Combinatorics Paper

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1 Introduction

Combinatorics is the mathematical study concerned with counting. Combinatorics uses concepts of induction, functions, and counting to solve problems in a simple, easy way. Combinatorics is extremely important in Mathematics because it allow for solutions to problems that can not be solved, or are very difficult to be solved any other way. One particular part of Combinatorics is a topic called Generating Functions. Generating functions are important because they give closed forms for recursions, which both saves time and provides useful information about the problem at hand. We will first discuss the principles of induction and proofs to set the foundation for Generating Functions. After, we will explore Generating Functions: how they work, what they do, how to solve problems, and their true significance. With these two idea, we will then discuss Induction and Generating Functions as a whole, connecting the two topics and understanding Combinatorics as one important branch of Mathematics.

All of the following examples and exercises are taken from the book by Miklos Bona, [1].

2 Induction

2.1 Principal of Induction

The Principle of Induction is pretty simple: we Want to show a property p holds for all natural numbers n (i.e. $p(n)$ holds for all n). Induction is used to prove an explicit formula holds true for a general sequence.

Definition 2.1. Induction is a mathematical proof and technique to prove a certain explicit formula holds true for a recurrence.

In the following theorem, we state mathematically the principal of induction that will be applied in the problems that follow this section.

Theorem 1. *Given a recurrence or explicit formula, we have to go through the following steps which is the Principle of Induction.*

- *Establish base case: show that $p(0)$ holds true.*

- *Inductive Hypothesis: assume that $p(k)$ holds true for an arbitrary k value.*
- *Inductive Step: Given inductive hypothesis, show $p(k+1)$ holds (i.e. $p(k)$ is true $\Rightarrow p(k+1)$ is true).*

2.2 Simple Examples

We begin with a series of simple problems that show some easy examples of how to apply induction. For example, if a closed form of sequence is provided, induction is very simple:

Example 2.1. Let $a_0 = 1$, and let $a_{n+1} = 3a_n + 2$, for all non-negative integers n. Prove that $a_n = 2 \cdot 3^n - 1$.

Solution. Given: $a_{n+1} = 3a_n + 2$, $a_0 = 1$, Prove: $a_n = 2 \cdot 3^n - 1$

Base Case: $n = 0$: $a_0 = 2 \cdot 3^0 - 1 = 2 \cdot 1 - 1 = 1$

As shown above, the base case holds true for the smallest value of n because if $n = 0$, then $a_n = 1$ which is equivalent to what we were given that $a_0 = 1$

Assume True For $n = k$: $a_k = 2 \cdot 3^k - 1$

Prove True For $n = k + 1$: $a_{k+1} = 2 \cdot 3^{k+1} - 1$

$3a_n + 2 = 2 \cdot 3^{k+1} - 1$

$3(2 \cdot 3^{k+1} - 1) + 2 = 2 \cdot 3^{k+1} - 1$

$2 \cdot 3^{k+1} - 3 + 2 = 2 \cdot 3^{k+1} - 1$

Therefore, the statement holds for all positive integers n. □

From this example, we see that when we know the formula for the solution, induction can easily blah blah blah... In the next example of applying induction, we are going to example a very different type of problem. In this problem, blah blah blah...

Example 2.2. For every natural number n, the integer $a(n) = n^3 + 11n$ is divisible by 6.

Solution. Prove: $(a(n) = n^3 + 11n)/6$=integer

Base Case: $n = 0$:$(a(0) = 0^3 + 11 \cdot 0)/6 = 0$

As shown above, the base case holds true for the smallest value of n because if $n = 0$, then $a(0) = 0$ and 0 is divisible by 6.

Assume True For $n = k$: $(a(k) = k^3 + 11k)/6$ = integer

Prove True For $n = k + 1$: $a_{k+1} = \frac{(k+1)^3+11(k+1)}{6}$=integer

$$a_{k+1} = (k^2 + 2k + 1)(k + 1) + 11(k + 1)$$

$$a_{k+1} = k^3 + 2k^2 + k + k^2 + 2k + 1 + 11k + 11$$

$$a_{k+1} = k^3 + 11k + 3k^2 + 3k + 12$$

Here, $k^3 + 11k + 3k^2 + 3k + 12$ can be simplified to just $3k^2 + 3k + 12$ because as proven above,$k^3 + 11k$ is divisible by 6 and by adding a number divisible by

6 to any number, that number's divisibility will not change. $3k^2 + 3k + 12$ can also be simplified to just $3k^2 + 3k$ for the same reason just stated.

