Knot Theory: Genus

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MIT PRIMES Circle
May 2024

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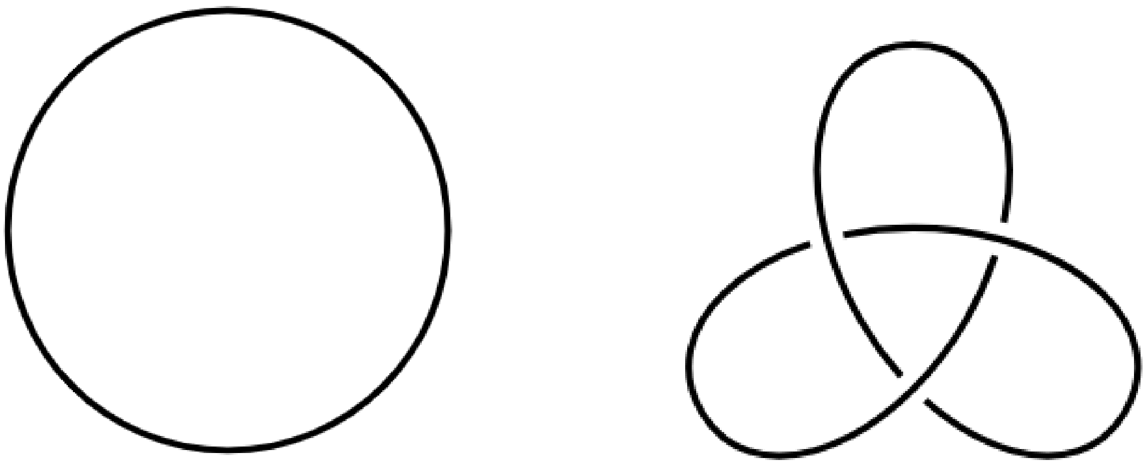
1 Introduction to Knots

1.1 Introduction

A knot is defined as a closed curve embedded in three-dimensional space or more simply, a string whose strands are twisted and crossed in any desired fashion and closed through connection. When drawing knots, the way in which strands cross each other are represented by creating a break in the strand that goes underneath (under strand) at where is crosses with the strand that goes above (over strand). Examples of knots include the unknot and the trefoil knot.

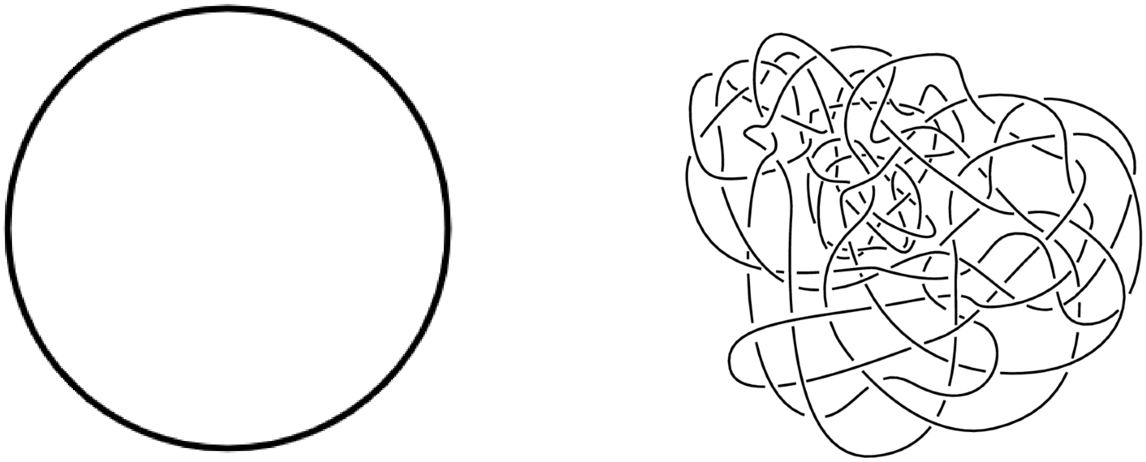


(a) Unknot (b) Trefoil Knot

Figure 1: Knot Examples

1.2 Knot Equivalence

Two knots are considered equal if one can be continuously deformed into the other without cutting or passing through itself. This means that if one knot can be transformed into the other through a series of continuous deformations (which include stretching, bending, twisting, and shrinking), then the two knots are considered equivalent or equal. Formally, if we have two knots $K1$ and $K2$ represented as smooth, simple closed curves in three-dimensional space, then $K1$ and $K2$ are considered equal if there exists a continuous deformation (an isotopy) from $K1$ to $K2$ such that at every point along the deformation, the curves remain smooth, simple, and closed, and there are no points where the curves intersect or pass through themselves. For example, the knots below are equal:



