Introduction to Probability

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Abstract

In this paper, we will be talking about probability. First, we will talk about the basic principle of counting, including permutations, combinations, and the binomial theorem. Then we will talk about sample space, the axioms of probability, and conditional probability. Finally, we will talk about independent events.

1 Introduction

If you roll a six-sided die, what is the probability that you roll a 4? In this problem, the probability would be 1/6 because out of 6 possible outcomes, you are choosing 1. Probability is a mathematical term used to talk about the likelihood of something happening, and it is the ability to understand and predict an outcome. In real-life scenarios, probability allows us to make informed decisions, whether it's predicting the weather, predicting game outcomes, assessing financial risks, or understanding the likelihood of medical outcomes.

2 The Basic Principle of Counting

2.1 Permutations

Definition: $n(n-1)(n-2)...3*2*1 = n!$ Permutations are a way, with a variety of variations, in which a set of numbers can be placed and ordered. It can mean two different things, an arrangement of its options in a sequence, or the process of changing that sequences order to another desired order.

n! is equal to 1*2...n where n has to be a positive integer. The reason why we use "n" is because the number is infinite(it represents any positive integer).

Permutations are used to find the different ordered arrangements possible. Order matters. For example, a question could ask:

How many different ordered arrangements of the letters a, b, and c are possible?

To solve this problem, we know that there are 6 different possibilities; abc, acb, bac, bca, cab, and cba. We can define this by saying that there are 6 possible permutations of a set of 3 objects(a, b, and c). The first "slot" of the letter arrangement has n choices. The second has n - 1 choices, and the third has n - 2 choices. This is exactly what is happening in the expression that's written on the definition for permutations. For example, the second letter has n choices to arrange them self, except for the "slot" that the first letter chose.

2.1.1 Permutations Example Problems

Permutations Example 1 When all letters are used, how many different letter arrangements can be made from the letters:

(a) Partying?
(b) Dancing?
(c) Acting?
(d) Singing?

Solution to Example 1: To solve for (a), we just need to do 8! because no letters repeat. 8! expanded is 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1. We are multiplying all the positions available for all 8 letters to find the possible arrangements. We then get 40320 different arrangements. However, for (b), "n" repeats twice. So, to solve this we would do 7!/2! to get 2520 as our answer. For (c), similar to (a), we would do 6! to get 720 arrangements, since all of the letters are unique. For (d) the letters "i", "n", and "g" repeat twice. So in order to solve for the letter arrangements for this word, we would do 7!/2!*2!*2! and we would get 630 arrangements as our answer. The reason why we are dividing by 2! in example problems (b) and (d) is because that's the number of times that a letter repeats. So we need to divide the total by 2! per letter so that no letter repeats.

Permutations Example 2: Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution to Example 2: We can arrange the mathematics books first, then the chemistry books, then the history books, and then the language books to get 4! * 3! * 2! * 1! arrangements. There are also 4! possible orderings of all 4 of the subjects. Therefore, we need to multiply all of those factorials together and would get 4! * 4! * 3! * 2! * 1! = 6912 arrangements possible.

2.2 Combinations

Combination Definition: $(n(n-1)...(n-r+1))/r! = n!/(n-r)!r!$

Combinations can be simply defined as a way of choosing items from a collection where the order of selection does not matter (whereas in permutations, order matters). For example, a problem can ask

How many different groups of 3 could be selected from the 5 items A, B, C, D, and E?

We can solve this using the combinations formula. There are 5 different ways to choose the first item, 4 to choose the next, and 3 to choose the last item. Therefore, there are 5 * 4 * 3 ways of choosing the different groups. However, since every group of 3 that is made will be counted 6 times(you can make 6 different groups using the letters provided), we would have to include 3 * 2 * 1 as the denominator of 5 * 4 * 3. We would then get the expression

$$(5 * 4 * 3)/(3 * 2 * 1) = 10$$

The total number of groups that can be formed with the given selection of items is 10.

2.2.1 Combinations Example Problem

Combinations Example 1: A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

Solution to Example 1: To divide the two categories of men and women, we can create two different fractions formed from the combinations formula. To choose 5 men to be paired off out of the 12 total, we can write the expression:

$$12!/5!(12-5)!$$

To solve for the 5 women to be paired off out of the 10, we can do the same thing:

$$10!/5!(10-5)!$$

Then, we multiply the two fractions together to get the final answer. The reason why we are multiplying is because the two options of men and women will be combined together. For each possible paring of men, there are as many pairings of women. The probability of the 5 men and 5 women getting chosen to be paired off should be multiplied. The final answer would be 199,584 possible pairings.

