Game Theory

Tanesha Nixel
Mentor: Sam A Packman

April 2024

Contents

1 Introduction

Game Theory is an area of math that allows us to see games' strategies, patterns, and outcomes. Players can actively make decisions according to the goal of the game while also keeping in mind the moves of the other player. Game theory can be used in other fields like economics, finance, and business to help them understand and learn more about strategic reasoning.

Combinatorial, normal play, and impartial games will be discussed in this paper. Combinatorial games are the broad branch that includes many games. Using the games SOS, Tic, and Pick-Up-Bricks and analyzing their game trees, we will further explain combinatorial games. Normal-play games are a subcategory of combinatorial games. We will define and explain what normal-play games are using Cut-Cake and Domineering. A subcategory of normal-play games is impartial games; this will be explained using Nim and the MEX Principle.

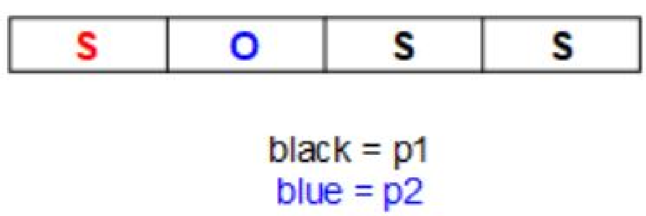
2 Combinatorial Games

Definition 1.0: [1] A combinatorial game is a 2-player game that includes:

1. A set of possible positions each player can move to.
2. A move rule so that each player can move to a specific position.
3. A win rule, indicating terminal or outcome positions.

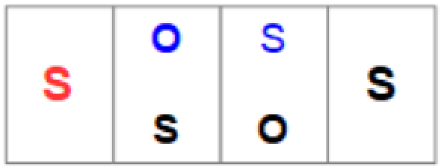
In this paper, the two players used to show combinatorial games will be Louise (L or left) and Richard (R or right). Either player is allowed to go first. Games like Checkers, Chess, Tic-Tac-Toe, etc. are all combinatorial games.

SOS:
This game is played by two players and each player can choose to place an S or an O on one of the blocks. The goal of the game is to spell out SOS. The starting position is with one S (in red) on the block furthest to the left.



In this case, player 1 can always win.

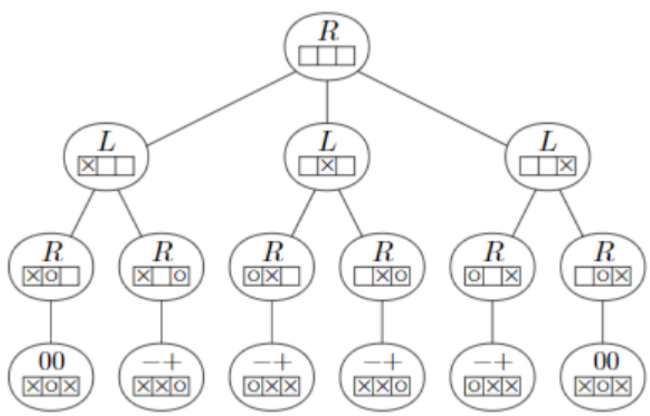
The winning strategy for player 1 in this example is to put an S in the right-most box:



This way no matter what the second player does the first player can win the game.

Game Trees:

Game Trees are used to show how a game can possibly play out. With this, we can see every possible way a player can move and the outcome of the game. The top node of the game tree, or the root node as it can also be called, shows how the game started. Each path from the root node to the bottom of the game tree represents a possible game. In the game Tic, the goal is to get two adjacent boxes with the player's symbol. Louise marks her boxes with an O and Richard marks his box with an X.



Theorem 1.1: Zermelo's Theorem: [1]
Every game tree is one of the winning types:

1. Louise has a winning strategy (+-)
2. Richard has a winning strategy (-+)
3. Both have drawing strategies (00)

Zermelo's Theorem can be proven using mathematical induction on game trees.

Mathematical Induction

This is used to find out what the outcome of a larger game would be and not have to play it, and all the possible combinations of it. More precisely, mathematical induction is used to prove a statement P(n) about a game of size n by proving two easier things.

Base Case: $P(0)$ is true.
Inductive Step: If $P(k)$ is true for all $k < n$ then $P(n)$ is true.

This means that we can use a base or small case and then use that to be able to prove a larger case.

Example:
We can use the mathematical induction with the game Pick-Up-Bricks. In the game Pick-Up-Bricks there is a pile of n bricks and each player can choose to take one or two bricks from the pile. The player to take the last bricks wins.

