Game Theory

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1 Introduction

Combinatorial games, a branch of game theory, allows us to further understand the topic of decision making and uses simple games to work out different strategies that result in various outcomes. It introduces rational thinking where a player makes decisions based on the outcomes it will bring them. This method of thinking can be applied to larger fields such as economics and finance for parties to maximize their own income.

In this paper, we will explore game theory in the branch combinatorial games through a broad view. This will be done through the aid of the game Pick-Up-Bricks and game trees. We will then take a deeper look into a category of combinatorial games, *Normal-Play Games*. To understand how they work, we will look at the example of Cut-cake. We will continue on to talk about the four different types of games, sum of games, and the properties of positions in normal play games. Afterwards, we will discuss impartial games and introduce the MEX Principle by analysing the game Nim. Lastly, we will apply the MEX Principle to the game of Shade.

2 Combinatorial Games

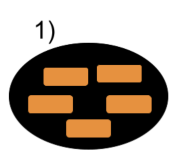
Definition 2.1. *Combinatorial games* are two player games that have:

- A set of positions players can move to
- A set of move rules that each player can use to move to a position
- A set of winning positions, which are terminal positions in the game

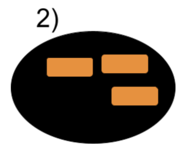
Some commonly known combinatorial games are Chess, Checkers, Tic-Tac-Toe, etc. Throughout our exploration of combintorial games, we will refer to Louise and Richard as the two players. They will also be referred to as L and R or left and right. There is no player who is immediately designated to go first.

Example: Pick-Up-Bricks is a game in which there is a pile of n bricks. When it is their turn, each player has the chance to remove either one or two bricks. Whichever player removes the last brick in the pile will be the one to win.

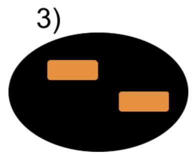
Let's say we are given a pile with five bricks(as shown below). Louise and Richard will play a sample game:



Louise may move first and remove two bricks, so the pile will be left with three bricks:



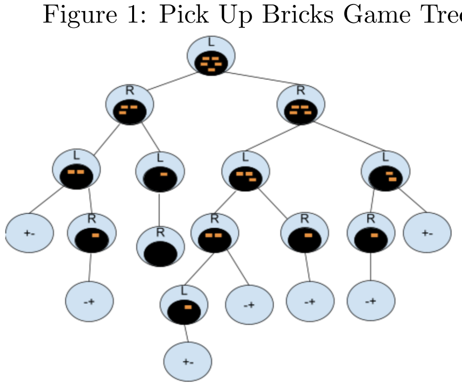
Richard may take his turn and remove only one brick. There will be two bricks left:



Louise can win the game by taking the remaining two bricks.

The possible moves and results of a simple game like Pick-Up-Bricks can be represented by a game tree. A Game Tree depicts every possible sequence of moves players can make in a game. Each branch represents a choice that players may make and the circular node represents the position to which it moved. Below is a sample game tree for a Pick-Up-Bricks game with three bricks:

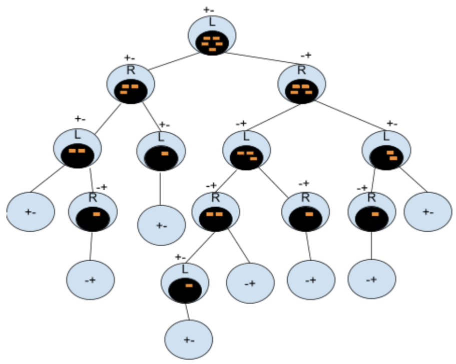
Figure 1: Pick Up Bricks Game Tree



The symbols +- and -+ represent the winner of the game. When the + is on the left, Louise wins, and when it is on the right, Richard wins. The notation for a tie would be 00.

