# Seifert Surfaces

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**Abstract**

This paper presents a comprehensive exploration of some of the key concepts in knot theory. It is organized into three main sections: an introduction to knots, followed by a discussion on surfaces, and concluding with an examination of Seifert surfaces. Beginning with a foundational overview of knots as closed loops in space, the paper explains their characteristics and classification methods. Subsequently, attention is directed towards surfaces. In this section, essential properties such as orientability, genus, and compactness are explored in detail. The Euler characteristic is then introduced as a fundamental invariant for surfaces, providing insights into their topology and aiding in surface classification. Finally, the paper delves into Seifert surfaces. In short, these are orientable surfaces closely tied to a given knot or link. We investigate their construction and properties through illustrative examples. Additionally this is when the significance of Seifert surfaces in understanding knot topology and calculating knot invariants is underscored. By sequentially addressing these three interconnected topics, the paper offers a comprehensive overview of knot theory, highlighting its practical applications and signficant role in topology.

# Contents

# 1 Introduction

Knot theory is the study of mathematical knots. It is a subsection of topology, or the study of properties of spaces that are invariant under constant deformation.

**Definition 1** (Knot)**.** A **knot** is a closed loop of string. ◇

It has no loose ends and does not intersect itself anywhere. In mathematics, knots are thought to have no thickness and are seemingly made of easily deformable rubber. A picture of a knot is called a projection, which supports us in visualizing the knot. You can imagine taking a 3-D knot and placing it onto a 2-D plane.

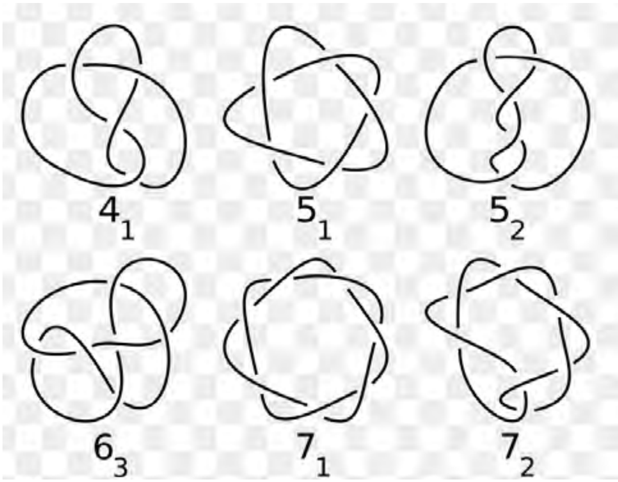


Figure 1: Different types of knots.

One characteristic of the projections of knots is the number of crossings they have. These are the places where a knot crosses itself. The simplest of knots is the unknot, also known as the trivial knot. It is completely unknotted, meaning it has zero crossings, and looks similar to a circle. The second simplest knot is the trefoil, which has three crossing points and can be described as the outline of three overlapping rings.

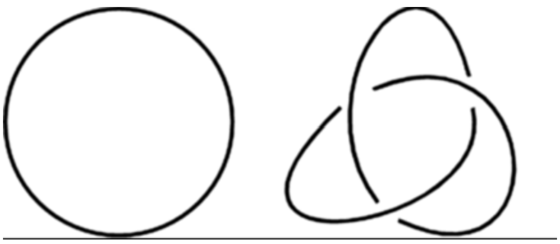

Figure 2: Unknot and Trefoil.

So now, you might be wondering what defines a knot. Even though the number of crossings can be used to describe a knot, they are not an invariant factor. For example, the following projections are all the unknot.

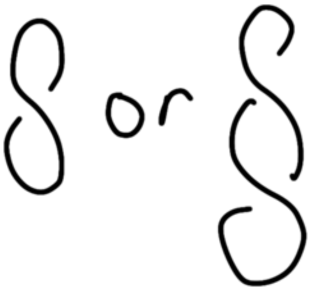


Figure 3: Projections of the trivial knot.

However, they all have more than zero crossings. They can easily be twisted or untwisted to get more or less crossings. Even though it is unchallenging that the previous images are all projections of the same knots, the majority of the time it is not that straightforward.

Figure 4: A knot with numerous crossings, taken from [Ada94].

This knot looks vastly different from the previous knot, but it is actually just another projection of the unknot. It would be inconvenient and difficult without a uniform process. So, how can you tell if two knots are the same or not? Well, we first have to understand surfaces to do so.

# 2 Surfaces

So what is a surface? A surface is like the glaze on a donut or the paint surrounding a mug. In these examples it is important to point out that a surface is not a solid object. However it can be thought of as the outermost layer of a physical object, hence the glaze on the donut and not the actual donut. Two of the most common surfaces are the sphere and the torus, which can be seen below. Another important characteristic of surfaces is their genus. In simple terms, a genus is just the number of holes a surface has. For example a sphere has a genus of 0 and a torus has a genus of 1.

**Definition 2** (Surface)**.** A **surface** is a shape that can have any point with a disk surrounding and containing it. ◇

This is a common and required property for all surfaces. In Figure 5 each point of the surface has a disk surrounding it. On the other hand, in Figure 6 these objects fail to be surfaces because there is at least one point that does not contain a disk.

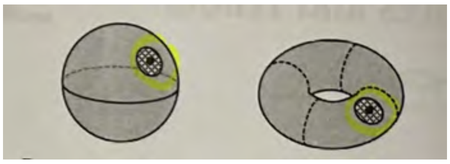

Figure 5: These are surfaces, the sphere and the torus, taken from [Ada94].

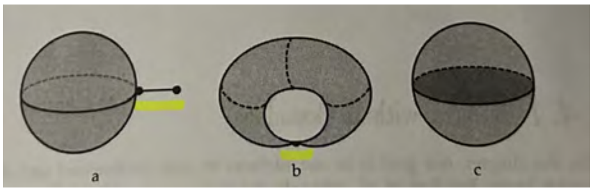


Figure 6: These are NOT surfaces, taken from [Ada94].

## 2.1 Isotopy and Homeomorphism

Additionally, we think of all surfaces as being made of rubber, meaning they are deformable. So for example, a rubber sphere is equivalent to a rubber cube because it could be contorted into one without any cutting or pasting. Likewise, if you can get from one surface to another using a rubber deformation, also known as an isotopy, then they are equivalent and therefore isotopic surfaces. Figure 7 shows three isotopic surfaces, revealing they are all equivalent. Figure 8 shows surfaces that are not isotopic because you cannot get from the first surface to the second without cutting and pasting an additional torus.

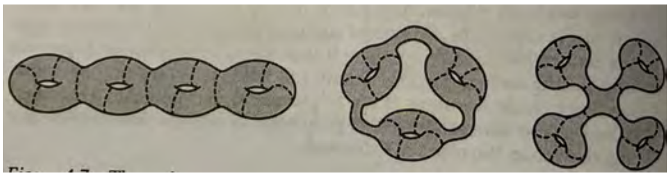

Figure 7: Isotopic surfaces, taken from [Ada94].

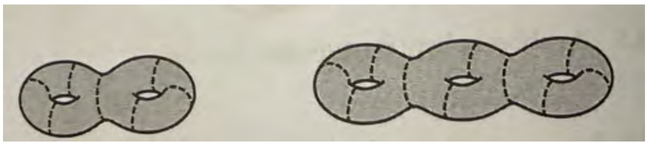

Figure 8: Non-isotopic surfaces, taken from [Ada94].

Two surfaces can also be homeomorphic to one another. For an isotopy, the surface is currently sitting in 3D space in some way. Isotopies preserve the way a surface lives in 3D space which relates to why knot theory exists. If we worked up to homeomorphism, we'd just get that everything is a circle. However, a homeomorphism between two surfaces is basically a continuous way to assign every point of one surface to another point in the other surface.

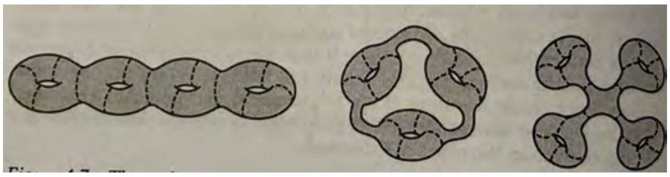

Figure 9: Homeomorphic surfaces, taken from [Ada94].