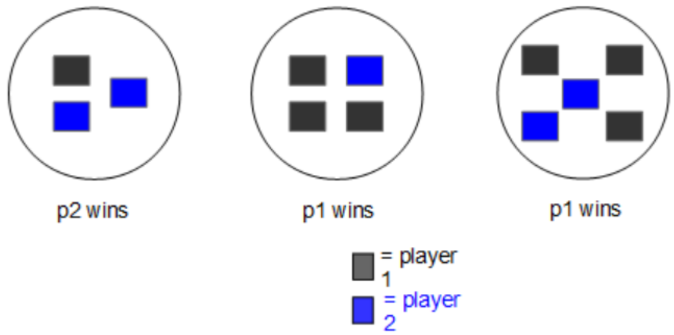
First, we can start with a base case where there is one brick.

In this game whoever starts will win the game.

We can then play a few more games and find out that for all of the games whose number of bricks is divisible by three the second player will win. If the number of bricks isn't divisible by 3, or any other number, the first player will win.

3 Normal Play Game

Definition: A normal-play game is a combinatorial game in which the win rule is that the last player to make a move wins.



There are two subcategories of Normal-Play games: Impartial, which we will cover later in this chapter, and Partizan.

Games that are normal-play games are Cut-Cake, Chop, and Chomp. These games have different types.

Types of Positions:

- Type L: This is when Louise has a winning strategy no matter who starts.
- Type R: This is when Richard has a winning strategy no matter who starts.
- Type N: This is when the next or first player has a winning strategy.
- Type P: This is when the second or previous player has a winning strategy.

Being able to determine what type of position a game is will help us know if there is a way to win as the first or second player. It is also important because it will determine who will win at the end of the game.

Proposition 1.0: If $\gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_m | \beta_1, \beta_2, ...\beta_n\}$, the type of γ is given in the chart below.

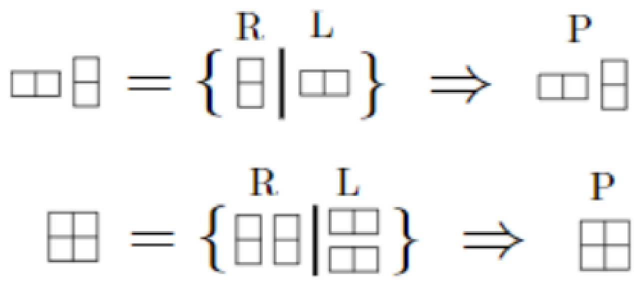
	Some of β_j is type R or P	All of β_1, ...β_n are types L or N
Some of α_1 is type L or P	N	L
All of α_1 … α_m are types R or N	R	P

Note that the α's are L's possible moves from γ, and the β's are R's possible moves from γ.

Cut-Cake

In this game, each player can make either a horizontal or vertical cut. Louise can only cut vertically while Richard can only cut vertically. The last player to make a move wins. We can find out the type of a Cut-Cake game by splitting it into this format:

$\gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_m | \beta_1, \beta_2, ...\beta_n\}$



Sum of Games

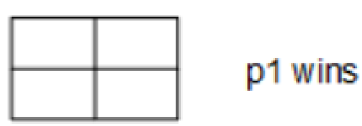
Definition: If α and β are positions in a normal play game, then $\alpha + \beta$ is a new position made up of α and β.

This means that the two games are combined and played as one game. A player can move from $\alpha + \beta$ to either $\alpha' + \beta$ or $\alpha + \beta'$. Each player can choose whether to play in the first game, α or in the second game, β.

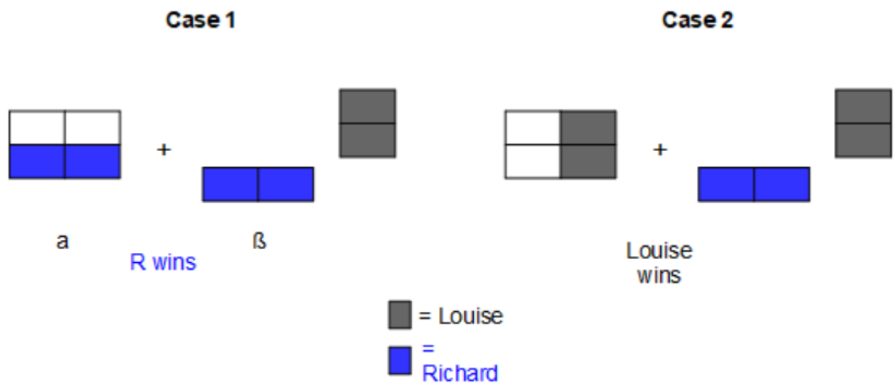
Proposition 1.1: If β is type P then α and $\alpha+\beta$ are the same types.