In a given game, if it is Louise's turn, she would rationally pick an option that leads her to win over one that leads Richard to win or tie. This would be the same for Richard as well. Using this and working backwards from the terminal nodes, we can now label the game tree based on which node the players would pick:

Figure 2: Pick Up Bricks Game Tree Labelled



To clarify, the symbols above the nodes represent who has the winning strategy at that given position. Since there is +- across the first node, which is the starting position of the game, the game when Louise has the first move would have a winning strategy for Louise.

Theorem 2.2. *Zermelo's Theorem: Every Game tree is one of the three following types:*

- *Louise has a winning strategy(+-)*
- *Richard has a winning strategy(-+)*
- *Both players have drawing strategies(00)*

Proof. We will prove this by induction. We will start with a game tree that is made up of only one node as the base case. In this case the winner is already determined because there are no moves to make so it will be type +- if Louise goes first and type -+ if Richard does. As the inductive step, we will also assume the theorem holds for all trees with less than n rows of nodes. Consider a game tree with $n$rows. All the nodes below the root node will form trees with n 1 or fewer rows. According to our assumption, each of these sub-trees have a winning strategy for Louise, a winning strategy for Richard, or a drawing strategy. For the sake of simplicity we will say Louise will make the first move; however, if Richard was to go first, the proof is the same in essence. In addition to this, there are three cases to consider when observing the nodes of the second row:

1. At least one node contains a winning strategy for Louise.
2. All nodes have a winning strategy for Richard.
3. There are no nodes for a winning strategy for Louise; however, there is at least a drawing strategy for both players.

For the first case, Louise would pick a winning strategy for herself, so then the game tree with n rows would have a winning strategy for Louise.

In the second case, Louise must pick a node with a winning strategy for Richard, as that is her only choice. This gives the overall game a winning strategy for Richard.

Lastly in the third case, Louise would prefer a drawing strategy over a winning strategy for Richard, so the game would have a drawing strategy for both players.

□

Games do not just have winning strategies just for Louise and Richard. There are two other situations as well. The Next Player(N) may have a winning strategy and the Previous Player(P) may have one as well.

Example: in the example of Pick-Up-Bricks with five bricks, the Next Player has the winning strategy. In the game tree diagram (1), Louise was pictured going first, but what if it was Richard who was first instead? It can be concluded that whichever player goes first has the winning strategy.

3 Normal Play Games

Definition 3.1. A *Normal Play Game* is a game that:

- Consists of a set of positions together that says, for each position, the positions that Louise and Richard can move to.
- The last player to move is the winner
- There are two types of normal play games, Partisan and Impartial.

Some examples of normal play games are Cut-Cake, Chop and Chomp. Games we will go into depth with as we go on.

Example: Cut-Cake is a game where each game position consists of uncut pieces of cake. Louise makes vertical cuts while Richard makes horizontal cuts, with each player having a different set of moves. There is a sharp contrast to Pick-Up-Bricks where each player had the same set of moves. The last player to make a move wins.

3.1 Types of Positions

Important concepts in normal play games are position notation and position type. Both of these concepts allow a player to determine a strategy that ends in a victory for them. Position notation is an equation that states the possible moves that each player can make, it is written in the format $\gamma = f\alpha_1, \alpha_2, \alpha_3, ...j\beta_1, \beta_2, \beta_3, ...g$, where $\alpha_1, \alpha_2, \alpha_3, ...$ represents the positions Louise can move to and $\beta_1, \beta_2, \beta_3$ represents the positions Richard can move to. Let us consider playing a normal play game starting from a certain position. If Louise plays first then according the definition of a normal play game, either Louise or Richard will win since normal play games cannot end in a draw. Similarly in the case of Richard moving first, one of the two players is guaranteed to have a winning strategy. And so we get the classification of positions of a normal play games; the following is a corollary of Zermelo's theorem.