(a) Unknot

(b) Haken Gordian Knot

Figure 2: Knot Equivalence Examples

Any knot that is not equivalent to the unknot is a non-trivial knot. An important condition of knot equivalence is that two knots are equal if and only if they are related to each other through a finite number of Reidemeister moves.

1.3 Equivalence and Reidemeister moves

Reidemeister moves are local modifications to a knot diagram that preserve the knot's topology. There are three types of Reidemeister moves:

1. Reidemeister Type I: This move involves changing the over-crossing of one strand to an under-crossing or vice versa by introducing a new strand that passes over or under both affected strands. This move changes the relative positions of the crossings in the diagram. This move preserves the knot's topology as it does not change which parts of the knot are connected to each other. Essentially, the knot can be "smoothed out" locally to ignore the twist without affecting its global structure.

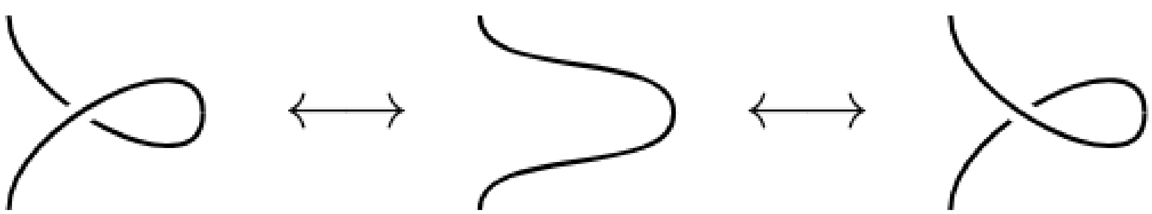


Figure 3: Reidemeister Move 1

2. Reidemeister Type II: This move involves sliding one strand underneath or over another to change the order of the crossings. This move can change the relationship between adjacent crossings. This move preserves the knot's topology as it does not change how the strands are connected in the larger structure of the knot. It simply changes the local arrangement of strands without affecting their endpoints or overall connectivity.

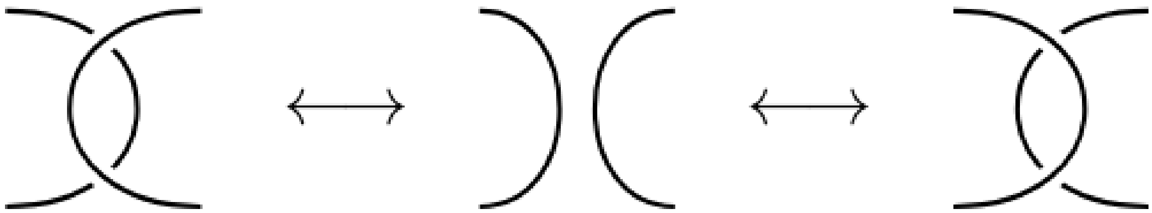


Figure 4: Reidemeister Move 2

3. Reidemeister Type III: This move involves passing one strand through a crossing to create two new crossings. This move can change the number and orientation of crossings in the diagram. This move preserves the knot's topology as it does not alter the sequence of connections in the knot. It merely adjusts the position of the crossings, keeping the same overall pattern of over and under crossings.