Example using Domineering:

This is a game where Louise can place a 2x1 domino over two unoccupied blocks and Richard can do the same but with a 1x2 domino. The last player to move will win.



In this game whoever starts will end up winning because they block the other player from being able to do anything. If we add another game, β, which is type P, α and $\alpha+\beta$ will be the same types.



This works because adding a type P game is like adding a zero, it doesn't affect the outcome of the games.

4 Impartial Games

Definition:

An Impartial Game is a game where both players have the same moves available. The only possible positions in an impartial game are Type N (the first player wins) or Type P (the second player wins). Pick-Up-Bricks is an example of an easier impartial game. We will explain and look at a more complicated impartial game, Nim.

Nim:

Similarly to Pick-Up-Bricks, Nim is a game that also has piles of stones. In Nim, there is more than one pile in a game. Each player can remove as many stones from a pile each turn. The last player to move wins.

In this game, the first player will win because the player can just take all of the stones from the pile. In any type of Nim game where there is only one pile, the first player will always win. When adding more piles the game becomes more complicated and there are many possibilities.

Binary expansion:

We can use binary expansion to later help us analyze and further understand the game of Nim. Binary expansion is when you expand numbers from its power of 2. There is only one way that each number can correctly be expanded, the way to find this is by starting with the largest power of two which goes into that number.

Example:

$45 = 32(2^5) + 13$
$= 32 + 8(2^3) + 5$
$= 32 + 8 + 4 + 1$

Balanced and Unbalanced Positions:

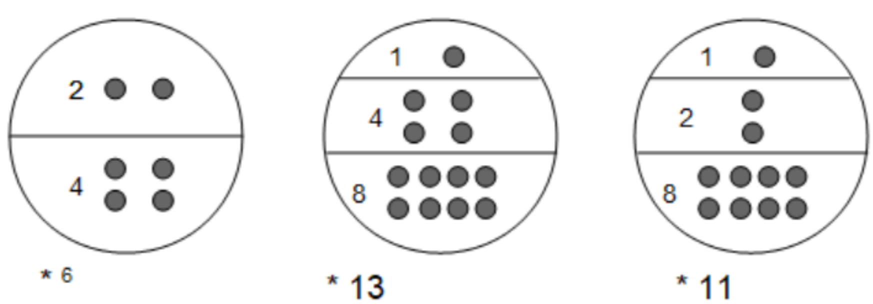
Proposition 1.3: Every balanced Nim position is type P and every unbalanced Nim position is type N. In both cases, the balancing procedure provides a winning strategy.

We can use the binary expansion of the numbers to find out if a game is balanced or unbalanced. For every pair of numbers in the binary expansion would get crossed out. If we have two piles, one containing 3 and the other containing 5, we would write this using the Nim sum:

$$3 \oplus 5 = (2 + \cancel{1}) \oplus (4 + \cancel{1})$$
$$= 2 \oplus 4$$
$$= 6$$

A game is considered balanced if there is nothing left after crossing these numbers out.

Example 1:

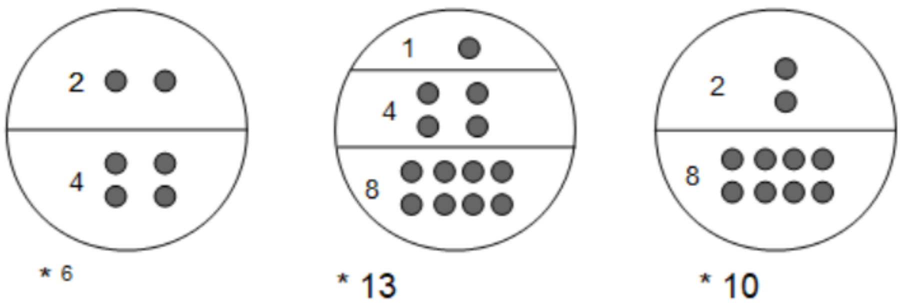


$$(2+4) \oplus (1+4+{}^*) \oplus (1+2+8)$$

In this example, the game is balanced because when using binary expansion we see that all of the numbers get crossed out. After all, they each have a pair in one of the other numbers. There is a pair of two, eight, four, and one. The winning strategy of the second player is to balance the game on each of their turns. For each balanced position, we see that the game will be a type P, meaning the second player will always win.

Example 2:

This game is unbalanced because not all of the numbers have a pair to go with, one does not have a pair while all the other numbers, two, four, and eight, all have pairs. Since this game is unbalanced, it is a type N game and the first player will always win by balancing the game on each of their turns.



$$(2+4) \oplus (1+4+8) \oplus (2+8)$$

References

[1] M. Devos and D. A. Kent (2016) *Game Theory: A Playful Introduction*, American Mathematical Society.