Corollary 3.2. *Every position in a normal play game is one of the following types:*

1. *Type L: Louise has a winning strategy no matter who goes first.*
2. *Type R: Richard has a winning strategy no matter goes first.*
3. *Type N: the next or first player has a winning strategy.*
4. *Type P: the previous or second player has a winning strategy.*

Determining the type of a position is crucial to proving whether a player has a winning strategy or not. Lets consider a game position γ in a normal play game and say that Louise starts at said position and has a move to position β that is of type L or P. By definition of each type, Louise has a winning strategy.

Proposition 3.3. *If $\gamma = f\alpha_1, \alpha_2, \alpha_3, ...j\beta_1, \beta_2, \beta_3, ...g$, the type of γ is given by the chart below:*

	some β_j *is type* R *or* P	*all of* $\beta_1, \ldots, \beta_n$ *are types* L *or* N
some α_i *is type* L *or* P	N	L
all of $\alpha_1, \ldots, \alpha_m$ *are types* R *or* N	R	P

Figure 3: determining the type of positions, figure taken from [1].

This proposition gives a direct way to determine a game position's type. We will illustrate this procedure by finding the types of some positions in a game of Cut-Cake.

In the figure below, whoever makes the first move gives only the opponent the ability to move. In other words Richard can only bring the position to a state of Type L and Louise can only bring the position to Type R. Positions like these would be classified as type P as they give the second player a winning strategy.

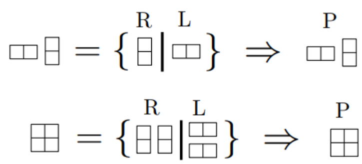
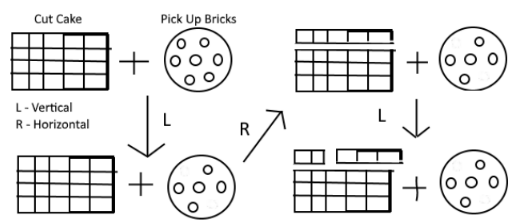


Figure 4: cut-cake positions example, figure taken from [1].

3.2 Sum of Positions

Definition 3.4. If α and β are positions in a normal play game, then $\alpha + \beta$ is a new position made up of the elements α and β. On each turn, a player moves from $\alpha + \beta$ to either $\alpha' + \beta$ or $\alpha + \beta'$.



Since we know how to add positions we need to understand how to develop a winning strategy for sums of positions.

Example: Let α be a Type R game and add it to a Pick-Up-Bricks game.

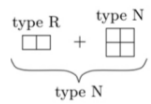
Figure 5: Game of α + Pick up Bricks

Since the Pick-Up-Bricks game is Type P, it doesn't change Richard's strategy and he can ignore the game entirely until Louise makes a move in that component. To which he will respond in that said component. As the game is normal-play, the last move wins so the next player plays in the other component.

Proposition 3.5. *If β is type P, then both α and $\alpha + \beta$ are the same type.*

This proposition is mostly intuitive. In order to prove this, we must prove that if either player has a winning strategy going first or second in the game α, then they must also have a winning strategy when the game β is added in. We will use an example of Louise playing second. Say Louise has a second player winning strategy in the game α. Since Richard is going first, he must make the next player move in either α or β. Louise may respond to this by making the corresponding move for the previous player in accordance to the winning strategy. This leads Louise to win both games.

How do we figure out the type of summed games? Well the game Domineering might help. Domineering is a normal-play game that is played by using some squares on a rectangular array. Louise places down a 2 1 domino over two unoccupied spaces, while Richard on his turn places a 1 2 domino over two unoccupied spaces. The last player to make a move wins. Consider the domineering positions in the figure below, one of the positions is type N since both players have moves available to them.



But once one player makes a move the other has no move. The other position is a type R because only Richard can make a move. If Richard were to make a move first then he would win the game since there are no longer any available moves for Louise, hence the game is Type R. If Louise were to go first then there would no longer be any available moves for Richard and she would win the game. Even so, not all N+R games are type N, if the game were to have a 2 2 position and a 1 4 position then the game would become type R because the 1 4 array has a greater advantage for Richard, if Louise were to make the first move then Richard can divert to the 1 4 array which would give him the last move, where he will win.