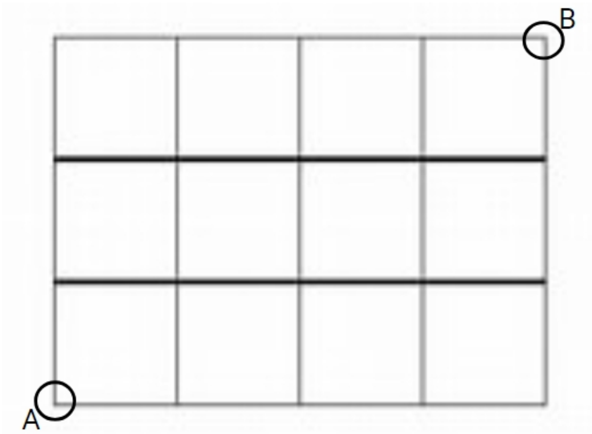
Another way to write these two expressions would be 12_C5 and 10_C5. Written above is the expanded expression for these two combination expressions.

Combinations Example 2: Consider a set of "n" antennas of which "m" are defective and "n - m" are functional and assume that all of the defectives and all of the functional are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution to Example 2: If no two defectives are allowed to be consecutive (next to each other), then there has to be spaces between the functional and defective antennas. That can be represented as "n - m + 1" possible positions between the antennas. Therefore, there would be $((n - m + 1)/m)$ possible orderings between the antennas in which there is at least on functional antenna between any two defective antennas.

2.2.2 The Grid Problem

Grid Problem 1: Consider the grid of points shown at the top of the next column. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?

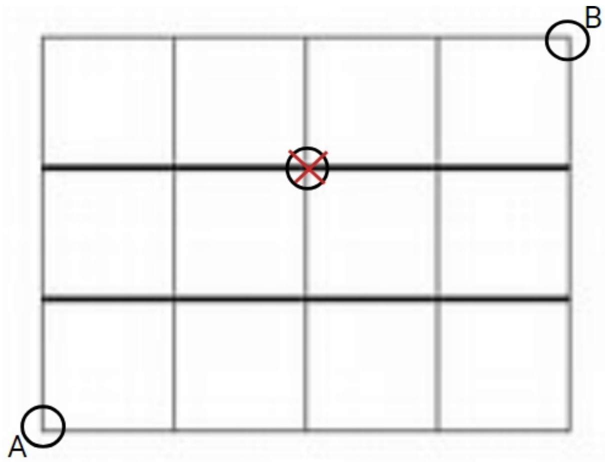


We can solve this problem using combinations. There are 3 steps up and 4 steps to the right, so the total number of moves needed is always 7. We have 7 moves, so we need to choose 3 moves to go up (alternatively, you can choose to move 4 spaces to the right. This is the same as 7_C4, which is equivalent to 35). We can solve this problem using the combination formula.

$$_7C_3 = 35$$

Therefore, by using the combinations formula, we can determine that there are 35 different paths from point A to point B from the grid.

Grid Problem 2: From the same grid as before, how many different paths are there from A to B that do not go through the point crossed in the following lattice?



In this problem, we need to avoid the point crossed on the grid but get from point A to B. In order to do this, we need to calculate and multiply two fractions together. One fraction defines the paths from point A to the avoided point. The other fractions defined the different paths from the avoided point to point B. By plugging in the points, we would get:

$${}_4C_2 *_3 C_2 = 12$$

The reason for writing 4 and 3 as the numerators of these fractions (once decomposed into factorials) is because of the number of steps required to take to get to different points on the grid. For example, the combination above can be further decomposed into:

$$4!/2!(4-2)! * 3!/2!(3-2)! = 12$$

The "r" value of the combinations is 2 because the 2 represents the horizontal steps taken to reach the desired points.

This problem is not completely solved yet, however, because what was calculated above were the paths that were not allowed because of the forbidden crossed point. So, in order to find the answer, we would have to subtract the total possible paths minus 12. That is

$$35 \text{ - } 12 = 23$$

Therefore, there are 23 different paths from point A to B that avoid the crossed point.

2.2.3 Binomial Theorem

The binomial theorem states that for any non-negative integer n and any real numbers a and b, the expansion of $(a+b)^n$ is given by:

$$(a+b)^n = \sum_{k=0}^{n} nka^{n-k}b^k$$

where nk represents the binomial coefficient, defined as:

$$nk = \frac{n!}{k!(n-k)!}$$

Example Problem 1 : Expand $(4x-3y)^4$

$$(4x-3y)^4 = 40(4x)^4 + 41(4x)^3(-3y) + 42(4x)^2(-3y)^2 + 43(4x)(-3y)^3 + 44(-3y)^4$$

After computing the binomial coefficients, the expansion simplifies to:

$$(4x-3y)^4 = 256x^4 - 144x^3y + 216x^2y^2 - 432xy^3 + 81y^4$$

3 Sample Space and Events

Definition: The set of all possible outcomes of an experiment (S).

