Intersecting Parallel Lines: Projective Geometry and its Applications

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April 2024

1 Abstract

In traditional Euclidean Geometry, parallel lines never intersect. In our perception of the world, however, parallel lines appear to converge at *vanishing points* infinitely far away. Projective geometry explores this possibility; at its core, examining the properties of points and lines, and how they behave under transformations of perspective. In this presentation, we present an exploration of analytic projective geometry, its sub-geometries, projective transformations, and its very useful applications in art, animation, and game design.

2 Intersecting parallel lines in Art

One of the most intuitive uses of projective geometry is within art. In fact, projective geometry was first developed during the Renaissance for use in art.

In art, artists employ the use of vanishing points, where parallel lines converge on a horizon to create the illusion of a three-dimensional world on a two-dimensional surface. Compositional techniques include : one point perspective, typically used when the subject of the art piece is viewed head on and a strong focal point is desired, two point perspective when a building is viewed from an edge, and three point perspective when a more dramatic angle is desired.

Figure 1: An example of a road drawn in one-point perspective

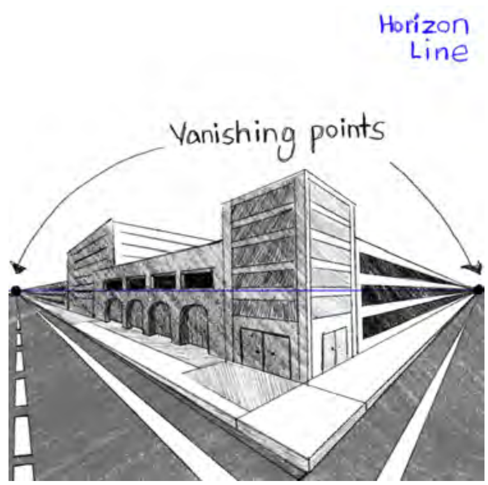

Figure 2: An example of buildings and roads in two-point perspective

Figure 3: If you were an earthworm, this would be how you saw the world

The three perspectives mentioned above are linear perspectives, and follow the same conventions as projective geometry. However, artists also introduce conventions beyond projective geometry by using curved lines in four, five, and six point perspectives. These perspectives are able to represent more complex fields of vision: four point perspective projects a cylindrical field of vision, five point perspective projects a hemispherical field of vision, and six point perspective projects a spherical field of vision, able to capture 360-degrees to create a sense of encircling the viewer.

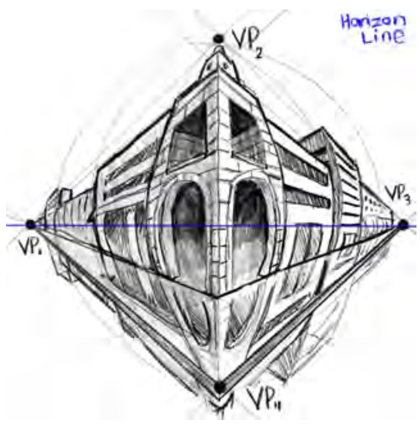

Figure 4: A street drawn in a four-point perspective.

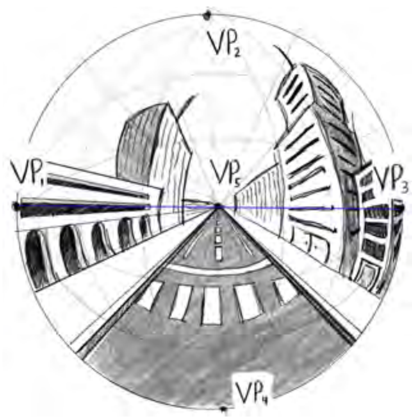

Figure 5: If you were a fish, you might see the world like this

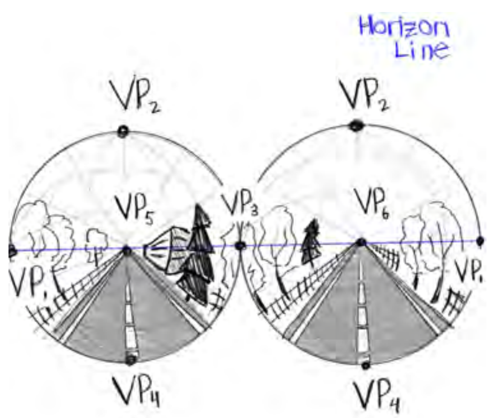

Figure 6: A street drawn in a six-point perspective.

3 Introduction

Before we begin our exploration, we must first define some terms

Definition 3.0.1. *A* ***mapping*** *is a function that assigns each element in a given set to a unique corresponding element in another set.*

Definition 3.0.2. *A* ***transformation*** *is a mapping of an input (such as a point, vector, or shape) to an output image.*

Since projective geometry originates from perspective drawings in Renaissance art, it does not preserve distances and is hence *non-metrical*. In projective geometry, we define all parallel lines intersecting at a point infinity.

Definition 3.0.3. *An* ***ideal point*** *is a point at infinity where parallel lines meet In a plane the ideal points form an ideal line, and in space they form an ideal plane or a plane at infinity.*

Axiomatically, we characterize this idea two distinct lines in the plane have at least one point on both lines. The definition of *persepectivity* formalizes this idea:

Definition 3.0.4. ***Perspectivity*** *with respect to a point is the mapping (such that each point has a unique relative in the resulting image) of points* A, B, C, D *on one line to the points* A', B', C', D' *on another line.*

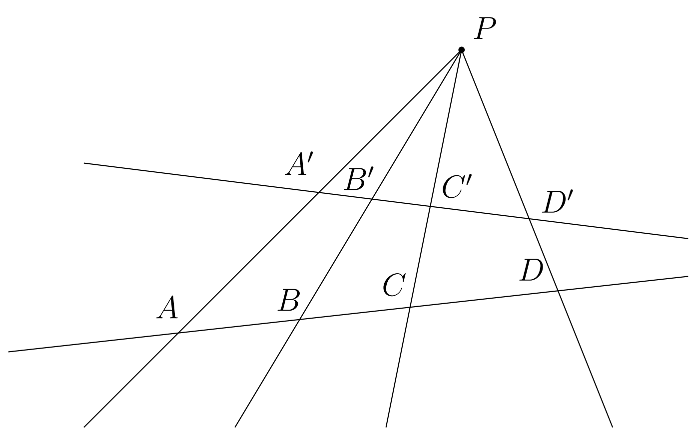

Figure 7: Points A, B, C, D and A', B', C', D' are related by a perspectivity with respect to point P.