Definition 3.6. Two positions α and α' in (possibly different) normal play games are *equivalent* if for every position β in any Normal Play Game, the two positions $\alpha + \beta$ and $\alpha' + \beta$ have the same type.

If two positions are equivalent, they will have the same type. There are some properties of equivalence that are fundamental to know. If α, β, γ are positions in a normal play game then:

- α α (reflexivity)
- α β implies that β α (symmetry)
- α β and β γ implies that α γ (transitivity)
- $\alpha + \beta$ $\beta + \alpha$ (commutativity)
- $(\alpha + \beta) + \gamma$ $\alpha + (\beta + \gamma)$ (associativity)

Another thing to note is if $\alpha = \alpha'$, then α and α' have the same type. However, when two positions have the same type that does not necessarily mean that they are equivalent.

When determining whether two or more positions are equivalent, type P acts as the number zero. If β is type P then $\alpha + \beta = \alpha'$. Any position added with a type P position results in a sum of positions equivalent to that of the starting position. In other words, if β is type P then $\alpha + \beta$ and α' have the same type.

4 Impartial Games

Impartial games are games where each player has the same set of moves. For example, Pick-Up-Bricks is an impartial game since the result is only dependent on who goes first, not who they are. We will be looking at another, more complicated impartial game, Nim.

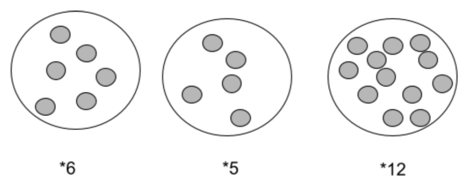
4.1 Nim

Nim is a game with multiple piles of stones and on each move, a player can choose a pile. They can then choose to remove from one up to all of the stones in a pile. The last player to move/empty all the piles is the winner. Each pile in Nim can be thought of as its own game, so the multiple piles can be though of as adding games together.

Before we prove the winning strategy for the game, it is necessary to mention a couple of things:

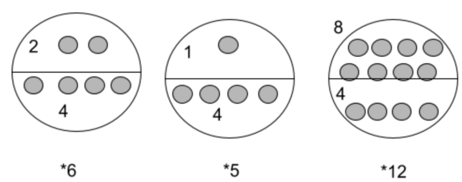
- Each power of two is one more than the sum of the previous powers of two
- Every natural number has a binary expansion
- The binary expansion of a natural number is unique

We will start by assigning each pile in Nim with a number in accordance with the number of stones in it. This will be called a *nimbers.* Nimbers can also be used to refer to a game with the integer components' amount of stones.

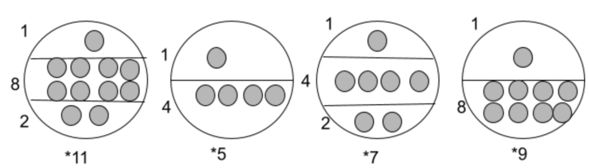


We will then split each pile into sub-piles based on its binary expansion. Define balanced as having an even number of each size of sub-pile.

Figure 6: Nim Unbalanced



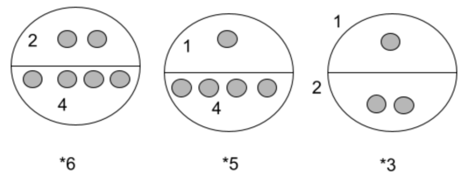
The game below is not balanced since there is one sub-pile of one, one sub-pile of two, three sub-piles of four, and one sub-pile of eight. At least one(in this case all) of the amounts of sizes of sub-piles are odd.



The game above is balanced because there are four sub-piles of one, two sub-piles of two, two sub-piles of four, and two sub-piles of eight. All of these numbers are even, so the game is balanced.