For example, the sample space to find the outcome of the gender of a newborn child is: S = (girl, boy). This is because in this situation there are only 2 total possible outcome, the baby being a girl or boy.

Another example is that sample space of the outcomes of flipping two coins is: S = [(h,h), (h,t), (t,h), (t,t)]. Similar to the example above, this is because when flipping 2 coins you have a combination of 4 possible outcomes(written above).

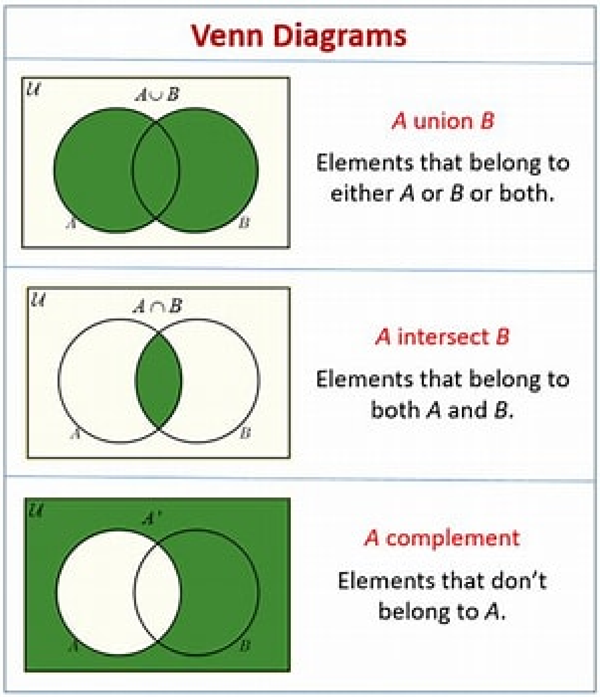
Any subset of E of a sample space is called an "event." An event is a set containing the possible outcomes of the experiment. For example, E = (h), then E is the event that the first coin flip lands on heads. Additionally, if event F = (t), then F is the event that the first coin flip lands on tails.

Definition: For any two of the events E and F of a sample space, a union of those two events can be defined as $E \cup F$.

$E \cup F$ is called the union of the event of E and the event of F. For example, the event that the first coin flip is a head or tail is: $E \cup F = (h,t)$.

Definition: Similarly, for two events E and F, we can also create a new event EF called the intersection of E and F. This shows the outcomes that are both in events E and F, which is sometimes written as E AND F. For example, if E = [(h,h), (h,t), (t,h)] for at least one head, and F = [(h,t), (t,h), (t,t)] is for at least one tail, then we can say: EF = [(h,t), (t,h)] which is the event in which one head and one tail occur.

Definition: For any event E, we can also write the complement of that event as E^c, which refers to all the outcomes in the sample space S that are not in E. Simply put, E^c will only happen if and only if E does not occur.



These images represent Venn diagram representations of different outcomes.

3.1 Sample Space Example Problems

Example Problem 1: Two football teams are playing. Let A be the event that the match ends in a draw, and let B be the event that the home team wins. Assuming that not more than 6 goals are scored in all, list the sample space and the following events: $A \cap B$. Let C be the event that the away team scores. List the elements of A $\cup$ C^C and $A^C \cap B \cap C$.

Solution: The sample space of this problem would be S = A0B0, A0B1, A0B2, A0B3, A0B4, A0B5, A0B6, A1B0, A1B1, A1B2, A1B3, A1B4, A1B5, A2B0, A2B1, A2B2, A2B3, A2B4, A3B0, A3B1, A3B2, A3B3, A4B0, A4B1, A4B2, A5B0, A5B1, A6B0. A0 represents no goals for the home team, A1 represents 1 goal for the home team, and so on. The same thing happens with B.

A $\cap B = N/A$.There is no possible way for the match to end in a draw and for the home team to win.

$A \cup C^C = (3,3),(6,0),(4,2),(5,1)$. These are the chances that the match ends in a draw OR everything but the away team scoring occurs in the 6 goals. The chances are greater in this example because we are considering two possible outcomes.