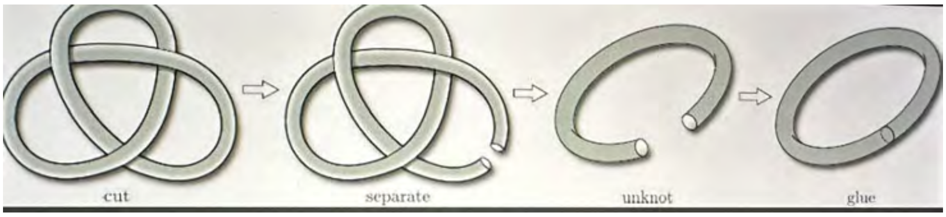


Figure 10: The process

## 2.2 Surface Characteristics

Now let's talk about important characteristics of surfaces. An important part of the surface is the boundary. The surface and boundary of a plane are different. The definition of boundary is a line that marks the limit of an area. For example, a normal torus doesn't have any boundary, but if the torus has a surface missing that limits the half disk of the surface, it is the boundary.

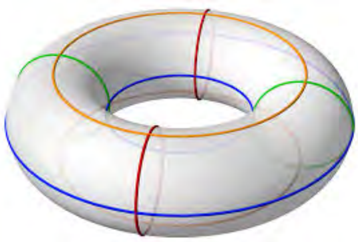

Figure 11: Torus without boundary

Figure 12: Torus with boundary

One quality is the orientability of a surface. An orientable surface is defined as one that has two distinct sides. Imagine painting two sides of a surface different colors, let's say red and blue. If the red paint never touches the blue paint, except along the boundary, the surface is orientable.

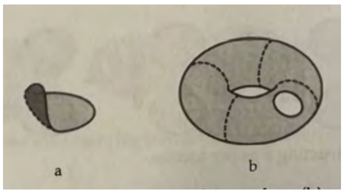


Figure 13: Orientable surfaces, a disk and a torus with boundary, taken from [Ada94].

In contrast, a nonorientable surface is one where both sides are connected. The surface cannot be painted two different colors because it is all the same side. As you can see in the diagram below, the surface on the left only has one side, whereas the surface on the right has two.

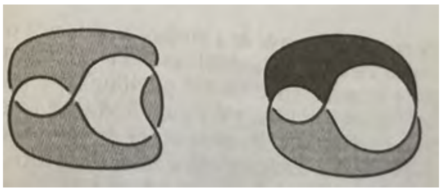

Figure 14: Nonorientable versus Orientable surfaces, taken from [Ada94].

An example of one of the simplest nonorientable surfaces is the Mobius band. Even though it looks to have two defined sides, the twist in the Mobius band causes it to only have one side. To visualize it, imagine you started painting one side of the band blue. As you continue painting the surface, you would see that the whole object has been painted blue.

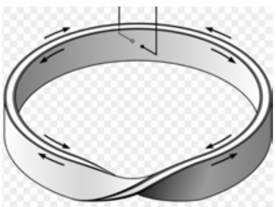

Figure 15: Mobius band

Additionally, any band with an even number of twists is orientable, whereas any band with an odd number of twists is nonorientable. This is because any band with an even number of twists can be painted with two different colors as well. For example, in the drawing below, we can see that there is a green and a yellow side for a band with two twists. Later on we will explain why a band with an even number of twists is just a different embedding of a cylinder, and a band with an odd number of twists is a different embedding of the Mobius band.

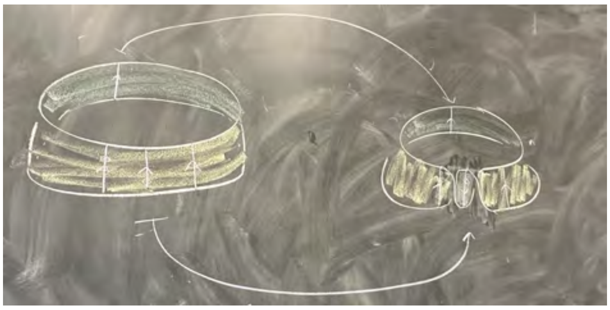

Figure 16: A two-twist band in an embedding of a Cylinder

Another element of surfaces is their compactness. A compact surface is one that is not infinitely large. It can be enclosed inside a finite object. For example, imagine placing a surface in 3D space. No matter how we place the surface, we can always encompass that surface in some sort of ball if it is compact. It is important to mention that compactness is a surface invariant. This means that any surface that is homeomorphic to a compact surface is also a compact surface. The genus $g$ surface is not compact because it is infinitely large and cannot be contained in a ball in 3D space.

Figure 17: Compact Surface

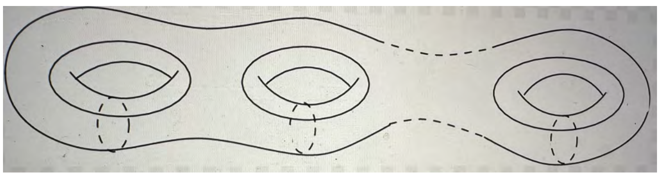

Figure 18: Non-compact Surface

# 3 Euler Characteristic

The Euler characteristic is a fundamental invariant used to distinguish between different surfaces. One way to compute the Euler characteristic is through triangulation. Triangulation involves decomposing a surface into a collection of triangles. The Euler characteristic $\chi$ is then given by the formula:

$$\chi = V - E + F \tag{3.1}$$

where $V$ is the number of vertices, $E$ is the number of edges, and $F$ is the number of faces (triangles) in the triangulation. An important property of the Euler characteristic is that it remains constant regardless of how the surface is triangulated, as long as the surface remains topologically equivalent.

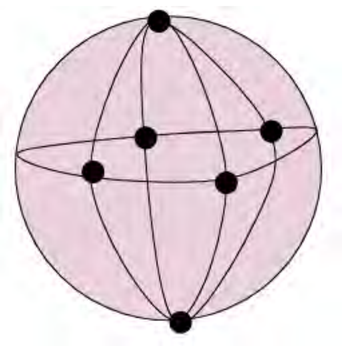

Figure 19: Triangulation of a Sphere 1

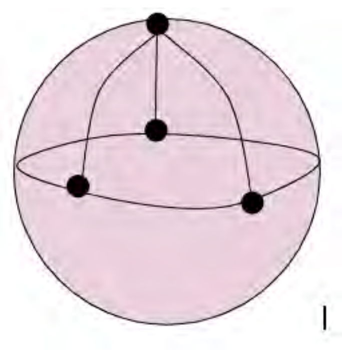

Figure 20: Triangulation of a Sphere 2

From Figures 19 and 20, we can observe that the vertices, edges, and faces differ between the two triangulations of a sphere. For the first triangulation, we have:

$$V = 6, \quad E = 12, \quad F = 8$$

Using the formula for the Euler characteristic, we get:

$$\chi = V - E + F = 6 - 12 + 8 = 2$$

For the second triangulation, we have:

$$V = 4, \quad E = 6, \quad F = 4$$

Applying the formula again, we get:

$$\chi = V - E + F = 4 - 6 + 4 = 2$$

This consistency confirms that the Euler characteristic is invariant under different triangulations of the same surface, in this case, a sphere.

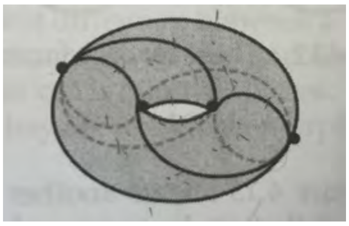

Figure 21: Triangulation of a Torus, taken from [Ada94]

Next, let's consider the torus. The triangulation shown in Figure 21 has:

$$V = 4, \quad E = 12, \quad F = 8$$

Calculating the Euler characteristic, we get:

$$\chi = V - E + F = 4 - 12 + 8 = 0$$

This demonstrates that the Euler characteristic for a torus is different from that of a sphere. Hence, we can distinguish between a sphere and a torus using their Euler characteristics. The sphere has an Euler characteristic of 2, while the torus has an Euler characteristic of 0.

In summary, the Euler characteristic is a powerful tool in topology that helps us differentiate between various surfaces by providing a numerical invariant that remains constant under different triangulations.

# 4 Euler Characteristic Via Genus

Over time, mathematicians have developed a more efficient way to determine the Euler characteristic of a surface. The Euler characteristic $\chi$ of a surface can be computed using the formula:

$$\chi = 2 - 2g \tag{4.1}$$

where $g$ is the genus of the surface. We will prove this claim by computing the Euler characteristic for a surface of genus $g$.

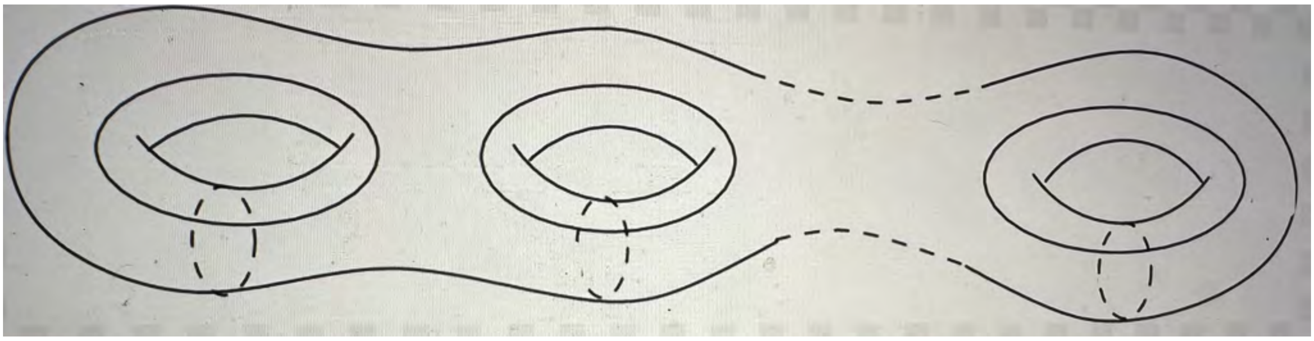

Figure 22: Genus $g$ surface

We use induction to prove this formula. The first step is to prove the base cases, which are $g = 0$ and $g = 1$.

A sphere is a surface with a genus of 0. If we plug 0 into the equation, we get an Euler characteristic of 2. This matches the Euler characteristic computed using triangulations of a sphere.

$$\chi(0) = 2 - 2(0) \tag{4.2}$$

$$\chi(0) = 2 - 0 \tag{4.3}$$

$$\chi(0) = 2 \tag{4.4}$$

A torus is a surface with a genus of 1. When we plug 1 into the equation, we get an Euler characteristic of 0. This is consistent with the Euler characteristic obtained from the triangulation of a torus.

$$\chi(1) = 2 - 2(1) \tag{4.5}$$

$$\chi(1) = 2 - 2 \tag{4.6}$$

$$\chi(1) = 0 \tag{4.7}$$

Having established the formula for $g = 0$ and $g = 1$, the next step is to assume it is true for $g$. We will create a triangulation for the genus $g$ surface and then extend it to $g + 1$ by adding an additional handle. This process will help us prove the formula holds for any $g$.

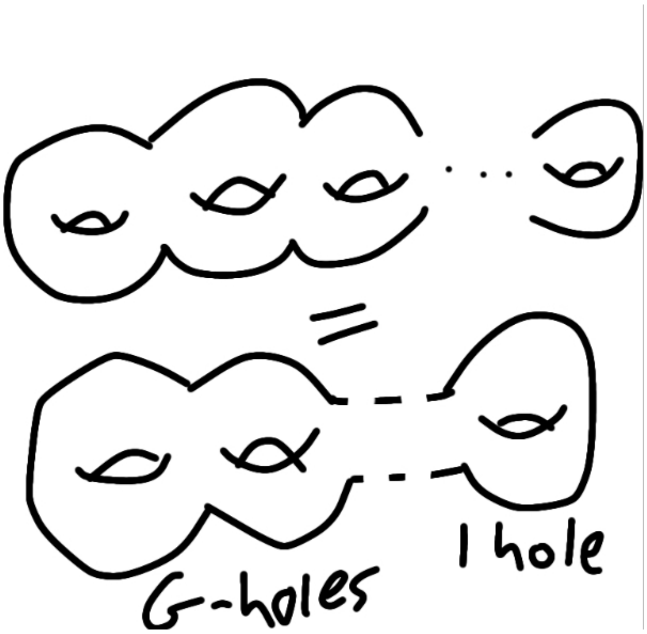

Figure 23: Genus $g$ surface

Figure 24: Genus $g + 1$ surface

Assuming the formula holds for a surface of genus $g$, we have:

$$\chi(g) = 2 - 2g$$

When we add an additional handle to the surface, the genus increases by 1. The new surface has genus $g + 1$. According to the formula, the Euler characteristic for the new surface is:

$$\chi(g + 1) = 2 - 2(g + 1)$$

Simplifying this, we get:

$$\chi(g+1) = 2 - 2g - 2 \tag{4.8}$$

$$\chi(g+1) = 2 - 2g - 2 \tag{4.9}$$

$$\chi(g+1) = 2 - 2g - 2 \tag{4.10}$$

$$\chi(g+1) = -2g \tag{4.11}$$

Thus, the Euler characteristic for the genus $g+1$ surface is consistent with the formula $\chi = 2 - 2g$. By the principle of mathematical induction, the formula holds for all $g$.

# 5 Seifert Surfaces

A Seifert surface is an orientable surface whose boundary coincides with a given knot or link. These type of surfaces can be used to study the properties of the associated knot or link. Many knot invariants can be easily calculated using a Seifert surface.

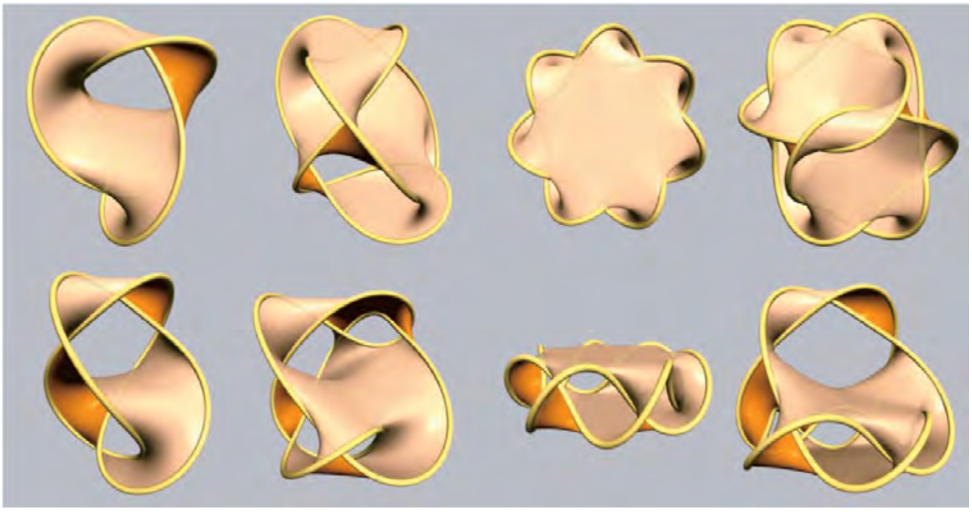

Figure 25: Examples of Seifert Surface

In mathematics there is a specific algorithm we can use to attain a Seifert surface. It can be seen in the image below. However, let's delve into each step to gain a deeper understanding. We will use Trefoil knot as our example.

**Step 1: Fix an Orientation**

The first step involves fixing an orientation for the knot. This orientation serves as a consistent direction along the knot's path, allowing for systematic resolution of crossings in subsequent steps.

Figure 26: Orient your Knot

**Step 2: Resolving Crossings**

Now we untangle the knot by adjusting each crossing to align with the chosen orientation. This involves rearranging the strands so they no longer overlap, maintaining a consistent directionality throughout the knot.

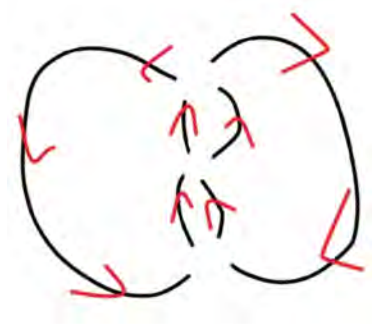

Figure 27: Resolve Crossings

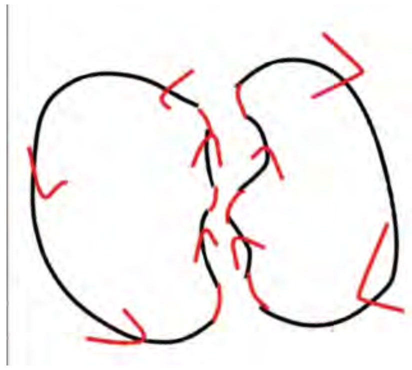

Figure 28: Further Resolving Crossings

**Step 3: Placing Disks**

Step 3 involves placing disks inside the spaces created by resolving the crossings. Each disk represents a bounded region within the knot. By enclosing these regions with disks, we define the necessary boundaries.

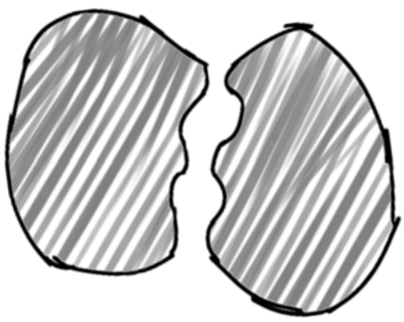

Figure 29: Fill with Disks

**Step 4: Connecting Disks**

Step 4 is about connecting the disks. We add bands between the disks associated with resolved crossings. These bands act like bridges, linking adjacent regions together. By adding bands, we ensure that the Seifert surface is connected, providing a consistent representation of the knot's structure.

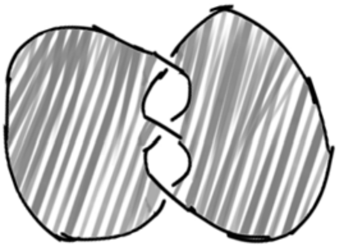

Figure 30: Connect with Bands

After using Seifert's algorithm the resulting surface is a single twisted band that loops around and through itself. This forms a single connected component which has has a genus of 1. If you recall from the previous pages, this means that the surface has one hole. To better understand the surface, imagine you have a torus. Now, think about cutting out a disk from this torus. The remaining surface is a punctured torus, which can be used as a Seifert surface for a trefoil knot.

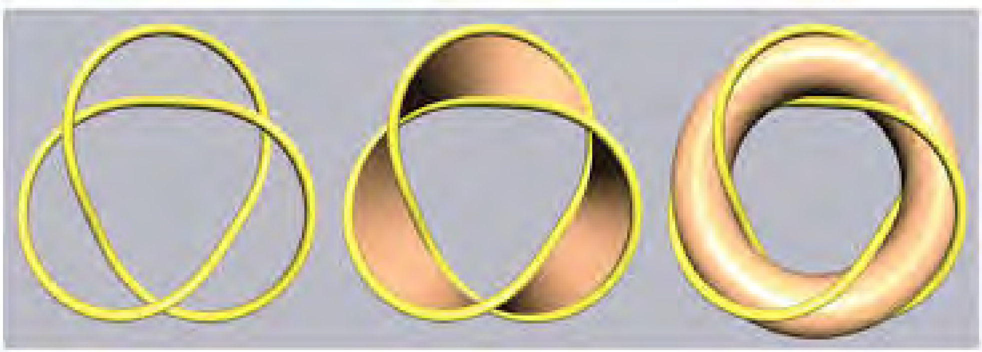

Figure 31: Seifert Surface of the Trefoil Knot

In conclusion, this paper has provided a comprehensive overview of knot theory, starting with an introduction to knots as closed loops in space and progressing through discussions on surfaces and Seifert surfaces. We have explored fundamental concepts such as orientability, genus, and compactness, and investigated the Euler characteristic as a powerful invariant for surface classification.

We hope that this paper has provided a glimpse into the rich mathematics of topology for you and we hope that you have enjoyed exploring the wonders of knot theory with us. We look forward to continuing to unravel the mysteries of knots. Futhermore, we are certain that by uncovering new insights and discoveries we will be able to progressively enrich our understanding of the world around us and you should as well!

# 6 Image Credits

- Seifert Surfaces for Knots and Links. Retrieved from https://thatsmaths.com/2015/01/08/seifert-surfaces-for-knots-and-links/
- Homology basis curves of a genus $g$ surface. Retrieved from https://www.researchgate.net/figure/Homology-basis-curves-of-a-genus-g-surface_fig3_244205582
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- From left to right: the unknot, trefoil knot, figure-eight knot, and cinquefoil knot. Retrieved from https://www.researchgate.net/figure/From-left-to-right-the-unknot-trefoi fig1_336858603
- Torus. Retrieved from https://en.wikipedia.org/wiki/Torus
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[Ada94] C.C. Adams, *The knot book*, W.H. Freeman, 1994.