Definition 3.0.5. *Points are collinear if they lie on the same line.*

Definition 3.0.6. ***Collineation*** *in the context of projective geometry, is a one-to-one mapping from one projective plane to another, such that the images of collinear points remain collinear after transformation.*

Additionally, in Euclidean geometry, distance is fundamental (we derive almost all associations through diatance). However, this property, in addition to angles, are not preserved in projective geometry. Instead, collineation ensures that lines remain as lines after transformations, preserving incidence relations (i.e., which points lie on which lines).

4 Cross Ratios

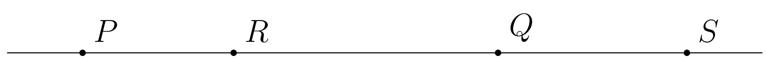

Figure 8: Cross ratio of $(P, Q; R, S)$

Cross-ratios, denoted $(P, Q; R, S)$, are the ratio of four distinct collinear points, as represented by the fraction:

$$\frac{\frac{PR}{QR}}{\frac{PS}{QS}}$$

R must be in between points P and Q for which the cross-ratio refers to. This property of betweeness arises the concept of *separation*, where any pair of points separate the point between them with its harmonic conjugate. Harmonic conjugates are two points that divide a line segment internally and externally in the same ratio.

4.1 Harmonic Set

Definition 4.0.1. *Four distinct points form a harmonic set, denoted $H(PQ, RS)$, if and only if they are collinear and their 6 lines (fulfilling the conditions illustrated below) form a complete quadrangle.*

Equivalently, P, Q, R, S form a harmonic set if $(P, Q; R, S) = -1$. (For the interested reader, this equivalence can be shown by applying Menelaus's and Ceva's theorems to triangle T_1PQ in Figure 9).

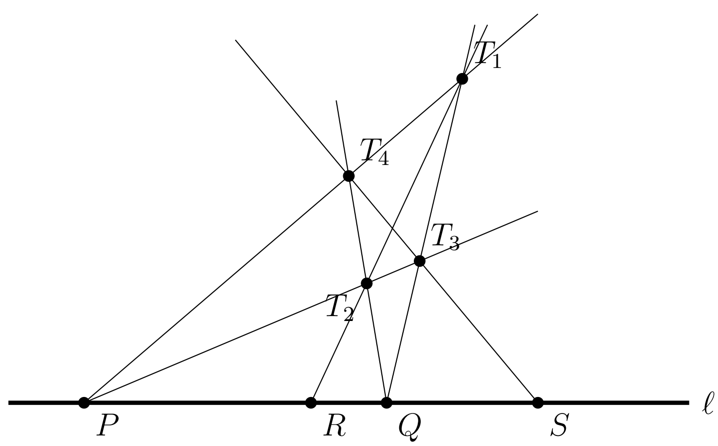

Figure 9: a quadrangle construction

Given 3 existing distinct collinear points, there exists an unique fourth one that can be determined through the construction in Figure 9. If P, Q, and R are 3 distinct collinear points on line ℓ, then the intersection of the harmonic conjugates – the two points that divide a line segment internally and externally in the same ratio – T_1, T_2, T_3, T_4 respectively, will each have 6 lines such that P, Q, R, and the fourth point S, are each on two of the lines. The shape formed by the intersections T_1, T_2, T_3, T_4, form a complete quadrangle.

Theorem 4.1. *Cross ratios of collinear points do not change under projections.*

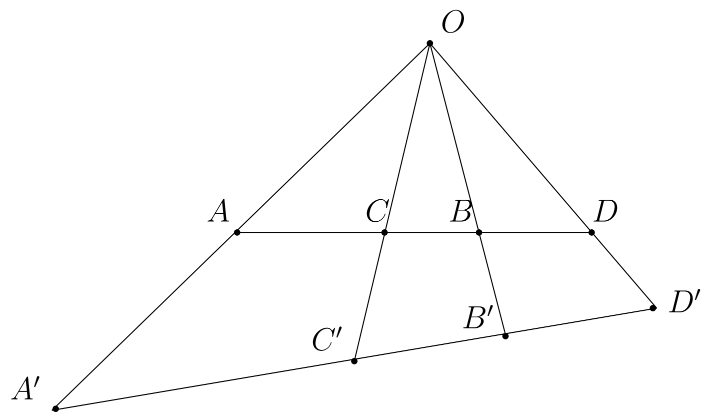

Figure 10: Proved by using the law of sines

Proof: The cross ratio of (A,B;C,D)

$$= \frac{\frac{AC}{CB}}{\frac{AD}{DB}} = \frac{AC \cdot DB}{CB \cdot AD} = \frac{AC \cdot DB}{(AB - AC) \cdot (AB + DB)}$$

$$= \frac{AC \cdot DB}{AB^2 - (AC \cdot AB) + (AB \cdot DB) - (AC \cdot DB)} = \frac{AB \cdot CD}{AC \cdot DB}$$