$A^C \cap B \cap C\ :\ (0,6),(1,5),(6,0),(4,2),(5,1),(2,4)$. We are asked to solve for the probability of everything but the match ending in a draw to happen, and the home team wins, and the away team scores.

Example Problem 2: A bistro owner was taking stock of 1101 orders received on a particular day. Four hundred and eleven people ordered only drinks, 231 only food, and 62 only sweets. Forty five had drinks and sweets but not food, 312 had food and drinks but no sweets, and 11 had sweets. However, the owner has a feeling that he has recorded one figure incorrectly. The other set of more precise data that he has is that the number of customers who chose drinks, food, and sweets were 797, 595, and 147, respectively. Translate the numbers above into probabilities and establish which number is incorrect.

Solution: To solve this problem, we need to first identify the different variables given.

Total customers = 1101

Only drinks = $411/1113 \approx 0.369$

Only food = $231/1113 \approx 0.208$

Only sweets = $62/1113 \approx 0.056$

Only drinks and sweets = $45/1113 \approx 0.040$

Only food and drinks = $312/1113 \approx 0.280$

Only sweets and food = $11/1113 \approx 0.010$

All three = $41/1113 \approx 0.037$

For the probabilities written below, the numerators are defined in the problem in the set of more precise data that the bistro owner has. We can use the resultants of these probabilities as a comparison with the owners' other collected data.

Drinks: $797/1113 \approx 0.716$

Food: $595/1113 \approx 0.535$

Sweets: $147/1113 \approx 0.132$

The incorrect number is the number of drinks and sweets because $P(D \cap S)$ equals 33 not 45.

4 Axioms of Probability

There are three axioms of probability:

$$1)\ 0 \leq P(E) \leq 1$$

This states that the probability that the outcome of the experiment is an outcome in E is some number between 0 and 1.

$$2)\ P(S) = 1$$

This states that with probability 1, the outcome will be a point in the sample space S.

$$3)\ P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

This states that for any sequence of mutually exclusive events, the probability of at least one of these events occuring is just the sum of their respective probabilities.

4.1 Example Problems

Example 1: When tossing a coin, what is the probability if a head were twice as likely to appear as a tail?

Solution to Example 1: If you toss a coin and we assume that a head is as likely to appear as a tail, then we would have: $P(H) = P(T) = 1/2$. However, if the coin were biased and a head were twice as likely to appear as a tail, then we would have: $P(H) = 2/3P(T) = 1/3$.

Example 2: What is the probability of rolling an even number on a 6-sided die?

Solution to Example 2: If a die is rolled and all 6 sides are equally likely to appear, then $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$. According to Axiom 3, the probability of rolling an even number would equal: $P(2, 4, 6) = P(2) + P(4) + P(6) = 1/2$.

5 Conditional Probability

Conditional probabilities are used to compute the desired probabilities more easily when no partial information is available.

Definition: $P(E|F) = P(EF)/P(F)$

This formula states that if the event F occurs, then, in order for E to occur, it is necessary that the actual occurrence be a point both in E and in F; that is, it must be in EF. Now, since we know that F has occurred, it follows that F becomes our new, or reduced, sample space; hence, the probability that the event EF occurs will equal the probability of EF relative to the probability of F.

5.1 Conditional Probability Example Problems

Problem 3.13: A recent college graduate is planning to take the first three actuarial examinations in the coming summer. She will take the first actuarial exam in June. If she passes that exam, then she will take the second exam in July, and if she also passes that one, then she will take the third exam in September. If she fails an exam, then she is not allowed to take any others. The probability that she passes the first exam is .9. If she passes the first exam, then the conditional probability that she passes the second one is .8, and if she passes both the first and second exams, then the conditional probability that she passes the third exam is .7.

PART A: What is the probability that she passes all three exams?

Solution: P(Passes all three exams) $= 0.9 * 0.8 * 0.7 = 50.4$ percent. We can find the probability of passing all three of those exams by multiplying all the probabilities of the college graduate passing the different exams.

PART B: Given that she did not pass all three exams, what is the conditional probability that she failed the second exam?:

Solution: We will use the formula to solve for conditional probability:

$$P(A|B) = P(A \cap B)/P(B)$$

A=Failing the 2nd exam

B=Failing all 3 exams

To find the probability of the college graduate failing all three exams, we can use complementary counting:

$$P(B) = 1 - P(\text{Pass all 3 exams}) = 1 - 0.504 = 0.496$$
$$P(A|B) = P(\text{Fail second exam and fail all 3 exams})/P(\text{Fail all 3 exams})$$

By plugging in the values from above, we can conclude that the probability that she failed the second exam is 71.3 percent.

5.2 Deriving Bayes' Theorem

Let E and F be events. Bayes' Formula states that:

$$P(E|F) = \frac{P(E)P(F|E)}{P(F)}$$

This says that the probability of event E given event F is equal to the probability of event E times the probability of event F given E divided by the probability of event F. Bayes' Formula essentially describes the probability of an event, based on prior knowledge of factors related to the event.

5.2.1 Example Problems

Example 3.19: A total of 46 percent of the voters in a certain city classify themselves as Independents, whereas 30 percent classify themselves as Liberals and 24 percent say that they are Conservatives. In a recent local election, 35 percent of the Independents, 62 percent of the Liberals, and 58 percent of the Conservatives voted. A voter is chosen at random Given that this person voted in the local election, what is the probability that he or she is; a) an Independent? b) a Liberal? c) a Conservative? d) What percent of voters participated in the local election?

Solution: The probabilities written below are given.

$$P(\text{Independent}) = 0.46$$
$$P(\text{Liberal}) = 0.30$$
$$P(\text{Conservative}) = 0.24$$
$$V = \text{voted}$$
$$P(V|I) = 0.35$$
$$P(V|L) = 0.62$$
$$P(V|C) = 0.58$$

We solve for part d first because in order to solve the other parts, we need to know the total number of people who voted in the election. We need to use a formula for independence which states that $P(E) = P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H)$. Then we need to plug in the numbers based off of what was given in the problem.

$$\text{d) } P(V) = P(V|I)P(I) + P(V|L)P(L) + P(V|C)P(C)$$
$$0.161 + 0.186 + 0.1392$$
$$P(V) = 0.4862$$
$$\text{a) } P(I|V) = (P(I) * P(V|I))/P(V)$$
$$(0.46 * 0.35)/0.4862$$
$$P(I|V) = 0.331$$

Using Bayes' Formula, we can plug in what we were given and what we solved for in part d into the formula. To solve parts b and c, we would also need to use Bayes' Theorem and plug in according to the probabilities that were given and solved for.

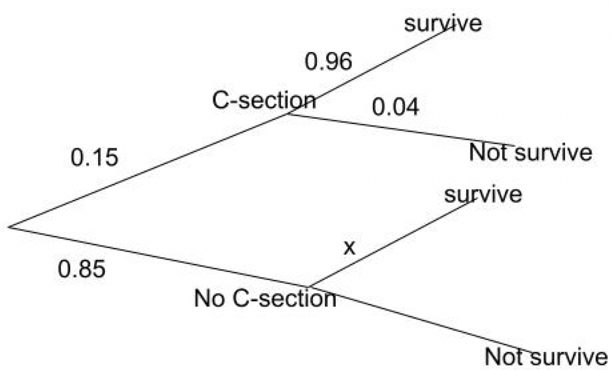
$$\text{b) } P(L|V) = (P(L) * P(V|L))/P(V) = (0.30 * 0.62)/0.4862$$
$$P(L|V) = 0.383$$
$$\text{c) } P(C|V) = (P(C) * P(V|C))/P(V) = (0.24 * 0.58)/0.4862$$
$$P(C|V) = 0.286$$

6 Independent Events

Definition: Two events E and F are said to be independent if $P(EF) = P(E)P(F)$ holds. Two events E and F that are not independent are said to be dependent.

6.1 Example Problem 3.17

Problem: Ninety-eight percent of all babies survive delivery. However, 15 percent of all births involved Cesarean (C) sections, and when a C section is performed, the baby survives 96 percent of the time. If a randomly chosen pregnant woman does not have a C section, what is the probability that her baby survives?



Solution: In order to solve this problem, you can create a tree diagram to visualize it. The two main branches would represent C-section and no C-section. Both of the main branches would each have two other branches that represent survive and not survive. In the problem, you're given that fifteen percent of all births involve C-section, so that would go on the main branch for C-section. You're also given that ninety-six percent of babies survive when C-section is performed.

With this knowledge, you can figure out that eighty-five percent of births don't involve C-section because if you subtract .15 from 1 you get .85 according to the complement rule. With this knowledge you can do $1 - .96$ to get .04, which represents the births that required C-section and resulted in death.

In order to solve the problem you can create the equation $0.98 = (0.15 * 0.96) + (x * 0.85)$. x represents the probability for when the birth doesn't involve C-section and the baby survives. You would multiply the probabilities that are given because they are independent events. After forming the equation, you can solve for x and get approximately 0.9835 which as a percentage is 98.35 percent and is your answer.

7 Acknowledgments

7.1 References

Ross Sheldon, *A First Course in Probability*, New York, 1984.

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