SURFACES

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Abstract. This paper explores the role of surfaces in knot theory, emphasizing their significance both within and beyond the field. We'll discuss the foundational concepts of knot theory, the importance of surfaces, and how they help in distinguishing knots. The paper also delves lightly into the construction and properties of Seifert surfaces, providing a comprehensive understanding of their applications.

1. Introduction

The purpose of this paper is to explore the concept of surfaces and their significance, both within knot theory and in broader mathematical and scientific contexts. Understanding what surfaces are—and why they matter—provides key insight into how knots behave in three-dimensional space.

Knot theory, a branch of topology, studies mathematical knots: closed, non-self-intersecting curves embedded in $\mathbb{R}^3$. These are not knots made of physical string but rather abstract loops that can be twisted, stretched, or bent—so long as they are not cut or allowed to pass through themselves.

Two knots are considered equivalent if one can be transformed into the other through a continuous deformation of space. These transformations are known as *ambient isotopies*, and they form the foundation for distinguishing different knots. Because knots can appear very different yet be topologically the same, a major question in knot theory arises: *How do we tell knots apart?*

To answer this, mathematicians have developed tools called **knot invariants**. These are properties—such as crossing number, tricolorability, and the Jones polynomial—that remain unchanged under ambient isotopies. Invariants allow us to classify knots and determine whether two knots are fundamentally different.

While knot theory is a deeply theoretical field, its implications stretch far beyond pure mathematics. It has important applications in biology, where DNA strands can knot and unknot themselves; in chemistry, where molecules can be entangled in complex ways; and in physics, especially in quantum field theory and the study of topological phases of matter.

As we will see, knots are not only confined to floating freely in space; they can also appear on or interact with surfaces. These surfaces can be familiar shapes like spheres and tori, or more complex ones like Möbius strips and Klein bottles. Exploring the nature of these surfaces—and their relation to knots—will help us better understand how knots function in the broader fabric of mathematical space.

An **ambient isotopy** of a knot is a deformation of the knot through three-dimensional space without allowing it to pass through itself. The term "ambient" emphasizes that the deformation occurs within the surrounding space ($\mathbb{R}^3$).

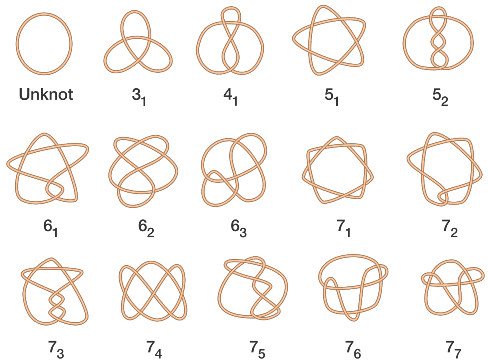

FIGURE 1. Different Knots

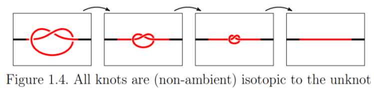

FIGURE 2. Example of Ambient Isotopy

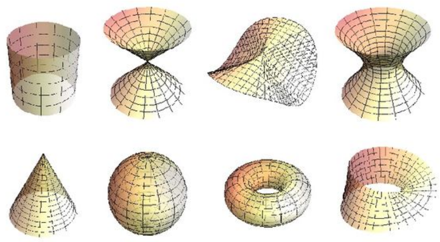

FIGURE 3. Different Surfaces

2. What is a Surface?

Considering the topic this paper is written on, it is important to figure out the exact meaning of what a surface is. In topology, a **surface** is a two-dimensional manifold embedded in three-dimensional space. That is, locally around any point, it resembles the Euclidean plane ($\mathbb{R}^2$), but globally, it can take on various shapes.

More generally, an n-**manifold** is a space where each point has a neighborhood homeomorphic to $\mathbb{R}^n$. For instance, zooming in on a point on a curve (a 1D object in $\mathbb{R}^3$) will make it appear locally like a straight line.

Surfaces are not limited to familiar objects like spheres or tori. They also include more exotic examples like the Möbius strip and Klein bottle. Adams describes these as "surfaces you can live on but not necessarily inside of."

Now that we know what surfaces are, we move on to the topic of link and knot components. When we are presented with the equation of $\mathbb{R}^3$ L, this means everything but the link or knot in the surface.

$\mathbb{R}^n$: The set of all n-tuples of real numbers. $\mathbb{R}^2$ represents the plane; $\mathbb{R}^3$ represents 3D space.

sectionFormalizing surfaces

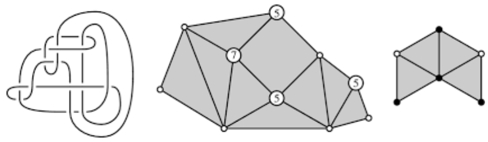

Figure 4. Figure of Triangulation

Triangulation refers to the process of dividing a surface into a set of non-overlapping triangles that completely cover the surface and fit edge to edge. This process is used to simplify the study of surfaces by breaking them down into simpler, more manageable triangular shapes.
Genus is the maximum number of closed curves that can be drawn on a surface without the manifold being disconnected.