Now we will talk about moving from an unbalanced to balanced position. First, we must find the largest power of two,2^n, in the binary expansion for which there is an odd number of piles. In the figure below with the unbalanced Nim game(6) this would be eight. We may pick any pile that has a size 2^n. Then, we would arrange this pile such that all sub-pile sizes that there are an odd amount of have an even amount.

This works because each power of two is one more than the sum of the previous powers of two. Here is an example on the unbalanced figure(6):



We have taken the pile on the right. Since, excluding the rightmost pile, there are two fours, a one, and a two. We want even numbers of all pile, so that means we need another pile of one and two. Therefore, we would remove nine stones from the rightmost pile, so we can divide the remaining three into a pile of one a two, which balances the game.

Whenever a move is made from an unbalanced position it can be turned to a balanced position and when a move is made from a balanced position, it must be unbalanced. The winning position of Nim is a balanced position since there are no sub-piles in each pile. Zero is an even number, so that means it is balanced. This is important because if a player first makes a move from an unbalanced position, they can always move to a balanced position on their turn while their opponent always moves to an unbalanced position. This would mean that when starting with a balanced game, the previous player would have a winning strategy and when starting with an unbalanced game, the next player would have a winning strategy.

To make it easier to find the winning strategy of a Nim game algebraically, we will introduce the concept of Nim-sums. Nim-sums are basically adding games of Nim together. A Nim-sum can be calculated by breaking each number of the sum into their binary expansion. Each power of two is paired with the same number and excluded. The leftover numbers can be summed to get the Nim-sum:

$$11 \quad 5 \quad 7 \quad 9$$
$$(\cancel{8}+\cancel{2}+\cancel{1}) \quad (\cancel{4}+\cancel{1}) \quad (\cancel{4}+\cancel{2}+\cancel{1}) \quad (\cancel{8}+\cancel{1})$$
$$11 \quad 5 \quad 7 \quad 9 = 0$$

For the unbalanced Nim game(6) the Nim-sum would be 15 because:

$$6 \quad 5 \quad 12$$
$$(\cancel{4}+2) \quad (\cancel{4}+1) \quad (8+4)$$
$$6 \quad 5 \quad 12 = 1+2+4+8 = 15$$

It is important to note that the Nim-sum is equal to zero if and only if the Nim game is balanced. This makes sense because the requirements for a balanced game is that there needs to be an even number of each size of group, which would cancel out when calculating the Nim-sum. From this we can say that if the Nim-sum is zero, the game has a winning strategy for the previous player, as it is balanced. If the Nim-sum is not zero, then there will be a winning strategy for the next player since the game would not be balanced.

4.2

Definition 4.1. The MEX or Minimal Excluded value for a set of nonnegative integers is the smallest number not included in that set.

Example: Given *set* $a = \{0, 1, 2, 5, 6, 7\}$, the MEX would be 3, as it is the smallest nonnegative integer not in the set.

Theorem 4.2. *The MEX Principle: Suppose* $a = \{a_1, a_2, a_3, ...\}$ *where* $a_1, a_2, a_3, ...$ *are the positions a player can move to from the position* a *in an impartial game. If each of* $a_1, a_2, a_3, ...$ *is equivalent to nimbers* b_1, b_2, $b_3, ...$, *then the position* a *is equivalent to the nimber* b *where* b *is the MEX of the set* $\{b_1, b_2, b_3, ...\}$

Proof. If a is equivalent to b, then each will have the same binary expansion, meaning the Nim-sum of the two games would be zero because each size pile can correspond to the same size pile in the other game canceling them all out. This would also mean the sum of games $a+ b$ would have a previous player winning strategy.

We will be looking at two cases: one where the first player moves in the game a and then in the game b.

Case 1: The next player moves in game a from the position of a to a_f. a_f cannot be b because it is the MEX in the set of possible positions to move to from a. Therefore, the game cannot be balanced as the not all the sizes of sub-piles will match up. When the previous player moves after, they are able to balance the game, so they have the winning strategy.