Similarly the cross ratio of $(A', B'; C', D') = \frac{A'B' \cdot C'D'}{A'C' \cdot D'B'}$

Applying law of sines to A, B, C, D:

$$DC = \frac{OC \cdot \sin \angle DOC}{\sin \angle ODC} \text{ etc. (the same for segments } AB, DB, AC)$$

Substituting in the cross ratio we find:

$$\frac{\frac{OB \cdot \sin \angle AOB}{\sin \angle OAB} \cdot \frac{OC \cdot \sin \angle DOC}{\sin \angle ODC}}{\frac{OC \cdot \sin \angle AOC}{\sin \angle OAB} \cdot \frac{OB \cdot sin \angle DOB}{\sin \angle ODC}} = \frac{\sin \angle AOB \cdot \sin \angle DOC}{\sin \angle AOC \cdot \sin \angle DOB}$$

Again applying law of sines to A', B', C', D':

$$D'C' = \frac{OC' \cdot \sin \angle D'OC'}{\sin \angle OD'C'} \text{ etc. (the same for segments } A'B', D'B', A'C')$$

$$\frac{\frac{OB' \cdot \sin \angle A'OB'}{\sin \angle OA'B'} \cdot \frac{OC' \cdot \sin \angle D'OC'}{\sin \angle OD'C'}}{\frac{OC' \cdot \sin \angle A'OC'}{\sin \angle OA'B'} \cdot \frac{OB' \cdot \sin \angle D'OB'}{\sin \angle OD'C'}} = \frac{\sin \angle A'OB' \cdot \sin \angle D'OC'}{\sin \angle A'OC' \cdot \sin \angle D'OB'}$$

Since $\angle A'OB' = \angle AOB$, $\angle d'OC' = \angle DOC$, $\angle A'OC' = \angle AOC$ and $\angle D'OB' = \angle DOB$

$$\frac{\sin \angle A'OB' \cdot sin \angle D'OC'}{\sin \angle A'OC' \cdot \sin \angle D'OB'} = \frac{\sin \angle AOB \cdot \sin \angle DOC}{\sin \angle AOC \cdot \sin \angle DOB}$$

As such, the cross ratio of $(A, B; C, D)$ = cross ratio of $(A', B'; C', D')$. *Hence, cross ratios are preserved under projections.*

Theorem 4.2. *Pascal's Theorem states that any hexagon on the circumference of a conic, with vertices A, B, C, D, E, F, (even with coincident points) have opposite pairs of sides such that their intersections are collinear.*

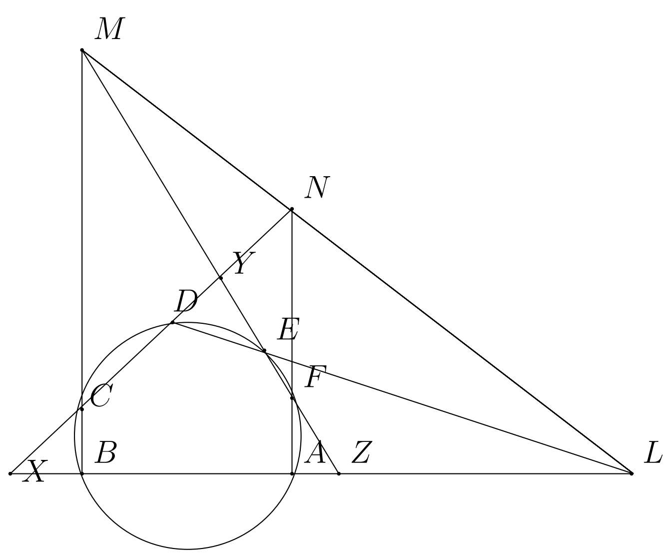

Figure 11: Collinear points constructed from Pascal's theorem

Proof: Let the intersection of BC and FE be point M, the intersection of DC and AF be point N and the intersection of DE and AB be point L. Then we construct triangle XYZ: Let AB and CD meet at X, CD and AF at Y and, EF and AB at Z. BM, AN, DL are transversals of triangle XYZ. Additionally, BC and YZ meet at point M, XY and AF meet at N, and DE and XZ meet at L. To prove that L, M, N are collinear:

BM: $\frac{ZB}{BX} \cdot \frac{XC}{CY} \cdot \frac{YM}{MZ} = -1$

AN: $\frac{ZA}{AX} \cdot \frac{XN}{NY} \cdot \frac{YF}{FZ} = -1$

DL: $\frac{XD}{DY} \cdot \frac{YE}{EZ} \cdot \frac{ZL}{LX} = -1$

Multiplying the above three equations, we get equation A:

$$\frac{ZB}{BX} \cdot \frac{XC}{CY} \cdot \cdot \frac{ZA}{AX} \cdot \cdot \frac{YF}{FZ} \cdot \frac{XD}{DY} \cdot \frac{YE}{EZ} \cdot (\frac{ZL}{LX} \cdot \frac{XN}{NY} \cdot \frac{YM}{MZ}) = -1$$

According to the intersecting secants theorem, that states that when two secants intersect at an exterior point, the product of the length of one secant segment and its external segment equals the product of the other secant segment and its external segment, we can deduce the following:

$ZB \cdot ZA = EZ \cdot FZ$
$XD \cdot XC = AX \cdot BX$
$YE \cdot YF = CY \cdot DY$

Substituting above equations into equation A: $\frac{ZL}{LX} \cdot \frac{XN}{NY} \cdot \frac{YM}{MZ} = -1$

Therefore, points M, N, L lie on the same line, i.e., a transversal of triangle XYZ.