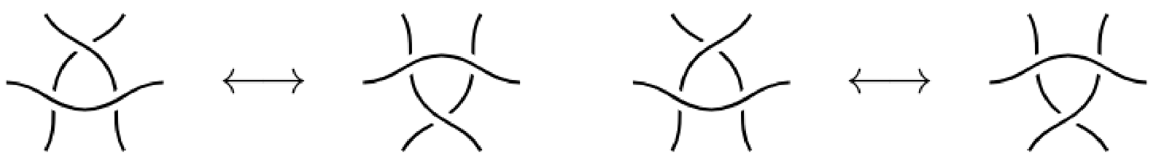


Figure 5: Reidemeister Move 3

In preserving a knot's topology, the Reidemeister Moves play an important role in identifying and defining properties of knots that are invariants.

1.4 Invariants

A invariant is a mathematical property or quantity associated with knots and links that remains unchanged under certain transformations or operations such as stretching, bending, and twisting. Invariants play a crucial role in distinguishing different knots and links from one another and in classifying them into different equivalence classes. Some common types of invariants in knot theory include:

1. Topological Invariants: these are properties of knots and links that are preserved under continuous deformations, such as stretching, bending, and twisting, but not under more drastic transformations like cutting and pasting. Examples include the number of components and the crossing number.

2. Algebraic Invariants: these are properties of knots and links that are derived from algebraic structures associated with their diagrams or representations. Examples include knot polynomials (such as the Alexander polynomial, Jones polynomial, and HOMFLY polynomial) and knot groups.

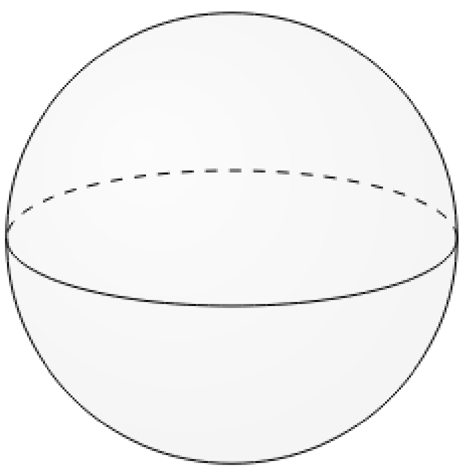
3. Geometric Invariants: these are properties of knots and links that are related to their embedding in three-dimensional space or in higher-dimensional spaces. Examples include the hyperbolic volume and the braid index.

One way to produce invariants for knots is by associating surfaces with them.

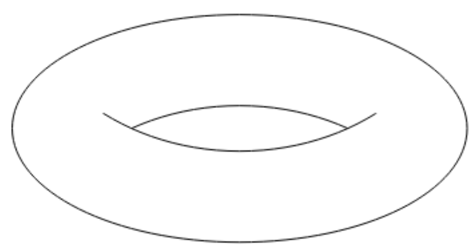
1.5 Surfaces and Genus

There are two main types of surfaces: surfaces without boundary and surfaces with boundary. An invariant that can be produced from surfaces is the genus. The rigorous definition of a genus is a non-negative integer equivalent to the maximum number of closed curves that can be drawn on a surface without the manifold being disconnected.

A surface without boundary is a two-dimensional manifold (surface) that is closed, meaning it has no boundary or edges. Formally, a closed surface is a compact, connected, and boundaryless two-dimensional manifold. More intuitively, a surface without boundary is a shape that is like a closed container, such as a sphere, torus, or any surface that doesn't have edges or openings. The genus of a surface without boundary is the number of holes on a surface without boundary.