The Euler Characteristic of a knot is a number associated with a knot that is invariant under continuous deformations of said knot; this definition of deforming a knot can also be classified as a Homeomorphism. χ =V-E+F
The genus is how to classify the Unknot from normal knots.
Boundary components are the surroundings that contain a disc in a surface, such as a sphere or torus.
Proof. Proof of $\chi = 2 - 2g - b$

- Let K1 and K2 be 2 knots, cut a little bit of each, and connect them using the pieces that were cut.
- Use the connect sum equation + induction
- Boundary components

□

3. Surfaces in Complements

In knot theory and the study of 3-manifolds, the concept of a surface within a knot complement plays a central role. When a knot is embedded in $\mathbb{R}^3$, we often examine the space that remains when the knot is removed—this space is known as the **knot complement**. Formally, for a knot K, its complement is $S^3 \setminus K$, or sometimes denoted $M(K)$, a 3-manifold with boundary.

A surface embedded in this complement allows us to probe the topology of the surrounding space. This becomes particularly powerful when using techniques such as **normal surface theory**, as introduced by Colin Adams. In a triangulated 3-manifold, normal surfaces intersect each tetrahedron in a standard way—through a finite collection of triangles and quadrilaterals. This simplification enables mathematicians to algorithmically search for and study surfaces in a knot complement.

Intuitively, a **complement** refers to "what's left" of the space after removing the knot. Surfaces within this complement can have different properties that help distinguish knots or understand their behavior. For instance, certain types of embedded surfaces can reveal whether a knot is fibered, determine its genus, or provide insight into its symmetry.

A critical classification of these embedded surfaces is based on whether they are *compressible* or *incompressible*. This distinction affects the rigidity and topology of the surface:

FIGURE 5. A knot complement using the Figure-8 knot

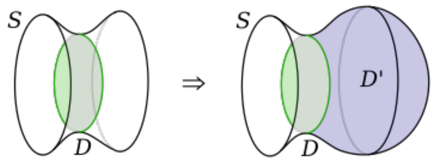

FIGURE 6. Compressing disk in an incompressible surface

Let S be a surface embedded in a 3-manifold M. Then S is called **compressible** if there exists a **compressing disk** D such that:

- $D \subset M$ (i.e., D lies entirely in the 3-manifold),
- The boundary ∂D lies on S,
- ∂D is essential in S (i.e., it does not bound a disk in S),
- The interior of D is disjoint from S.

If no such disk exists, the surface is called **incompressible**.

Incompressible surfaces are especially important because they preserve essential topological information about the manifold. These surfaces cannot be simplified further through compressions and often correspond to fundamental features of the knot.

For example, a Seifert surface—a surface whose boundary is a given knot—is often studied in the knot complement. When this surface is incompressible, it implies the knot has a minimal genus and allows for deeper classification via invariants like the Alexander polynomial.

To frame it geometrically, consider that $M(K)$ (the knot complement) is a 3-manifold with a boundary. This boundary is usually a torus, denoted $\partial N(K)$, since the neighborhood around a knot in S^3 resembles a thickened loop. When we examine surfaces embedded in $M(K)$, they intersect this torus boundary only along their own boundary curves.

Understanding which surfaces exist in the complement—and how they behave—helps to connect knot theory with other fields of topology, including the study of 3-manifold invariants.

Example: Seifert Surface of the Unknot. A disk bounded by the unknot in S^3 is a Seifert surface. This surface is **incompressible**, since any loop on the disk that bounds a disk in the 3-manifold already bounds one in the surface itself.

4. Seifert Surfaces

Referring to the brief mention of Seifert surface above, a Seifert surface for a knot K is a compact, connected, orientable surface S properly embedded in $S^3 \setminus \mathrm{int}(N(K))$, with boundary. They don't tend to be unique, meaning a knot or

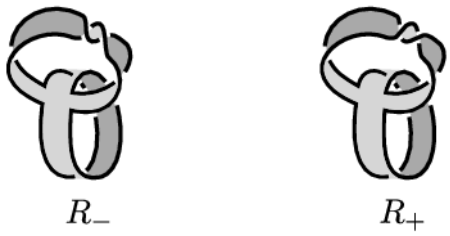

FIGURE 7. Two Seifert Surfaces for the Unknot

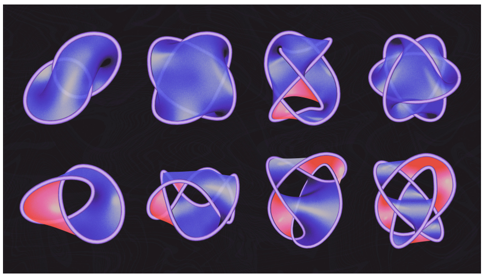

FIGURE 8. Seifert Surface

link can have many different Seifert surfaces. Such surfaces can be used to study the properties of the associated knot or link. For example, many knot invariants are most easily calculated using a Seifert surface. To construct a Seifert surface, you need to use Seifert's Algorithm. To start, assign an orientation to the knot. You then resolve the crossings, so for each crossing in a diagram of K, replace it with two arcs that do not intersect, resulting in a collection of disjoint simple closed curves known as **Seifert circles**. You connect these Seifert circles by attaching bands at the locations of the original crossings, ensuring that the orientation of the knot is preserved. This process yields an orientable surface whose boundary is the original knot. By constructing these surfaces, we're able to compute knot invariants such as genus.

REFERENCES

[1] Colin C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, American Mathematical Society, 2004.

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