Theorem 4.3. *Desargues' Theorem states that if two triangles are in perspective from a point, then they are perspective from a line.*

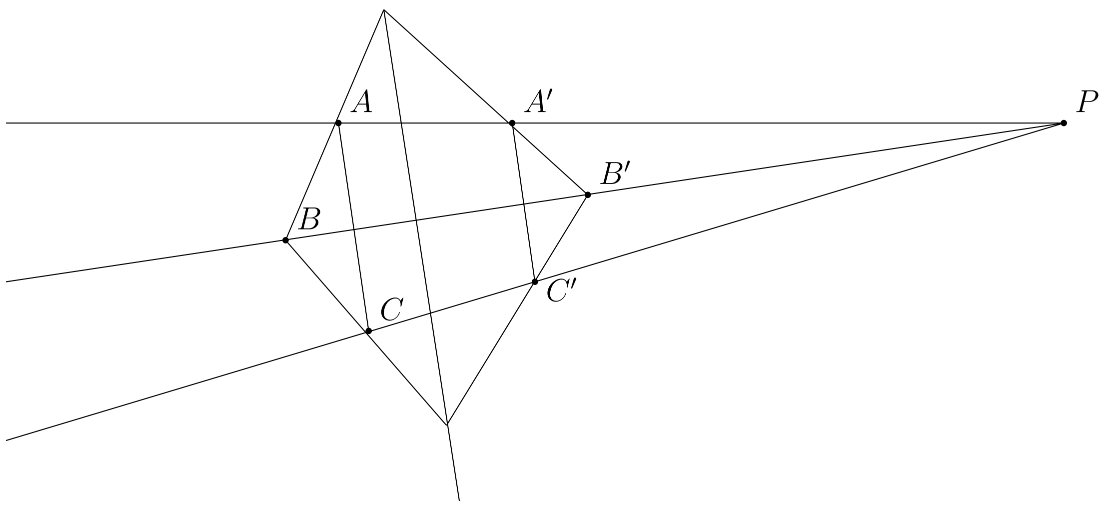

Figure 12: Two triangles in a Desargueian configuration

Proof: Let ABC and $A'B'C'$ be two triangles in the (projective) plane. Let P be the common intersection of AA', BB', CC'. Hence, there are scalars α, β, γ and α', β', γ' such that:

$$\alpha A - \alpha A' = P \beta B - \beta B' = P \gamma C - \gamma C' = P$$

Then we can derive:

$$\alpha A - \beta B = \alpha' A' - \beta' B' \beta B - \gamma C = \beta' B' - \gamma' C' \gamma C - \alpha A = \gamma' C' - \alpha' A'$$

From this, we can conclude that point $\alpha A - \beta B$ on line AB also lies at $\alpha' A' - \beta' B'$. Similarly points $\beta B - \gamma C$ and $\gamma C - \alpha A$ lie at $\beta' B' - \gamma' C'$ and $\alpha A = \gamma' C' - \alpha' A'$ respectively. Since we already defined ABC and $A'B'C'$ be two triangles in the (projective) plane, we can say that:

$$(\alpha A - \beta B) + (\beta B - \gamma C) + (\gamma C - \alpha A) = 0$$

The lines AA', BB', CC' intersect in a single point if and only if the intersections of corresponding sides $(AB, A'B')$, $(BC, B'C')$, $(CA, C'A')$ lie on a single line.

As such, the three intersection points are collinear.

4.2 Homogeneous Coordinates

Naturally, we require a new system to accommodate for the variations of projective geometry. *homogeneous coordinates* are a system of coordinates, introduced by Augustus Möbius, to represent points in the projective space.

Definition 4.3.1. *Formally: Homogeneous coordinates are determined by:* $\mathbb{P}^n = \{X_0, ..., X_n\}$*:* $X_0, ..., X_n$ *are not all 0 and* $(X_0, ...X_n) = \lambda(X_0, ...X_n)$*,*

where λ is the mathematical symbol for a scale factor and n is the dimension of projective space. This means that all scalar multiples of a point are the same point in projective geometry, as we are not concerned about distances.

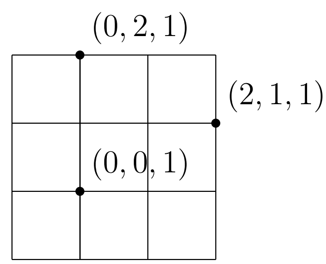

Figure 13: The origin in the projective plane

They have an additional coordinate to include representation of points at infinity. $(x, y, 1)$ would represent a finite point while $(x, y, 0)$ would represent a point at infinity in the projective plane. In this case, $(0, 0, 0)$ does not exist in the projective plane and the origin is defined as $(0, 0, 1)$.

They are essential when defining points during transformations in projective space.

5 Projective Transformations

The usefulness of homogenous coordinates becomes apparent in terms of *transformations.* A transformation is a one-to-one function from a space onto itself: the points are columns vectors and lines are row vectors. Thus projectivities and collineations correspond to invertible matrices.

Definition 5.0.1. *Projectivity is represented by an invertibel* 2×2 *matrix. Two matrices differing by a non-zero constant represent the same projectivity.*

Theorem 5.1. *Cross Ratios, harmonic sets, and separations are preserved under projectivities.*

Proof: Let $X_i = (u_i, v_i)$ for $i = 1, 2, 3, 4$ be collinear points for the projectivity $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. The cross ratios of $R(X_1, X_2, X_3, X_4)$ and $R(MX_1, MX_2, MX_3, MX_4)$ are equal since the determinants of the coordinates of the points are equal. Harmonic sets and separations are defined in terms of the cross ratio; hence, they are also preserved under projectivities.

5.1 Collineation

Definition 5.1.1. *Collineation of the projective plane is represented by an invertible* 3×3 *matrix. Two matrices represent the same collineation if one is a non-zero scalar multiple of the other.*

Example: If $\begin{bmatrix} 6 & -1 & 0 \\ 9 & 1 & 0 \\ 3 & -1 & 3 \end{bmatrix}$ illustrates a collineation of points, where would $U = (1, 1, 1)$ get projected to?