Case 2: The next player moves in game b to position b^{θ}. Position b^{θ} must be in the set $a = fa_1, a_2, a_3, ...g$ since b is the MEX, so is the smallest number not in set, so b^{θ} is in the set, (b^{θ} is smaller because the possible moves in Nim game reduce the amount of stones in the pile). Therefore, there must be an a_s that is equivalent to b^{θ}. The previous player may make a move in game a to the position a_s, which would balance the game, meaning the previous player a winning strategy in this case as well.

Since both cases work out, the MEX, b would always be equivalent to a. □

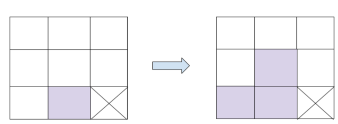
To apply this to all impartial games, we must prove that every position in an impartial game can be assigned a nimber. This is called the Sprague-Grundy Theorem

Theorem 4.3. *Sprague-Grundy Theorem: A given position in any impartial can be assigned a nimber.*

Proof. We will prove this by induction. In the case where the game is at a winning position, the previous player would have the winning strategy since the next player has no moves to make. This will be the base case. We can proceed to assume that this theorem holds true for all positions $f\alpha_1, \alpha_2, \alpha_3...g$ that are possible to move to from α with any number of moves. So, each element in the set $f\alpha_1, \alpha_2, \alpha_3...g$ would have an equivalent nimber: $f\ \alpha_1,\ \alpha_2,\ \alpha_3...g$. This set of nimbers(Nim games) would have a MEX of β, which would in turn be equivalent to the position α. □

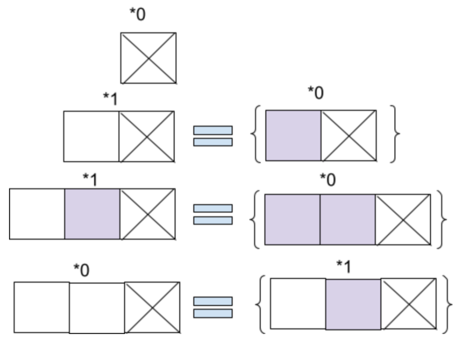
4.3 Shade

Shade is an original game where you are given an $m\ \ n$ board with the bottom right corner is filled, and the objective of the game is to fill in the board. On their turn, each player may chose a square that is already shaded in(if it is the first turn of the game, they must chose the bottom right corner). From this square, they are allowed to shade in any of the squares touching this square, not including the diagonals. An example move is as follows:



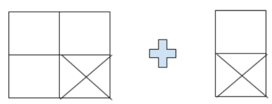
This game is impartial, so we may use the MEX principle on it. The base case is a 1   1 board where the only square is already shaded in. Since the next player cannot make a move, it is the previous player's win. Therefore, we can assign it a nimber of *0. Given a filled 1   2 board, there are no moves that the next player can make, meaning it is the previous player's win, so it's nimber is 0. From an unfilled 1   2 board, the only move would result in a filled 1   2 board. Using the MEX principle, we can say that a 1   2 board has a nimber of 1. Using this process we can get as follows:

Figure 7: Caption

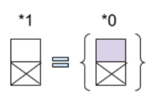


We can use this in a broader sense while summing games of Shade. To find the values of such games, we would find their Nim-sum.

Example: A 2 2 game of Shade added to a 2 1 game of Shade.



First, we will need to analyze each game separately. We will start with a 2 1 board. There is only one move to make from an empty 2 1 board, and that is to a full board. A full board has a nimber of 0 because the next player would have no moves, so it is the previous players win. It can be concluded, by using the MEX Principle, that a 2 1 board has a nimber of 1.



In a full 2 2 board, there are no moves for the next player to make, in other words, the previous player wins. Therefore, the position with a full board would have a nimber of 0. From this we can say that any position that only has a move to the winning position has a nimber of 1. We can continue using the MEX Principle until we have reached an empty board, giving a 2 2 board a nimber of 2: