the game tree of a game of Pick-Up-Bricks with 5 bricks.

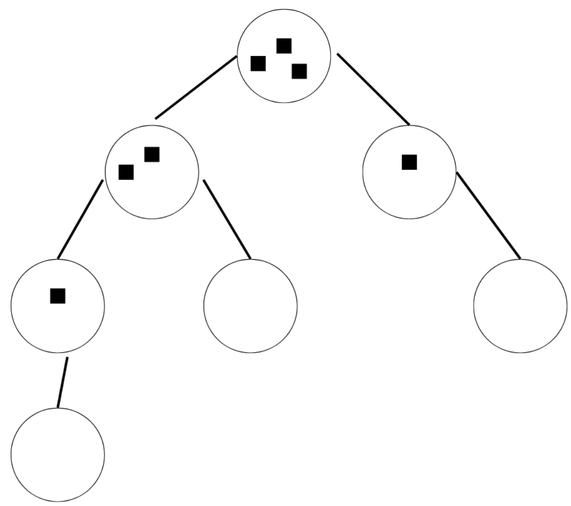

Figure 1. Game tree for a game of Pick-Up-Bricks with 3 bricks

**Theorem 2.5** (Zermelo's)**.** *For every position in any combinatorial game, one and only one of the following cases is true for each player:*

1. *They have a strategy that guarantees their win (a "winning strategy").*
2. *The other player has a strategy that guarantees their win.*
3. *They have a strategy that guarantees a tie or a win (a "tying strategy"), and so does the other player.*

*Proof.* Without loss of generality, assume that Daniil is the next player to move. Suppose inductively that Zermelo's theorem is true for each position to which Daniil could move ("subsequent positions"), noting that Auden will be the player moving from the subsequent positions. Therefore, there either exists a subsequent position for which Daniil has a winning strategy, or there does not. In the case that he doesn't, there either exists a subsequent position for which Daniil has a tying strategy, or there does not. These three possibilities will be classified into one of the above cases in the paragraphs below.

If Daniil has a winning strategy moving from one of the subsequent positions, then he can move to that position and thus has a winning strategy moving from his current position. This falls under case 1 of the previous list.

Similarly, if Daniil has a tying strategy moving from one of the subsequent positions, but not a winning strategy, then he has a tying strategy from the current position. If none of the subsequent positions give Daniil a winning strategy, then by our inductive assumption, Auden must either have a winning or tying strategy from all of them, meaning that regardless of the position that Daniil moves to, Auden has a winning or tying strategy. Since Daniil has a tying strategy, Auden cannot have a winning strategy, so Auden must have a tying strategy (because it has been established that Auden has either a winning or a tying strategy). Since both players have tying strategies, case 3 of the previous list is true for Daniil.

If none of the subsequent positions allow Daniil a winning or tying strategy, then by our inductive assumption, Auden has a winning strategy from all of the subsequent positions (so Daniil can only move to positions that give Auden a winning strategy), and thus Auden has a winning strategy from the current position, which is concurrent with case 2 of the previous list.

Since all possible combinations of future positions fall under one of the 3 aforementioned cases, Zermelo's theorem is true for a position if it is true for all positions that could result from it, confirming our inductive hypothesis.

By the definition of a combinatorial game, each terminal position must either result in a tie or a win for Auden or Daniil, so Zermelo's theorem is true for terminal positions. Since all combinatorial games must end in terminal positions, terminal positions serve as a base case for our inductive hypothesis, and therefore Zermelo's theorem is true. □

**Definition 2.6** (Normal-Play Games)**.** A *normal-play game* is a combinatorial game in which the last player to move wins. Note that this prevents ties. For example, Pick-Up-Bricks is normal-play, as the last player to take a brick wins.

**Definition 2.7** (Impartial Games)**.** An *impartial game* is a normal-play game wherein, for each position, the moves available to Auden are the same as those available to Daniil. Chess is not an impartial game because the players can have different possible moves. Pick-Up-Bricks, on the other hand, is an impartial game, as either player can take either one or two bricks if permitted by the size of the pile.

## 3. The MEX Principle and Its Applications

This section mostly follows Chapter 3 of [1], but its definitions and theorems about positions, their summation, and their equivalence rely upon results in Chapter 2.

**Definition 3.1** (Positions and Their Types)**.** A *position* in an impartial game is of *type P* if the last player to have moved has a winning strategy, and of *type N* if the next player to move has a winning strategy. Note that all impartial games are either type N or type P: by Zermelo's theorem, either Auden or Daniil must have a winning strategy, as no ties can occur since impartial games are normal-play. In an impartial game, the moves available to each player are the same, so the winning player can be identified by whether they just moved or are about to move (since moves alternate).

**Procedure 3.2** (Summation of Positions)**.** A *sum* of two positions $\alpha$ and $\beta$ is a new game made of two components, both of which are not necessarily from the same game. On their next move, a player can pick one component and take a move in it. For example, the next player in the position $\alpha + \beta$ can choose to move in the $\beta$ component and bring it to $\beta'$, resulting in a new position $\alpha + \beta'$. For example, if we sum two one-brick Pick-Up-Bricks positions, the resultant position is type P, because the first player can only make a single move in one component (taking one brick), and the following player can take the single move (taking one brick) in the other component to win the game.

**Definition 3.3** (Equivalence)**.** We define two positions $\alpha$ and $\beta$ to be equivalent when:

$$\alpha + \mu \text{ has the same type as } \beta + \mu$$

for all positions $\mu$. We mark two positions equivalent with the symbol $\equiv$.

Note that when two positions $\alpha$ and $\alpha'$ are equivalent, they have the same type, because we can add a position $\gamma$ which has no moves to both (keeping the positions effectively the same), and by the definition of equivalence, they will have the same type.

**Theorem 3.4.** *Summing a Position to Itself Results in a Type P Position.*

*Proof.* When one sums a position $\alpha$ to itself, a new position $\alpha + \alpha$ is created. Suppose the first player moves in one component and brings the position to $\alpha' + \alpha$. Then the optimal move for the second player is to move in the $\alpha$ component with the same move as the first player to bring the position to $\alpha' + \alpha'$. The second player can continue "mirroring" the first player in the opposite component until the first player takes the last move in one component. Then, the second player will take the last move in the other component and win. □

**Theorem 3.5.** *Adding a Position of Type P to Another Position Does not Change Its Type.*

*Proof.* Suppose that we have a position $\alpha$ (of any type) and a position $\beta$ of type P. Then, we can create cases based on the type of $\alpha$. First, we can approach the case where $\alpha$ is type P, wherein the winning strategy for the second player is to respond to the first player by moving in the same component, because the second player has a winning strategy going second in both components (and they will make the last move in both). Since the second player has a winning strategy, $\alpha + \beta$ is type P.

Approaching the case where $\alpha$ is type $N$, we want to prove that $\alpha + \beta$ is type N, or that the first player has a winning strategy. Suppose that Auden goes first: the optimal move for Auden is to move in the $\alpha$ component as it is type N. Then there are two type $P$ positions, the sum of which we know is type $P$ by the previous paragraph, so the type remains unchanged. □

**Corollary 3.6.** *A Position is Equivalent to itself Plus a Type P Position.*

*Proof.* We want to prove that if $\beta$ is type P, and $\alpha$ is any position, then $\alpha + \beta \equiv \alpha$. Previously, we proved that adding a position to a type P position preserves its type. Thus for any position $\mu$, the positions:

$$(\alpha + \mu) + \beta \text{ and } \alpha + \mu \text{ have the same type.}$$

Since for any position $\mu$, adding $\alpha + \beta$ and $\alpha$ has the same effect on the type, $\alpha + \beta \equiv \alpha$. □

**Corollary 3.7.** *All Type P Positions Are Equivalent.*

*Proof.* Suppose we have the positions $\alpha$ and $\beta$, both of which are type P. If we add a position $\mu$ then:

$$\alpha + \mu \equiv \mu \equiv \beta + \mu$$

by repeatedly applying the previous corollary. Since this holds for any position $\mu$, by the definition of equivalence, $\alpha$ and $\beta$ are equivalent. □

**Game 3.8** (Nim)**.** The game *Nim* consists of some number of piles which contain some number of sticks (the number of sticks does not have to be the same across piles). Players take turns taking up to all of the sticks from a single pile. The last player to take a stick wins, so the game is normal-play, and since the same moves are available to each player, it is also impartial.

**Procedure 3.9** (Subpile Division in Nim)**.** Recall that each positive integer has a unique binary expansion. For every pile in a game of Nim, we can assign that pile a binary expansion based on its size. For example, in a game of Nim with piles of 59, 78, and 45 sticks, the binary expansions for each of the piles are:

$$59 = 32 + 16 + 8 + 2 + 1 = 2^5 + 2^4 + 2^3 + 2^1 + 2^0$$
$$78 = 64 + 8 + 4 = 2^6 + 2^3 + 2^2$$
$$45 = 32 + 8 + 4 + 1 = 2^5 + 2^3 + 2^2 + 2^0$$

One can then *subdivide* a pile into various *subpiles*, each the size of a different term in the pile size's binary expansion.

**Definition 3.10** (Balanced Positions)**.** We define a position in Nim to be *balanced* when every power of two shows up an even number of times across the binary expansions of the size of all of the piles of sticks. Therefore, the position in Procedure 3.9 is unbalanced because there is only one power of $2^6$.

**Theorem 3.11.** *Balanced Positions in Nim Are Type P, and Unbalanced Positions are Type N.*

*Proof.* We want to prove that a balanced position in Nim is type P, and an unbalanced one is type N. To do this, we need to show that a player can always balance an unbalanced position and that a balanced position will always go to an unbalanced one. Repeating this process will eventually produce a position with no moves, which is balanced and type P because the next player to move has no moves available. This would mean that the previous move (unbalanced) was type N, and the move before (balanced) was type P, etc.

First, we want to prove that a balanced position always goes to an unbalanced one. In a balanced position, every pile can have either 1 or 0 of a subpile corresponding to a power of 2 due to the structure of binary expansions. Thus, when a player chooses from which pile sticks will be removed, they will always remove exactly one instance at least one power of two, making the whole position unbalanced (as some power of two went from an odd number to an even number of instances across piles).

To prove that an unbalanced position can always be balanced, we assume that the largest power of 2 that appears an odd number of times across piles is $2^n$. The next player can imagine picking up (not actually removing) $2^n$ sticks from a pile with $2^n$ in its binary expansion. Then, for all powers of 2 which occur an odd number of times across the binary expansions of all piles, the player can imagine "putting back" sticks numbering that power of two. Then the player can actually take (as their real move) the sticks that they imagined picking up but did not imagine putting back. This removes one pile of size $2^n$ (ensuring that there is an even number of subpiles of size $2^n$) and "puts back" one pile for all other powers of two that occur an odd number of times across the binary expansions of all piles, ensuring that all powers of 2 occur an even number of times. This process will always work, because the player has to put back at most every single power of two less than $2^n$, or:

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^{n-3} + 2^{n-2} + 2^{n-1}$$

which is less than $2^n$, as we can prove that:

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^{n-3} + 2^{n-2} + 2^{n-1} = 2^n - 1$$

because we can write

$$2^0 + 2^1 + \cdots + 2^{n-1} = 2(2^0 + 2^1 + \cdots + 2^{n-1}) - (2^0 + 2^1 + \cdots + 2^{n-1})$$

and

$$2(2^0 + 2^1 + \cdots + 2^{n-1}) - (2^0 + 2^1 + \cdots + 2^{n-1}) = 2^n - 1$$

as in the above equation, every power of two is in the parenthetical expression on the left except for $2^0 = 1$, and every term except $2^n$ is between the right set of parentheses. □

**Definition 3.12** (Nimbers)**.** We define a *nimber* as a single pile in Nim, denoted with $*$, so the nimber $*3$ represents a single pile with 3 sticks in Nim. We have proven that balanced positions in Nim are type P, so they are equivalent to $*0$ or a pile with 0 sticks in Nim. Unbalanced positions, on the other hand, are equivalent to nonnegative nimbers.

**Definition 3.13** (Nim Sums)**.** We define the *nim sum* of a position in Nim to be the sum of powers of 2 which appear an odd number of times in the binary expansions of the piles in a position, written as $a_0 \oplus a_1 \oplus \cdots \oplus a_{n-1} \oplus a^n$ where $a_0 \ldots a_n$ refers to the sizes of the piles (if there are $n$ piles in the position). Thus, the nim-sum of the position $*59 + *78 + *45$ is:

$$\begin{aligned} 59 \oplus 78 \oplus 45 &= (32 + 16 + 8 + 2 + 1) \oplus (64 + 8 + 4) \oplus (32 + 8 + 4 + 1) \\ &= 2 + 8 + 16 + 64 \\ &= 90 \end{aligned}$$

because there are an odd number of 2, 8, 16, and 64 in the binary expansions of the subpiles.

**Theorem 3.14.** *For a position in Nim we can prove that:*

$$*a_1 + *a_2 + *a_3 + \cdots + *a_n \equiv a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_n$$

*Proof.* If we represent the nim sum of the position with $*b$, then we know that by the definition of nim sum the position:

$$*a_1 + *a_2 + *a_3 + \cdots * a_n + *b \equiv *0$$

because the position is balanced. We proved that adding a position to itself is type P, so the position:

$$*a_1 + *a_2 + *a_3 + \cdots * a_n + *b + *b \equiv a_1 + *a_2 + + \cdots * a_n + *0 \equiv a_1 + *a_2 + \cdots + *a_n$$

And we can show that:

$$*a_1 + *a_2 + *a_3 + \cdots * a_n + *b + *b \equiv *0 + *b \equiv *b$$

And by combining the results of the two above equations:

$$*a_1 + *a_2 + *a_3 + \cdots * a_n \equiv *b$$

□

**Definition 3.15** (MEX)**.** The *MEX* (Minimal EXcluded Value) of a set is the smallest nonnegative integer not included in the set. Thus MEX($\{0, 1, 2, 3\}$) = 4.

**Theorem 3.16** (MEX Principle)**.** *The MEX principle states that, for a position* $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *in an impartial game where* $\alpha_i \equiv *a_i$ *for* $1 \le i \le n$*,* $\alpha \equiv *b$ *where* $*b$ *is the MEX of* $\{a_1, a_2, \ldots, a_n\}$*.*

*Proof.* We wish to show that $\alpha + *b \equiv *0$, because then we can add $*b$ to both sides to get $\alpha + *b + *b \equiv *b$, and the sum of a position and itself is type P and equivalent to $*0$, reducing our congruence to $\alpha \equiv *b$.

If the first player moves to component $\alpha_j$ from $\alpha$, the position reduces to $*a_j + *b$ (as $\alpha_j \equiv a_j$). By the definition of MEX and the theorem statement, $a_j \neq b$. Thus, using a nim sum, we can reduce this position to $*(a_j \oplus b)$, which is necessarily positive (as $a_j \neq b$, so they must have different terms in their binary expansions), giving a winning strategy to the next player to move, which is the second player. Therefore, if the first player moves from component $\alpha$, $\alpha + *b$ is type P and $\alpha + *b \equiv 0$.

If the first player moves from component $*b$, then the resulting position will be $\alpha + *a_j$ for some $a_j < b$ (because any move in nim requires taking at least one stick), as $b = \text{MEX}(\{a_1, a_2, \ldots, a_n\})$, so all numbers less than $b$ are in $\{a_1, a_2, \ldots, a_n\}$. $\alpha$ can be moved by the second player to $\alpha_j$. The position would then be $\alpha_j + a_j$, but $\alpha_j \equiv *a_j$, and the sum of two equivalent positions is a position of type P, so the second player has a winning strategy. This means that $\alpha + *b$ is also type P if the first player moves in $*b$. Since this is also true if the first player moves in $\alpha$, $\alpha \equiv *b$. □

**Corollary 3.17** (Sprague-Grundy Theorem)**.** *Every position in an impartial game is equivalent to a nimber.*

*Proof.* To prove this we can introduce a new concept called *depth*, the largest possible number of moves to get from a given position to a terminal position. For example, the depth of a Pick-Up-Bricks game with 1 brick is just 1, as there is a maximum of one move that can be made. We can induct on the depth of positions in impartial games: for our base case, the depth is 0 (and the position is terminal), so $\alpha \equiv *0$. Suppose that the Sprague-Grundy theorem is true for a position with depth $k - 1$. Then for a position with depth $k$, the position $\alpha$ can be written as $\{a_1, a_2, \ldots a_n\}$, where every single one of the possible future positions has a smaller depth, by the definition of depth. We can apply the MEX principle on $\{a_1, a_2, \ldots a_n\}$ to reduce it to a single nimber, confirming our inductive hypothesis. □

## 4. $S$-Pick-Up-Bricks

Chapter 3 of [1] provides the first several definitions in this section, and also poses Theorem 4.4 as a problem. Its solution and further results are our own.

**Game 4.1** ($S$-Pick-Up-Bricks)**.** The game *S-Pick-Up-Bricks* is similar to normal Pick-Up-Bricks, but in their turn, a player can remove only the number of bricks such that that number is in the set $S$. For example, a position with 2 bricks in a game of S-Pick-Up-Bricks where $S = \{3, 4\}$ is type $P$, because the next player cannot make a move, as they can only remove either 3 or 4 bricks. In other papers, S-Pick-Up-Bricks is often referred to as a subtraction game.

**Definition 4.2** (Nimber Sequence)**.** The *nimber sequence* of a game of $S$-Pick-Up-Bricks is the sequence of nimbers (with the starting term being *term 0*, the following term being *term 1*, etc.) equivalent to a position of bricks in $S$-Pick-Up-Bricks numbering that of the term. For example, the nimber sequence of normal Pick-Up-Bricks ($\{1, 2\}$-Pick-Up-Bricks) is $(*0, *1, *2, *0, *1, *2 \ldots)$ by the MEX principle.

**Lemma 4.3** (Determining the Repeating Portion of a Nimber Sequence)**.** *In a game of $S$-Pick-Up-Bricks, let the largest number in $S$ be $l$. If any $l$ nimbers (at terms $k$, $k+1$ $\ldots k+l-1$, $k+l$) are the same as any $l$ nimbers occuring later in the nimber sequence (at terms $k+p$, $k+p+1$ $\ldots k+l+p-1$, $k+l+p$), the sequence repeats every $p$ nimbers.*

*Proof.* If the largest number in $S$ is $l$, a nimber is only based on the previous $l$ nimbers by the MEX principle. Therefore, the nimber sequence will generate from $k+l+p+1$ the same way that it did from $k+l+1$, implying the existence of another period after $p$ nimbers.

However, the period could in fact be smaller than $p$ (a number that divides $p$), so it should be determined for all integers dividing $p$ (WLOG $q$) if, when partitioned into sections of size $q$, all such sections are equal (the smallest $q$ for which this is true is the true period of the nimber sequence). □

**Theorem 4.4.** *The nimber sequence of every game of $S$-Pick-Up-Bricks is eventually periodic.*

*Proof.* We can prove that every game of $S$-Pick-Up-Bricks' nimber sequence has an eventual period. Let's say that $S = \{a_1, \ldots, a_n\}$, $a_1 < \cdots < a_n$. When applying the MEX principle to determine the nimber for the $k$th term in the nimber sequence, or $N_S(k)$, we only look at the nimbers of previous terms in the nimber sequence. The MEX principle (which is assumed to treat nimbers the same as numbers) states that

$$N_S(k) = \text{MEX}(\{N_S(k-a_1), \ldots, N_S(k-a_n)\})$$

In particular, we only care about the window $[k-a_n, k-a_1]$, which is an interval of length $a_n$, as any possible move can remove a minimum of $a_1$ bricks and a maximum of $a_n$ bricks.

By Lemma 4.3, whenever we see two repeated intervals of length $a_n$, we know the sequence has become periodic. The nimbers in the sequence are all at least 0 and at most $|S| = n$, as when given $n$ inputs, the MEX function will output at most $n$ due to "gaps" between numbers in its input set. This means that the number of different intervals of length $a_n$ is $(n+1)^{a_n}$. The first $a_n + a_n(n+1)^{a_n}$ elements of the sequence can be split into $(n+1)^{a_n}+1$ non-overlapping intervals of length $a_n$. The pigeonhole principle implies that we have two identical intervals (as there are $(n+1)^{a_n}$ unique intervals). The distance between two identical intervals is at most $a_n(n+1)^{a_n}$, as their length is $a_n$. This means that the maximum period of a nimber sequence is $a_n(n+1)^{a_n}$. In other words, there exists $p, M \leq a_n(n+1)^{a_n}$ such that $N_S(x+p) = N_S(x)$ for all $x \geq M$. □

**Definition 4.5** (Immediately Periodic)**.** A nimber sequence is *immediately periodic* if the $n$ infinitely repeating nimbers of the sequence are the first $n$ nimbers. Note that not all nimber sequences are immediately periodic: When $S = \{2, 4, 7\}$, the repeating portion begins at the 8th term.

**Theorem 4.6** (Period of Sets of Size 1)**.** *If $S$ only contains one positive integer $n$, then the period of $S$ is $2n$, and the repeating portion is $n$ zeroes immediately followed by $n$ ones.*

*Proof.* The smallest amount of bricks that can be subtracted from the pile is $n$, so the first $n$ nimbers in the sequence (0 through $n-1$ bricks) must be terminal positions and therefore equivalent to $*0$.

In a nimber sequence, whenever $n$ nimbers in a row are zeroes, the next $n$ nimbers (at terms $p, \dots, p+n-1$) must be ones, as each of their nimbers are equal to $*\text{MEX}(N_S(k-n)) = *\text{MEX}(0)$ for $p-1 < k < p+n$.

Whenever $n$ nimbers in a row are ones, the next $n$ (at terms $p, \dots, p+n-1$) must be zeroes, as each of their nimbers are equal to $*\text{MEX}(N_S(k-n)) = *\text{MEX}(0)$ for $p-1 < k < p+n$.

Since the first $n$ nimbers are $*0$, the nimber sequence must therefore oscillate between a sequence of $n$ zeroes and a sequence of $n$ ones, with a period of $2n$. □

**Theorem 4.7** (Period of Sets of Size 2)**.** *Suppose $S$ consists only of two positive integers, $a$ and $b$. Then the period of $S$ is $a+b$, unless $b$ is an odd multiple of $a$, in which case the period is $2a$.*

*Proof.* Let $P_{a,b}(k)$ be the position of $k$ bricks in $\{a, b\}$-Pick-Up-Bricks. To show that $a, b$ has a period every $a+b$ nimbers, it can be shown that $P_{a,b}(k) \equiv P_{a,b}(k+a+b)$, as this implies a repetition of equivalent nimbers every $a+b$ nimbers in the nimber sequence. To show that $P_{a,b}(k) \equiv P_{a,b}(k+a+b)$, $P_{a,b}(k)+\gamma$ and $P_{a,b}(k+a+b)+\gamma$ must have the same type for any game $\gamma$. If one player (Daniil, WLOG) has a winning strategy in $P_{a,b}(k)+\gamma$, they can play according to that strategy in $P_{a,b}(k+a+b)+\gamma$, ignoring the fact that the pile of bricks has an extra $a+b$ bricks: when the opponent (Auden, WLOG) makes their first move in $P_{a,b}(k+a+b)$, Auden must subtract one of the two nimbers in $S$ ($a$, WLOG). Daniil can, in turn, subtract $b$, pretend that the past two moves did not happen, and continue playing according to his strategy in $P_{a,b}(k)+\gamma$ (it is possible that he made a move before Auden's first move, and in that case, he can just make his first move pretending as if the extra $a+b$ bricks were not in the pile). Therefore, Daniil still has a winning strategy, and the type of the position is maintained, so $P_{a,b}(k) \equiv P_{a,b}(k+a+b)$, and the nimber sequence must repeat every $a+b$ nimbers. Note that this does not mean the period is $a+b$, it only means that the period evenly divides $a+b$ (as there must occur a repetition every $a+b$, but there could be repetitions within this repetition).

Suppose WLOG that $b > a$. This means that until the $b$th term in the nimber sequence, $a$ is the only number that affects the nimber sequence. By the previous theorem, the nimber sequence will consist only of zeroes and ones until this point. Note that $b > \frac{a+b}{2}$, so if the true period of the nimber sequence is smaller than $a+b$, a repetition must start before the $b$th term (otherwise it would not evenly divide $a+b$). Therefore, if there is a nimber in the nimber sequence at or after the $b$th term that is not found in the first $b$ terms, the sequence's period cannot be smaller than $a+b$. Unless $b$ is an odd multiple of $a$, the number 1 must be contained between the $(b-a)$th and $(b-1)$th (inclusive) terms, as that region contains $a$ different nimbers, not all of which could be 0 unless $b-a$ was an even multiple of $a$ (due to the fact that before the $b$th term, the sequence consists of $a$ zeroes followed by $a$ ones over and over again), implying that $b$ is an odd multiple of $a$. This means that one of the $b$th through $(b+a-1)$th terms must be equivalent to the MEX of both a one (in between the $(b-a)$th and $(b-1)$th terms) and a zero (as $b$ terms before $b$ through $a+b-1$ would be terms 0 through $a-1$, which are zeroes), so one of them must be a two. There could not be a two before the $b$th term (as this was generated as if $b$ was not in the sequence), so there could not possibly be any periods smaller than $a+b$. Therefore the period of the nimber sequence of $S = \{a, b\}$ is $a+b$ if $b$ is not an odd multiple of $a$.

If $b$ is an odd multiple ($n$ times the size) of $a$, it can be shown that it is as if $b$ is not in $S$. Assume inductively, for a term in the sequence $k$, that all nimbers before $k$ could be generated with $S = a$. Since the period is $2a$, nimbers a multiple of $2a$ apart will be the same, so $N_S(k) = MEX(k - a, k - na) = MEX(k - a)$, since $n - 1$ is even and therefore $(n - 1)a$ is a multiple of $2a$. Since the first $na$ nimbers will be generated as if $b$ was not in $S$ (as $b = na$), the inductive hypothesis has a valid base case ($k = na$), so the inductive hypothesis will always hold, so it will be as if $b$ is not in $S$. Since it is as if $b$ is not in $S$, the period of $S$ is $2a$ if $b$ is an odd multiple of $a$. □

**Theorem 4.8** (Period of Sets $\{p, p+1, \ldots, q-1, q\}$)**.** *The nimber sequence for $\{p, p+1 \ldots q-1, q\}$-Pick-Up-Bricks has a period of $p + q$.*

*Proof.* Let $P_{p,p+1,\ldots,q-1,q}(k)$ be the position of $k$ bricks in $\{p, p+1 \ldots q-1, q\}$-Pick-Up-Bricks. Note that a player in $\{p, p+1 \ldots q-1, q\}$-Pick-Up-Bricks can cause $p+q$ bricks to be removed after 2 moves, as every number in $\{p, p+1 \ldots q-1, q\}$ has a corresponding number equal to the difference between it and $p+q$. As detailed in the first paragraph of the previous theorem, when $p+q$ bricks can be removed every two moves, $P_{p,p+1,\ldots,q-1,q}(k) \equiv P_{p,p+1,\ldots,q-1,q}(k+p+q)$ (the size of the $S$ doesn't matter, the logic of the first paragraph of the previous theorem only shows that the removal of $p + q$ bricks can be ignored without referencing the the size of $S$). This means that the nimber sequence repeats every $p + q$ nimbers, but it must also be shown that there is no repetition after fewer than $p + q$ nimbers.

The MEX of the $k$th term in the sequence, $p + q > k > p - 1$, is equal to MEX($\{N_S(k - q), N_S(k - (q - 1)), \ldots, N_S(k - (p + 1)), N_S(k - p)\}$). Note that $k - q$ must be less than $p$ since $p + q > k$, so $N_S(k - q)$ must equal 0, as the first $p$ nimbers are terminal positions. This means that no nimber in the nimber sequence after the first $p$ nimbers – before the $(k + q)$th nimber – can be equivalent to $*0$ (because $*0$ is part of the set that serves as an input to the MEX function). Since the first $p + q$ nimbers must repeat again, no repeated section can exist that excludes a 0, but since no zero occurs again until the $p + q + 1$th term, no period of size less than $p + q$ can exist. Since we know that the nimber sequences repeats after $p + q$ nimbers, and after no less than $p + q$ nimbers, the period must be $p + q$. □

**Theorem 4.9** (Sets Containing Odd Numbers' Nimber Sequences Have a Period of 2)**.** *If $S$ contains only odd numbers, than the nimber sequence for $S$-Pick-Up-Bricks is* $(0, 1)$ *infinitely repeating.*

*Proof.* The set has a period of 2 because all of the possible moves remove an odd number of bricks. A position with 0 bricks is equivalent to $*0$ and is type P. Since every possible move from a position with an even number of bricks results in a position with an odd number of bricks, and vice versa, positions with an even number of bricks are type P and equivalent to $*0$. Since positions with an odd number of bricks can only go to positions with an even number of bricks, their nimber is based on the MEX of only zeroes. Therefore all positions with an odd number of bricks are equivalent to $*1$. This means that the nimber sequence will alternate between $*0$ and $*1$ (starting with $*0$) infinitely. □

**Theorem 4.10.** *If $S$ is a set consisting of positive integers, and $S$-Pick-Up-Bricks' nimber sequence consists solely of a repeating portion that contains only distinct nimbers, then $S \cup X$ has the same nimber sequence if $X$ does not contain the period of $S$.*

*Proof.* Let the period of $S$ be $n$. Let $S = \{s_1, s_{2,p-1}, s_p\}$, and $X = \{x_1, x_2, \ldots, x_{q-1}, x_q\}$. The $k$th term in $S \cup X$-Pick-Up-Bricks is equivalent to MEX($N_S(k - s_1), N_S(k - s_2), \ldots N_S(k - s_{p-1}), N_S(k - s_p), N_S(k - x_1), N_S(k - x_2), \ldots N_S(k - x_{q-1}), N_S(k - x_q)$. Note that $k - x_j \neq k \pmod{n}$ for $0 < j < q+1$ by the theorem statement. Since every group of $n$ consecutive terms in $S$-Pick-Up-Bricks' nimber sequence does not contain the same number twice, $N_S(k) \neq K_s(k - x_j)$. The only for the $k$th term in $S \cup X$-Pick-Up-Bricks' nimber sequence to be different from the the $k$th term in $S$-Pick-Up-Bricks' nimber sequence is for one of the elements in the set serving as the input to the respective MEX to be equal to the $k$th term in $S$-Pick-Up-Bricks' nimber sequence. However, we have already shown that this will not occur, so the two nimber sequences must be the same. □

**Theorem 4.11** (There Exists an Infinite Number of Sets $S$ With Not Immediately Periodic Nimber Sequences)**.** *All sets $S = \{2, 4, 7, 7 + 3 \cdot z_1, 7 + 3 \cdot z_2, \ldots, 7 + 3 \cdot z_{n-1}, 7 + 3 \cdot z_n\}$, where $\{z_1, z_2, \ldots, z_{n-1}, z_n\}$ is a set containing nonnegative integers, are not immediately periodic. Since $n$ can be infinitely large, there exists an infinite number of sets $S$ with not immediately periodic nimber sequences.*

*Proof.* Note that, for two sets $A$ and $B$, MEX($A$) = MEX($A \cup B$) as long as $MEX(A) \notin B$. It must be shown that $S$ has the same nimber sequence regardless of $n$; in other words, it must be shown that, no matter how many multiples of three plus seven are added to $S$, its nimber sequence remains the same. Therefore, it must be shown that for the $p$th term in the nimber sequence of $S$, that the nimber at the $p - (7 + 3 * q)$th position in the sequence (if it exists), for all $q \in \mathbb{Z}$ and $q > 0$, is not equal to $p$.

Note that when $S = 2, 4, 7$, its nimber sequence is $(0, 0, 1, 1, 2, 2, 0, 3, 1, 0, 2, 1, 0, 2, 1, 0, 2 \ldots)$. The periodic portion of the sequence begins on the 8th term, and the smallest number that can be added to $S$ in accordance with the theorem statement is 10, so only $p > 9$ (or the repeating portion of the sequence) must be considered. When $p \geq 8$, observe that when $p \equiv 2 \pmod{3}$, the $p$th nimber in the sequence is 1, if $p \equiv 0 \pmod{3}$, the nimber is 0, and if $p \equiv 1 \pmod{3}$, the nimber is 2. It therefore must be proven that the nimber at the $p - (7 + 3 * q)$th position in the sequence must not be equal to 1 if $p \equiv 2 \pmod{3}$, 0 if $p \equiv 0 \pmod{3}$, and 2 if $p \equiv 1 \pmod{3}$.

If $p - (7 + 3 * q) \geq 8$, this is clearly true, as $p - (7 + 3 * q) \equiv p - 1 \pmod{3}$, so the nimber at the $p - (7 + 3 * q)$th term will not be the same as the nimber at the $p$th term (because the nimber sequence repeats every 3 terms starting at the 8th term).

To show that the same is true for $p - (7 + 3 * q) < 8$, we can show the following: based on each nimber's position (mod 3) in the repeating portion of the nimber sequence, the $p - (7 + 3 * q)$th term is not 1 when $p - (7 + 3 * q) \equiv 1 \pmod{3}$, not 0 when $p - (7 + 3 * q) \equiv 2 \pmod{3}$, and not 2 when $p - (7 + 3 * q) \equiv 0 \pmod{3}$. With simple casework, we observe that when $p - (7 + 3 * q) \equiv 1 \pmod{3}$ and $p < 8$, the $p - (7 + 3 * q)$th terms are zero, two, and three (so not one); when $p - (7 + 3 * q) \equiv 2 \pmod{3}$, the $p - (7 + 3 * q)$th terms are one, two, and one (so not zero); and when $p - (7 + 3 * q) \equiv 0 \pmod{3}$, the $p - (7 + 3 * q)$th terms are zero, one, and zero (so not two). Therefore, when $p - (7 + 3 * q) < 8$, the $p$th term is also not equal to the $p - (7 + 3 * q)$ term, proving this theorem.

□

## 5. Questions and Conjectures

**Conjecture 5.1.** *Let the small-augmentation of a set $S$ be $S$ but with its smallest number increased by one (if this results in two of the same number in the set, remove duplicates). If the largest number in $S$ is $l$, and if $S$ can be transformed into $\{l\}$ within $a \leq \lceil \frac{n}{2} \rceil$ small-augmentations, it will have a period of $2l - a$.*

**Question 5.2.** In $S$-Pick-Up-Bricks, is there a more efficient way to determine whether a sequence is not immediately periodic other than determining at what point the repeat occurs by observing all nimbers in the sequence until there is an evident repeat (see Lemma 4.3)?

**Question 5.3.** More generally, in $S$-Pick-Up-Bricks, is there a more efficient way to determine the period of a sequence than examining the possible moves with the MEX principle (see Lemma 4.3)?

**Acknowledgements.** This paper is a result of MIT PRIMES Circle, a program that enables local high school students to explore higher mathematics. We thank Mary Stelow and Marisa Gaetz, the directors of PRIMES Circle, for our acceptance into this program, the interesting choice of topic and book, and their helpful feedback on drafts of this paper. Also, we thank Honglin Zhu, our advisor, for helping us to understand the chapters of our textbook, challenging us with interesting problems (including that of the periodicity of $S$-Pick-Up-Bricks), and helping us write a strong paper.

## References

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