Solution: Multiplying the matrices out, we get the 3×1 matrix:

$$\begin{bmatrix} 6 + (-1) + 0 \\ 9 + 1 + 0 \\ 3 + (-1) + 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$$

However, since a scalar multiple of any point in homogeneous coordinates is the same as its simple form, point U would be projected to the point $(1, 2, 1)$.

5.2 Conics

Projective geometry doesn't distinguish between circles, ellipses, parabolas, and hyperbolas. These are *conic sections*, which are the curves obtained accordingly after intersecting a cone with a plane. When transforming different types of lines in the projective plane, different conic sections are obtained after the mappings. Circles remain circles. Tangent lines of the circle become parabolas because it retains a point of tangency with the original circle. Secant lines in the circle become hyperbolas, as hyperbolas have two branches, and each branch corresponds to one of the two points of intersection with the original circle.

Definition 5.1.2. *The ideal line is designated as a projective line made up of all ideal points on the projective plane.*

This ideal line handles parallel lines by allowing us to treat them as intersecting at a single point.

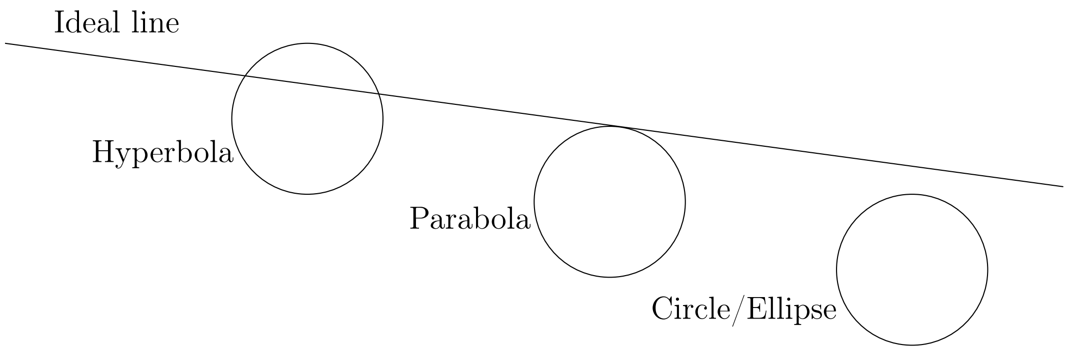

Figure 14: Conics in relation to the ideal line

Hyperbolas intersect the ideal line at two points, parabolas intersect it in one point, and circles and ellipses do not intersect it.

5.3 Inversion

Definition 5.1.3. *Inversion maps circles or lines to other circles or lines with respect to a circle. The inverse of a point P with respect to the inversion circle O is P', determined by the equation $\overline{OP} \cdot \overline{OP'} = r^2 P'$. P' is collinear to line OP.*

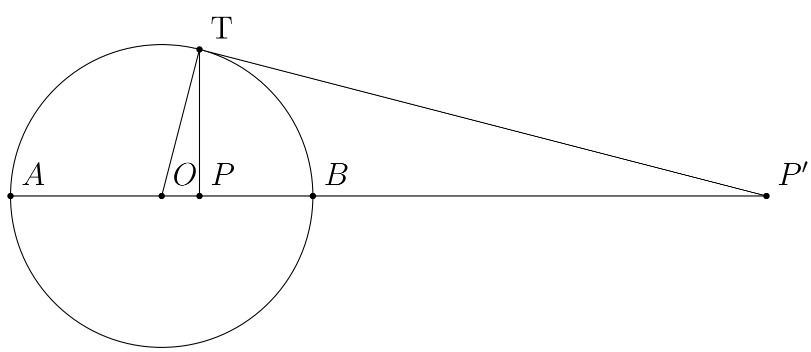

Figure 15: Cross-ratios are preserved under inversion

Theorem 5.2. *If P is a point in the diameter AB of a circumference with center O, and P' is the inverse of P with respect to this circumference, then $(A,B;P,P') = -1$, i.e., four distinct points A,B,P,P' form a harmonic set.*

Proof:

$$\frac{\frac{AP}{PB}}{\frac{AP'}{P'B}} = \frac{\frac{r+OP}{r-OP}}{\frac{OP'+r}{OP'-r}} = \frac{(OP\cdot OP') - (OP\cdot r) + (OP'\cdot r) - r^2}{-(OP\cdot OP') + (OP\cdot r) - (OP'\cdot r) + r^2}$$

$$(OP\cdot OP') = r^2$$

$$\frac{-(OP\cdot r) + (OP'\cdot r)}{(OP\cdot r) - (OP'\cdot r)} = -1$$

6 Projective Space

Homogenous coordinates and collineations can be extended to higher dimensional projective space.

6.1 Coordinate system

Projective Space adds an *ideal plane* to the Euclidean Points (x,y,z). Hence, homogeneous coordinates require an extra dimension as well. (x,y,z,t) is a general 3-dimensional point in Projective Geometry. The Ideal points have homogeneous coordinates $(x,y,z,0)$.An n dimensional projective space $\mathbb{P}^n$ translates into a subspace of $n+1$ dimensional vector space space $\mathbb{R}^{n+1}$. Hence, the 1-dimensional vector space is a point. The line is a 2-dimensional vector subspace and the plane is a 3-dimensional subspace and so on.

6.2 Duality

The concept of duality, unique to projective geometry, states that for every theorem that refers to a line, there is a dual for this theorem that refers to a point. This concept is derived from the fact that on every line there are infinite points, and for every point, there are infinite lines intersecting it.

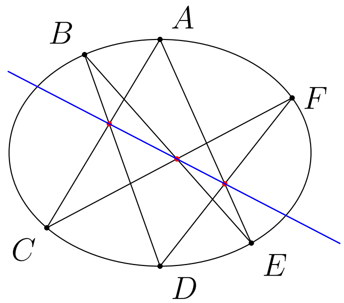

Figure 16: Pascal's theorem

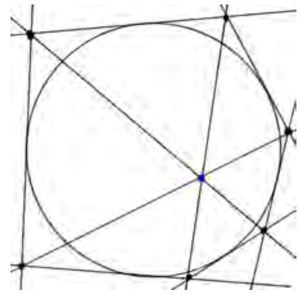

Figure 17: Pascal's inverse: Brianchon's Theorem

Remarkably, Pascal's theorem, as proved above, has a dual theorem involving points; Brianchon's theorem is a theorem stating that when a hexagon is circumscribed around a conic section, its principal diagonals (those connecting opposite vertices) meet at *one* single point, called a *Brianchon point*. It is beautiful to see how the two theorems encapsulate how points and lines have identical properties in projective geometry.

7 Sub-geometries

Mathematicians originally thought of projective geometry as an extension of euclidean geometry.

7.1 Sub-geometries of Projective Geometry

In 1859, Arthur Cayley showed that Euclidean Geometry is a sub-geometry of projective geometry. Felix Klein built upon this to show other relations.

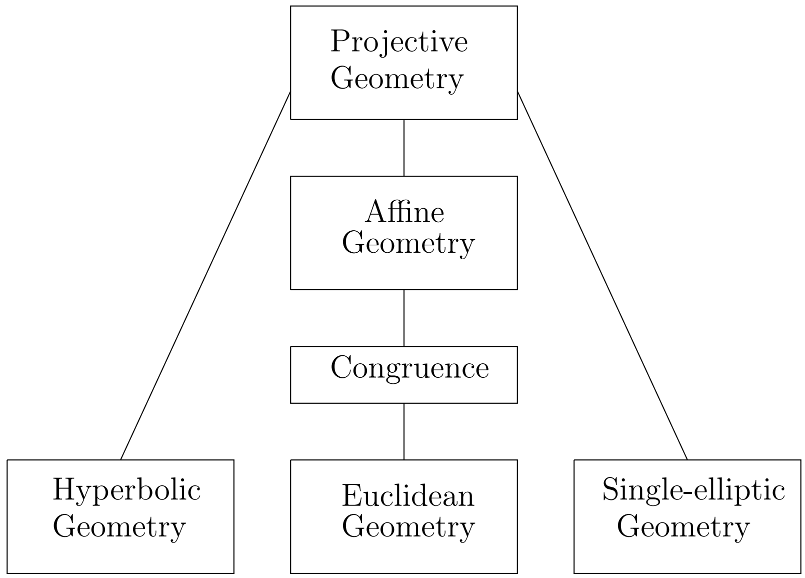

Figure 18: Relation of Projective Geometry's subgeometries

Some geometries encompassed by projective geometry include hyperbolic geometry, affine geometry, Euclidean geometry, and single elliptic geometry. In order for one geometry to be a sub-geometry of another, it must fulfill two requirements:
1) First, all transformations of the sub-geometry must also be transformations within the encompassing geometry.
2) Second, undefined terms in the sub-geometry must be able to find a definition within the encompassing geometry.

A good example of the second requirement is distance within the hyperbolic and single elliptic planes. Distance, a relation involving two points A and B, could be defined in the hyperbolic and single elliptic planes relative to other points Ω and Λ on a line using the cross ratio. To obtain Ω and Λ, Cayley used the intersection of a line with a fixed conic – the *Absolute Conic*.

7.1.1 Hyperbolic Geometry as a sub-geometry

The hyperbolic distance between A and B is based on their cross ratio with the intersections Ω and Λ of the line they determine with the conic. In the *Klein model*, the unit circle would be the absolute conic. Since collineations preserve cross-ratio, the distance formula is preserved under collineations that are hyperbolic.

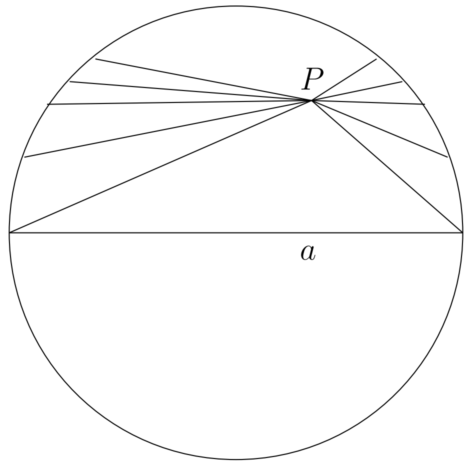

Figure 19: Klein model of hyperbolic geometry

The absolute conic is given by the equation $x^2 + y^2 - z^2 = 0$. The point (x, y, z) is $x^2 + y^2 < z^2$, i.e., it is interior to the absolute conic. The line is a set of points interior to the absolute conic that are on a projective line $[a, b, c]$. The intersections of the line with the absolute conic are the *omega points* of the line. The *distance* between A and B is

$$d_H(A, B) = |\log(R(A, B, \Omega, \Lambda))| = \left|\log\left(\frac{A\Omega}{A\Lambda} \div \frac{B\Omega}{B\Lambda}\right)\right|$$

Where XY is the Euclidean distance between X and Y, and Ω and Λ are the two omega points of line AB. By *hyperbolic isometry* we mean a collineation that leaves the absolute conic *stable*.

Example: Verifying that the adjacent points Q, Q_i have the same distance between them. X- coordinates of the points are $P_0 = 0, P_1 = \frac{1}{3}, P_2 = \frac{3}{5}, P_3 = \frac{7}{9}, P_4 = \frac{15}{17}, P_5 = \frac{31}{33}, \Omega = -1$, and $\Lambda = 1, P_{-i} = -P_i$, $\Omega = -1$, and $\Lambda = 1$

Solution: The Euclidean distances between the points are the differences of their x-coordinates of the points.

$$\left(\frac{P_0\Omega}{P_0\Lambda}\right) \div \left(\frac{P_1\Omega}{P_1\Lambda}\right) = \frac{1}{1} \div \frac{\frac{4}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Similarly, all the corresponding products equal (1/2). The absolute values of the logarithms are equal. Hence all the distances are the same.

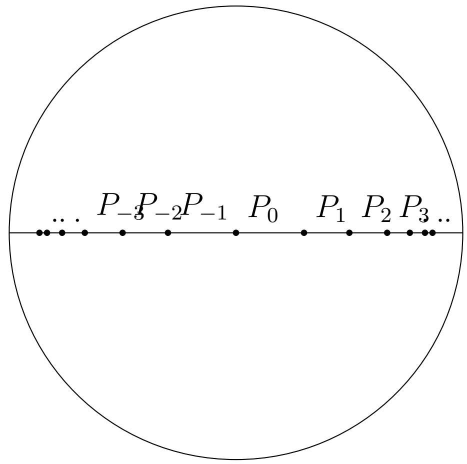

Figure 20: Equidistant points in the Klein model of the hyperbolic plane

Hyperbolic isometries are a group of transformations: The general matrix form of hyperbolic isometries M must take C to itself, by the theorem in projective transformations $M^{1T}CM^{-1} = \Lambda C, for \Lambda \neq 0$.

Definition 7.0.1. *We define the h-inner product of two vectors* $(r, s, t) \cdot h(u, v, w)$ *to be* $ru + sv - tw$.

Definition 7.0.2. *We define the h-length of a vector* (r, s, t) *to be* $(r, s, t,) \cdot h(r, s, t,)$.

Definition 7.0.3. *We define two vectors to be h-orthogonal if and only if their h-inner product = 0*

To find conditions on a 3×3 invertible matrix M so that M is a hyperbolic isometry, we must have $M_{1T}CM^{-1} = \Lambda C$ for some $\Lambda \neq 0$:

We multiply by M^T on the left and M on the right, we get $C = M^T \Lambda CM$. Then we get $\frac{1}{\Lambda}C = M^TCM$. If we write P for the first column, Q for the second column, and R for the third column of M:

$$M^TCM = M^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M = \begin{pmatrix} P \cdot HP & P \cdot HQ & P \cdot HR \\ Q \cdot HP & Q \cdot HQ & Q \cdot HR \\ R \cdot HP & R \cdot HQ & R \cdot HR \end{pmatrix} =$$

$$\frac{1}{\Lambda}C \begin{pmatrix} \frac{1}{\Lambda} & 0 & 0 \\ 0 & \frac{1}{\Lambda} & 0 \\ 0 & 0 & -\frac{1}{\Lambda} \end{pmatrix}$$

Hence, we can conclude that the columns must be $h - orthogonal$ to each other to give 0 off the main diagonal. Additionally, the first two columns have the same *h-length*, which is the negative of the *h-length* of the third column. Therefore, a *projective collineation is a hyperbolic isometry* if it can be written as a 3×3 matrix whose columns are $h - orthogonal$ to each other with the first two columns having the same *h-length* and the third column having the negative *h-length* of the first two.

7.1.2 Single Elliptic Geometry as a Sub-geometry

Unlike spherical geometry, single-elliptic geometry has no parallel lines because they all intersect when the antipodal points identify with each other. The absolute conic contains no projective points or lines: $x^2+y^2+z^2 = 0$. The only real solution for this conic is $(0, 0, 0)$ which is not a projective point. Single-elliptic as a subgeometry is characterized by the absence of parallel lines and the inclusion of a finite number of points at infinity. In this context, the points at infinity added to the single-elliptic are homeomorphic to projective space.

7.1.3 Affine and Euclidean Geometries as Sub-geometries

Intuitively, the projective plane can be thought of as an Euclidean plane with the ideal line $[0, 0, 1]$ added. As affine geometry does not account for angle or distance metrics in particular, we can say that affine geometry is a sub-geometry of projective geometry. The absolute conic in this case is $z^2 = 0$. The point is a point not on the absolute conic: $(x, y, 1)$. The line is any other line except the absolute conic: $[a, b, c]$. In all the cases, *affine transformations* are collineations that leave the absolute conic stable.

7.2 Sub-geometries of Projective Space

Euclidean, hyperbolic, and single-elliptic geometries of n- dimensions are sub-geometries of the projective geometry of the same number of dimensions. 3-dimensional hyperbolic space can briefly be considered related to *Minkowski geometry* used in the theory of relativity.

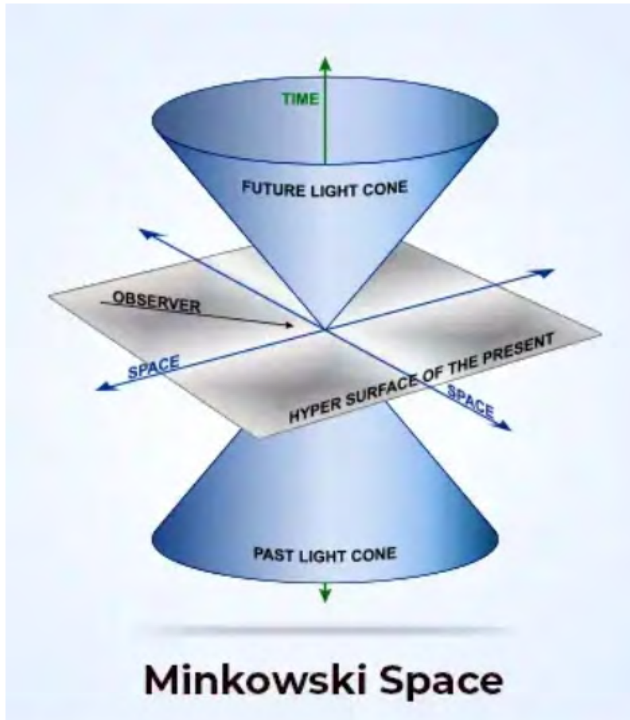

Figure 21: Lorentz Transformations are symmetries of the Minkowski Space

In 3-dimensions, Cayley's absolute conic becomes an absolute quadric sphere, example: ellipsoid, hyperbloid, cone, elliptic parabloid etc. The points of a 3-dimensional hyperbolic space are the points in the interior of the unit sphere $x^2 + y^2 + z^2 - t^2 = 0$, which is taken to be the absolute quadric surface. Two 4×4 matrices represent the same 3-dimensional hyperbolic isometry if and only if they differ by a non-zero scalar.

If we wanted to find conditions on a 4×4 invertible matrix M so that M is a hyperbolic isometry, we must have $M^{-1T}CM = \gamma C$ for some $\gamma \neq 0$. Similar to finding the conditions on $3 \cdot 3$ invertible matrix M so that M is a hyperbolic isometry, we can also find conditions on a $4 \cdot 4$ matrix.

Hence a 4×4 nonsingular matrix represents a hyperbolic isometry iff any two of its columns are h-orthogonal, the first three have the same h-length, and the last column has the opposite h-length.

Minkowski geometry – that contains the *Lorentz transformations* – is a sub-geometry of $\mathbb{P}^4$. More is explained in the section 7.3

8 Applications

Projective geometry has proven it's worth in classical geometry, CAD systems, relativity theory, and other applications.

8.1 Computer Aided Design

Perspective geometry is used in Computer Aided Design through the use of matrices. A generalized version of the matrix used in CAD is as follows:

$$\begin{pmatrix} a & a & a & t_x \\ a & a & a & t_y \\ a & a & a & t_z \\ p_x & p_y & p_z & 1/r \end{pmatrix}$$

Where the $3 \cdot 3$ matrix with a represents affine transformations like rotation, reflection, shear dilation etc. t is used to make translations by adjusting the coordinates of the origin, p is used to adjust the perspective of the lines, and $1/r$ is the scaling ratio. The scaling ration provides additional flexibility lacking in the affine transformation.
For example, a cube with no perspective can be represented by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Whereas the same cube with one-point perspective is represented by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/5 & 0 & 0 & 1 \end{pmatrix}$$

And would be represented in two-point perspective by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/5 & -1/5 & 0 & 1 \end{pmatrix}$$

And finally, in three-point perspective would be:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/5 & -1/5 & -1/5 & 1 \end{pmatrix}$$

As mentioned above, projective geometry is not oriented; however, no perspective view of a real object will ever turn it inside out. The CAD was built to not allow any negative scalars. Despite the complex sounding nature of projective geometry and all of the matrices involved, these calculations are really just a way to model the distortion of space. Video game graphics rely heavily on such calculations. From rendering lifelike landscapes to creating immersive virtual environments, projective geometry is the backbone of modern computer graphics.

8.2 Lorentz Transformations

The *Michelson-Morley experiments* had a significant impact on contradicting the *additivity of velocities.* Regardless of the direction light was sent to, the velocity of the light always stayed the same. Later experiments proved that the speed of light is constant in vacuum. Henri Poincare and Hendrik Lorentz determined the $Lorentz\ Transformations$, a theoretical group of symmetries corresponding to Galileo's *principle of relativity.* About 15 years later, Albert Einstein developed the theory of relativity, where he dropped the measurement of space, time and additivity of velocities.

Suppose Observer A finds the difference in time between two events to be Δt_A and the differences in the x, y, z directions to be $\Delta x_A, \Delta y_A, \Delta z_A$ respectively. Observer B has measurements $\Delta t_B, \Delta x_B, \Delta y_B, \Delta z_B$ respectively.

The Theory of relativity guarantees: $\Delta(x_A^2+y_A^2+z_A^2-t_A^2 = x_B^2+y_B^2+z_B^2-t_B^2)$

Hermann Minkowski developed a 4-dimensional geometry using the above as a distance formula. The Lorentz transformations are the symmetries of Minkowski space, i.e., *special relativity.* The above value is closely related to the equation of the absolute quadric surface for hyperbolic space. Transformations are preserving a constant $k = x_A^2 + y_A^2 + z_A^2 - t_A^2$ clearly leave $x^2 + y^2 + z^2 - t^2 = 0$ stable, and hence are *hyperbolic isometries.*

Additionally, Minkowski geometry needs $(x, y, z, t, 1)$ – an extra coordinate to allow the movement of the origin. Hence Minkowski geometry is a subgeometry of $\mathbb{P}^4$.

A Lorentz transformation is a collineation in $\mathbb{P}^4$, where the bottom row is $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and the upper left 4×4 submatrix is a hyperbolic isometry. The first tree columns having $h - length = \pm 1$ and the fourth column has the $h - length = \pm 1$

9 Acknowledgements

We would like to thank Mary, Marisa, and the MIT PRIMES Circle program for giving us this opportunity for us to go beyond what we imagined we could do. We are also deeply grateful to our mentor Luis A. Modes for teaching us so much geometry, as well as guiding and supporting us through this amazing journey, while always bringing interesting and fun anecdotes to each meeting. We are extremely thankful for everyone involved in this project for their unwavering support.

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