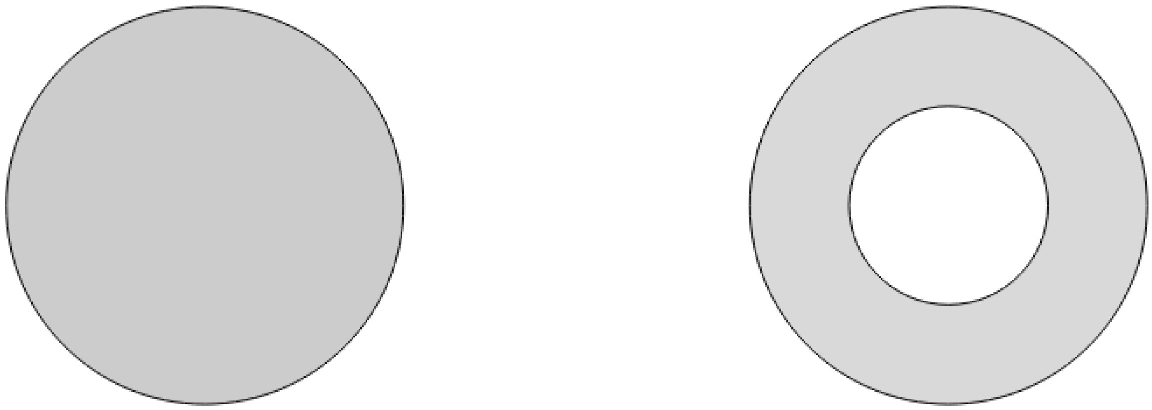
(a) Sphere Genus 0



(b) Torus Genus 1

Figure 6: Genus and Surfaces Without Boundary

A surface with boundary is a two-dimensional manifold (surface) that may have a boundary consisting of one or more disjoint simple closed curves. These curves are called the boundary components of the surface. Formally, a surface with boundary is a topological space that locally looks like the Euclidean plane ($\mathbb{R}^2$) near each point, but globally may have a more complicated structure, possibly with holes or handles. The boundary of the surface consists of those points that do not have neighborhoods homeomorphic to an open disk. The genus of a surface with boundary can be found through capping off.

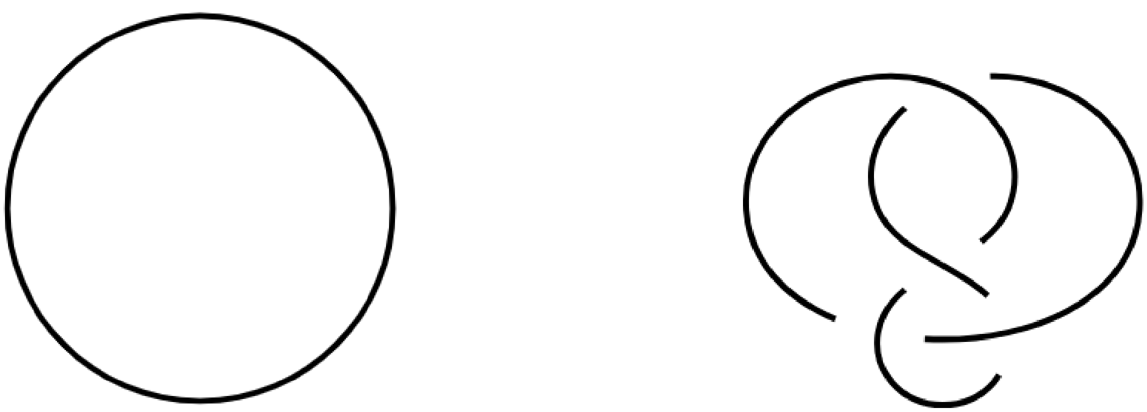


(a) Disk Genus 0

(b) Annulus Genus 1

Figure 7: Genus and Surfaces With Boundary

A Seifert Surface is a special type of surface with boundary associated with a knot or link in three-dimensional space. It is a smooth, orientable surface with one boundary component whose boundary is the given knot or link. Seifert surfaces were introduced by the German mathematician Herbert Seifert in the 1930s as part of his work on knot theory. The genus of a knot is the least genus of any Seifert surface for that knot.



(a) Unknot Genus 0

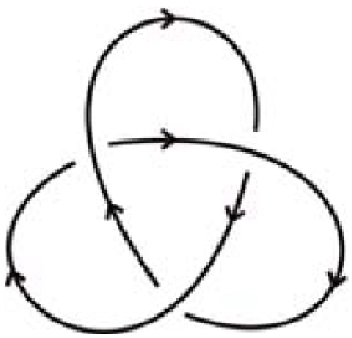
(b) Figure Eight Knot Genus 1

Figure 8: Genus, Knots and Seifert Surface

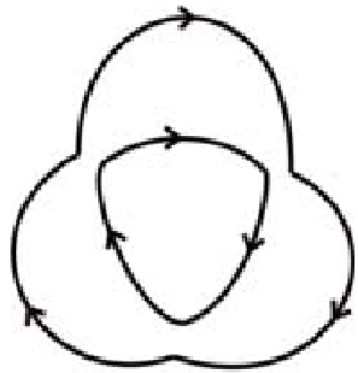
2 Knots to Seifert Surfaces

There are three main steps to convert knots into Seifert Surfaces

1. After assigning and orientation to the knot, eliminate all crossings in the knot diagram to create Seifert Circles



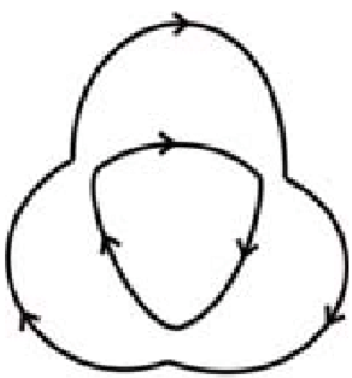
(a) Trefoil Knot with Orientation



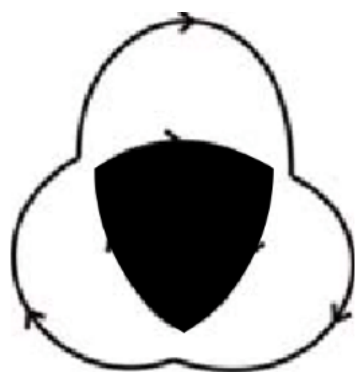
(b) Split Trefoil Knot: Seifert Circles

Figure 9: First Step

2. Place the circles at different heights so that each circle bounds a disk in the plane at different heights to prevent intersection



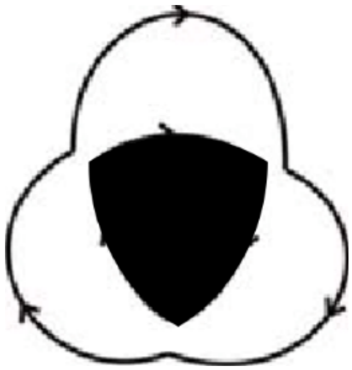
(a) Split Trefoil Knot: Seifert Circles



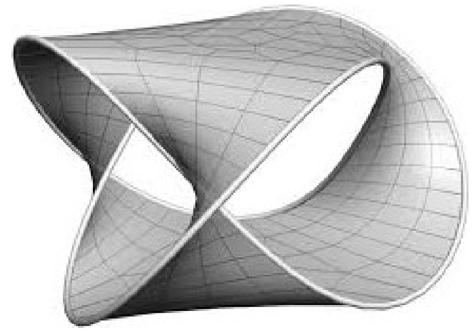
(b) Seifert Circles at Different Heights

Figure 10: Second Step

3. Connect disks to one another at crossing of knot by twisted bands. The result is a surface with one boundary component: boundary component is the knot



(a) Seifert Circles: Different Heights



(b) Trefoil Knot Seifert Surface

Figure 11: Third Step

3 Genus and Euler Characteristic

3.1 Introduction to Genus and Euler Characteristic

To further understand genus, it is useful to be familiar with another invariant of surfaces, known as the Euler characteristic. Finding the Euler characteristic of a surface involves triangulation, a process of drawing connected vertices and edges on a surface so that it is entirely split up into triangles. For a given triangulated surface, if V is the number of vertices, E is the number of edges, and F is the number of faces, then the Euler characteristic, X, is defined as

$$X = V - E + F$$

A triangulation of the torus , which has genus 1, shown below has 4 vertices, 12 edges, and 8 faces. Therefore, the Euler characteristic of a torus is

$$X(T) = 4 - 12 + 8 = 0$$

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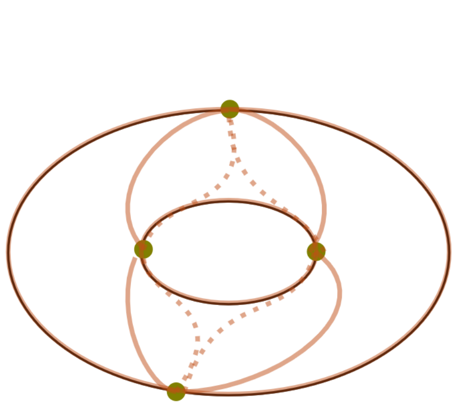
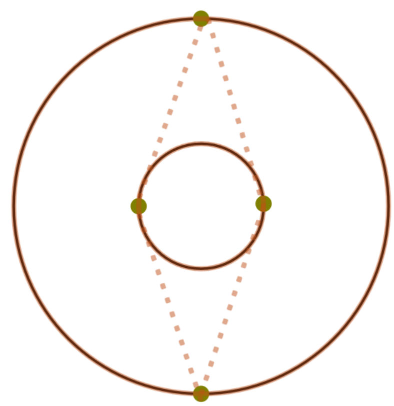


(a) Triangulation of Torus



(b) Triangulation of Torus Top View

Figure 12: Triangulation of a Torus

Note that different triangulations of the same surface yield the same characteristic.

3.2 Connected Sum

Instead of calculating Euler characteristic of surfaces of multiple genera by hand, we can utilize the connected sum. To perform this operation with two tori, remove a disk from each torus before gluing them together along these boundaries. If the tori are already triangulated, this is akin to removing the interior of one triangle from each torus such that when they are glued together, the vertices and edges of the empty triangles are glued together:

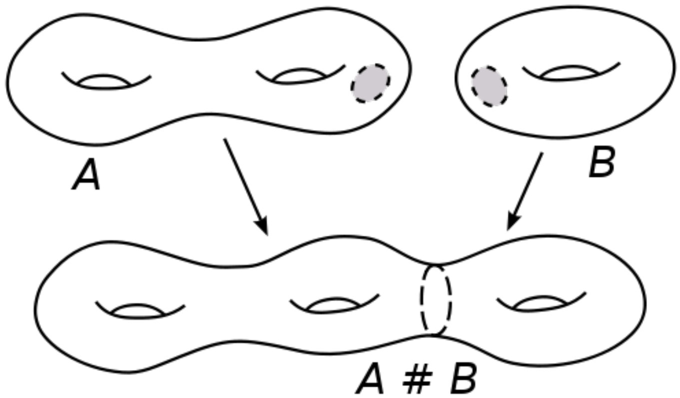


Figure 13: Connected Sum

This results in a surface of genus 2 with three fewer vertices, three fewer edges, and two fewer faces. Note that the three fewer vertices and three fewer edges cancel out according to the definition of the Euler characteristic. As a result, the Euler characteristic of the new surface is:

$$X(S) = X(T_1) + X(T_2) - 2.$$

This is true for the connected sum of any two surfaces without boundary. A new formula for the Euler characteristic of a surface without boundary can be proven using the connected sum, as shown below.

Statement: Let $P(g)$ be the proposition that the Euler characteristic X of a surface without boundary of genus g is $X(\sigma) = 2 - 2g$ for all positive integers.

Base Case: A Torus

$$V = 4$$

$$E = 12$$

$$F = 8$$

$$g = 1$$

$$X = V - E + F$$

$$\Rightarrow X = 4 - 12 + 8 = 0$$

$$X(\sigma) = 2 - 2g$$

$$\Rightarrow X(0) = 2 - 2(1) = 0$$

$$0 = 0$$

The base case holds.

Induction: Assume that $X(\sigma)$ holds for a surface of genus $g = k$. That is, for a surface S_k with genus k, $X(S_k) = 2 - 2k$. Consider a surface, $S_{(k+1)}$, of genus $g = k + 1$ that is obtained by adding a handle (torus) to S_k.

$$X(S_{(k+1)}) = X(S_k) + X(T) - 2$$

$$= 2 - 2k + 0 - 2$$
$$= 2 - 2k - 2$$
$$= 2 - 2(k+1)$$
$$\Rightarrow X(S_{(k+1)}) = 2 - 2(k+1)$$

Conclusion: If $X(\sigma)$ holds for $g = k$, it also holds for $g = k + 1$. Since the formula holds for the base case $(g = 0)$ and if it holds for $g = k$, it also holds for $g = k + 1$, by mathematical induction, the formula $X(\sigma) = 2 - 2g$ is true for all surfaces without boundary, where g is the genus and X is the Euler characteristic.

A similar formula for surfaces with boundaries can be derived from $X(\sigma) = 2 - 2g$. For every boundary component that is added to a surface, its genus decreases by one. This is intuitive from the rigorous definition of genus, as there is one less closed curve that can be drawn before disconnecting the surface. Therefore, a formula for the Euler characteristic of a surface with b boundaries in relation to its genus can be derived:

$$X(\sigma) = 2 - 2g - b$$

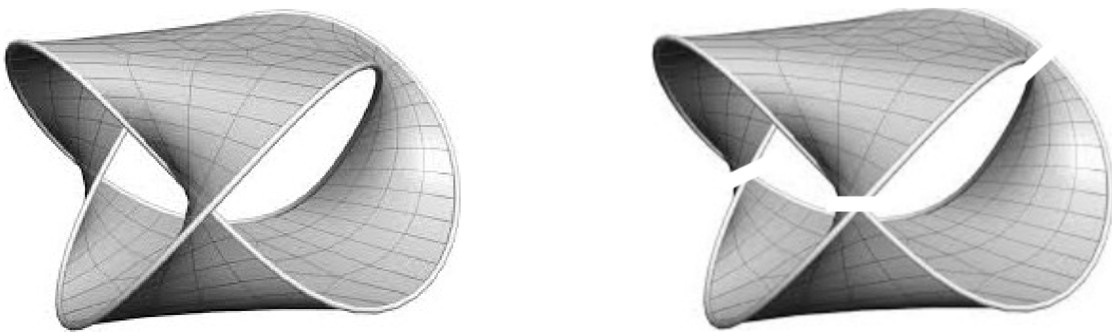
4 Genus and Seifert Surfaces

4.1 Euler Characteristic of a Seifert Surface

Relations can also be made between the Euler characteristic, genus, and Seifert surfaces. Considering the fact that a Seifert surface is a surface with one boundary component, the Euler characteristic of a Seifert surface can be calculated in relation to its genus through this formula:

$$X(\sigma) = 2 - 2g - 1 = 1 - 2g$$

Another formula can be derived for the Euler characteristic of a Seifert surface that does pertain to its genus. For instance, let S be a Seifert Surface constructed from a knot K, c be the number of crossings, and s be the number of Seifert circles. Now, suppose an edge and two vertices are added to the surface across each crossing, such that each band is cut in half. Notice that this divides the surface into valid faces as shown below.



(a) Image of Original Seifert Surface (b) Divided Seifert Surface

Figure 14: Divided Seifert Surfaces

Because there are two vertices inserted across each crossing, it is apparent that:

$$V = 2c$$

As for edges, there is one added across each crossing in addition to four emerging from each crossing. However, notice that two of those four aforementioned edges are shared by another crossing. To account for this:

$$E = c + (4 - 2)c = 3c$$

Finally, note that there is one face per Seifert surface:

$$F = s$$

The Euler characteristic of this Seifert surface may now be written as:

$$X = V - E + F = (2c) - (3c) + (s) = s - c$$

4.2 Formula for the Genus of a Seifert Surface

A formula for the genus of a Seifert surface in relation to the number of Seifert circles and the number of crossings of its corresponding knot can be derived from combining the equations for the Euler characteristic of a Seifert circle presented above:

$$X(\sigma) = 1 - 2g$$

$$X = s - c$$

$$\Rightarrow 1 - 2g = s - c$$

$$2g = 1 - s + c$$

$$g = \frac{1 - s + c}{2}$$

Therefore, the formula for the genus of a Seifert surface is: $g = \frac{1-s+c}{2}$. It is important to note that since the Seifert surface corresponds to a knot, this formula only applies to certain Seifert surfaces as different Seifert Surfaces of a knot can yield different genera. This is why computing the genus of a knot is not trivial. This formula can be utilized to deduce the properties of groups of knots, which in turn, can connect different groups.

5 Genus of Knots

5.1 Genus of Twist Knots

Twist knots are a specific group of knots that include the trefoil and figure eight knots:

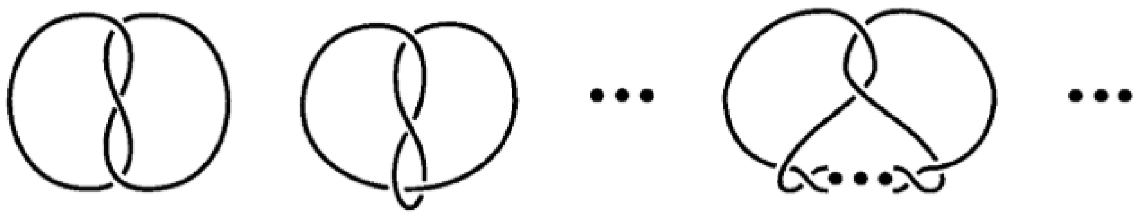


Figure 15: Twist Knots

A property of twist knots is for a twist knot with c crossings and s Seifert circles, $s = c - 1$. The proof for this property is:

Statement: For all twist knots, $s = c - 1$.

Base Case: Trefoil Knot

$$c = 3$$
$$s = 2$$
$$2 = 3 - 1$$
$$\Rightarrow s = c - 1$$

The base case holds.

Induction: Assume that the statement, $s = c - 1$, holds for $c = k$. That is, for a twist knot with k crossings $s = k - 1$. Consider a twist knot with $k + 1$ crossings that is obtained by adding a crossing.

It is intuitive to that every time a crossing is added to a twist knot, the number of Seifert circles increase by 1 as adding a crossing creates a new loop in the knot, which in turn requires a new Seifert circle to accommodate it in the Seifert surface. Therefore, the number of Seifert circles for a twist knot with $k + 1$ crossings is:

$$s = (k - 1) + 1$$
$$\Rightarrow s = (k + 1) - 1$$

Conclusion: If $s = c - 1$ holds for $c = k$, it also holds for $c = k + 1$. Since the property holds for the base case ($c = 3$) and if it holds for $c = k$, it also holds for $c = k + 1$, by mathematical induction, the property $s = c - 1$ is true for all twist knots where c is number crossings and s is the number of Seifert circles.

This property, when combined with the formula for the genus of certain Seifert surfaces in terms of c and s, can be applied to find a characteristic of twist knots that pertain to their genus:

$$s = c - 1$$
$$g = \frac{1 - s + c}{2}$$

$$g = \frac{1-(c-1)+c}{2}$$
$$g = \frac{1-c+1+c}{2}$$
$$g = \frac{2}{2}$$
$$g = 1$$

Therefore, a property of twist knots is that all twist knots have genus 1.

5.2 Genus of Prime Knots

Another group that knots can classified under is prime knots. Prime knots are non-trivial knots that cannot be expressed as the knot sum of two non-trivial knots (K_1 and K_2 are nontrivial knots):

$$K \neq K_1 \# K_2$$

On the other hand, composite knots are knots that can be expressed as the knot sum of two more nontrivial knots:

$$K = K_1 \# K_2$$

It is important to note that the unknot is neither a prime knot, nor a composite knot. A theorem for the composition of knots related to genus, Additivity of knot Genus, states:

$$g(K) = g(K_1 \# K_2) = g(K_1) + g(K_2)$$

This theorem can be used to deduce a characteristic of genus 1 knots that connects prime knots and twist knots: Assume knot K has genus 1:

$$g(K) = 1$$
$$\Rightarrow g(K_1 \# K_2) = 1$$
$$\Rightarrow g(K_1) + g(K_2) = 1$$

The only non-negative integers 1 is the sum of is 0 and 1.

Case 1:

$$g(K_1) = 0$$
$$g(K_2) = 1$$

Case 2:

$$g(K_1) = 1$$
$$g(K_2) = 0$$

In both Case 1 and Case 2, at least one knot must have a genus of 0. This implies one of the knots, K_1 or K_2, must be the unknot. However, K_1 and K_2 must be a non-trivial knot and the unknot is not a non-trivial knot. Therefore, a knot with genus 1 cannot be decomposed into two nontrivial knots. Hence, a knot with genus 1 must be a prime knot.

It has been proven that all genus 1 knots must be prime knots. This statement can be connected to a statement proven in the previous section: all twist knots have genus 1. Through combining these two statements, it can be deduced that all twist knots are prime knots, connecting these two groups of knots.

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