Additionally, $k^2 + k$ is always an even number because if k is an odd integer, then it would be $odd^2 + odd = odd + odd = even$. Similarly if k is an even integer, then it would be $even^2 + even = even + even = even$.

Thus, given $3(k^2+k)$ the statement holds true for all positive integers because k^2+k always equals an even number as shown above and any even number times 3 is divisible by 6. □

2.3 More Complex Examples

In the examples above, the applications of induction were very straight forward with closed forms provided. However, what if these closed forms weren't given, and the problem were more intuitive. We begin win an example that shows how application of induction is more "hidden."

Example 2.3. At a tennis tournament, every two players played against each other exactly one time. After all games were over, each player listed the names of those he defeated, and the names of those defeated by someone he defeated. Prove that at least one player listed the names of everybody else.

solution. Let n denote the number of players

We will first go through the inductive process.

base case: n=2 It works!

hypothesis: Assume when there is n players, there is someone who lists everyone and random person i.

step: If person "i" beats person $n + 1$ when person i is still the person to list all payers. If $n + 1$ beats i, then i is on the list of $n + 1$.

There are 2 conditions for which person i doesn't list person 1:

1. If I defeats i

2. If I defeats all players that i defeated.

However, statement 2 is contradictory because according to the inductive hypothesis, player 1 is supposed to have the least number of wins; thus, person i must have listed player 1.

If we do this kind of intuitive thinking for rest of players, the problem is solved. Person i, some random person, will have the listed the names of everybody else, so at least one person *must* have listed all the other players. □

We see in the example of above that we didn't need to manipulate any numbers; we were simply thinking through the problem using induction. The next example explores a more difficult example with manipulating equations.

Example 2.4. Let $a_0 = 1$, and let $a_{n+1} = 10a_n - 1$.Prove that for all $n \geq 1$, $a_n = \frac{8(10)^n+1}{9}$

Solution. base case: let n=0, $a_0 = \frac{8*(10)^0+1}{9} = 1$ The base case works.

Inductive hypothesis: $a_{k+1} = 10a_k - 1$ and $a_k = \frac{8(10)^k+1}{9}$
Inductive Step:
$a_{k+1} = \frac{8(10)^{k+1}+1}{9} = \frac{10*8*10^k+1}{9} = 10(\frac{8*10^k}{9})$
$a_k - \frac{1}{9} = \frac{8*10^k}{9}$
Then, $10(\frac{8*10^k}{9}) = 10(a_k - \frac{1}{9}) + \frac{1}{9}$
Simplifying this further, we get that the expression equals $10a_k - 1$ which is exactly what a_{k+1} equals to. Thus, this problem has been proved by induction. □

We can also apply induction to examples that require both intuitive thinking and mathematical manipulation.

Example 2.5 (Exercise 2.4 from [1])**.** Prove that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Solution. To actually solve this, we will use an interesting approach. This approach is modules. Mods are extremely helpful with divisibility, which is perfect for the problem at hand. We will have to think about divisibility intuitively and solve with mods.

base case: let $n = 3$, then $3/3 = 1$, whole number. So this works!
let $k = a_0 + a_1 + \cdot + a_k$.
If a number,n is divisible by 3, it means that its $0mod(3)$. $mod(3)$ means that n and the sum of the digits of n will be equivalent to the same number, $mod(3)$. If a number has $0mod(3)$, it is divisible by 3. If it has $1mod(3)$ or $2mod(3)$, then the number is not divisible by 3. There are two states that come from this:

1. If the $mod(3)$ for n is 0, then n and the sum of its digits will be divisible by 3.
2. If the $mod(3)$ doesn't equal 0, then neither n nor the sum of its digits will be divisible by 3.

□

Induction allows us to walk through elementary principles and prove them as shown here.

3 Generating Functions

Induction is a powerful way to prove an explicit formula given such formula or deriving such formula. However, if this formula isn't given or is hard to determine, we need to use generating functions.

3.1 What is a Generating Function

As shown above, induction is a classical way to solve recurrences that would be very hard without a the closed form. Generating functions allows the derivation of a closed form by using the elements of a sequence as the coefficients of a power series. However, in order to do so we must neglect convergence and allow the series to diverge. First, we start by introducing the necessary background knowledge for generating functions.

3.2 Background

Since generating functions derives a closed form by using elements of a sequences as coefficients of a power series, we must introduce the definition and application of a power series.

A power series is an infinite sequence of a certain form - a summation of polynomial x. Because generating functions deal with *formal* power series, we can ignore convergence because generating functions solves for coefficients and not specific values of x. Because of this, we can state the following theorem.

Theorem 2. *For all values of x,*

$$f(x) = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

Proof. First, we define the function S to be $f(x)$.

$$S = 1 + x + x^2 + x^3 + \cdots$$

Then, we multiple both sides by x.

$$Sx = x + x^2 + x^3 + x^4 \cdots$$

Then we subtract the two equations and factor out S on the left.

$$(1 - x) = 1$$

Lastly, we solve for S

$$S = \frac{1}{1-x}$$

□

Definition 3.1. Let $f(x) = \frac{1}{a^2-2ab+b^2}$, where a and b are the roots of the quadratic, then the *partial fractions* for this function are $f(s) = \frac{A}{a-b} + \frac{B}{a-b}$, where A and B are coefficients.

Example 3.1.

$$\frac{1}{2x^2+3x+1} = \frac{A}{2x+1} + \frac{B}{x+1}$$

Solution. To solve this problem, we will use simple testing values to solve for A and B. Let x=0 and x=1. This will yield:

$$1 = A + B$$
$$\frac{1}{6} = \frac{A}{3} + \frac{B}{2}$$

Now, we have a system of equations with 2 unknowns and 2 equations. This can now be easily solved. The solution is that A=2 and B=-1.

Thus,
$$\frac{1}{2x^2+3x+1} = \frac{2}{2x+1} - \frac{1}{x+1}$$

□

3.3 Simple Examples

Example 3.2. Let $a_{n+2} = 3a_{n+1} - 2a_n$ if $n \geq 0$, and let $a_0 = 0$ and let $a_1 = 1$. Find an explicit formula for a_n.

Solution. recursive formula: $a_{n+2} = 3a_{n+1} - 2a_n$, $n \geq 0$

Let $G(x) = \sum\limits_{n=0}^{\infty} a_n x^n$.

Then, multiply both sides of the recurrence relation by x^{n+2} and take the sum across all natural numbers n:

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = 3\sum_{n=0}^{\infty} a_{n+1}x^{n+2} - 2\sum_{n=0}^{\infty} a_n x^{n+2}$$

Substitute in $G(x)$:

$$G(x) - a_o - a_1 x = 3xG(x) - 2x^2G(x)$$

$$G(x) - 3xG(x) + 2x^2G(x) = a_0 + a_1 x$$

$$G(x)(1 - 3x + 2x^2) = x$$

$$G(x) = \frac{x}{2x^2 - 3x + 1}$$

Using partial fraction decomposition we then find that

$$\frac{x}{2x^2-3x+1} = \frac{A}{2x-1} + \frac{B}{x-1}$$

.

We now have to use simple testing values to solve for A and B. If x=2 and x=3, then...

$$2 = A + 3B$$
$$3 = 2A + 5B$$

We now have a system of equations that can be solved to find the two unknowns, A and B. By solving the system we find that $A = -1$ and $B = 1$.

Thus,

$$G(x) = \frac{-1}{2x-1} + \frac{1}{x-1}$$

And if we multiply this by negative one we find that

$$G(x) = \frac{1}{1-2x} - \frac{1}{1-x}$$

... and can use the series expansion that we previously learned to continue solving.

$$G(x) = \sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty} x^n$$
$$G(x) = \sum_{n=0}^{\infty}(2^n - 1)x^n$$

Based on how we defined $G(x)$ at the beginning, we can then substitute and further simplify

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(2^n - 1)x^n$$

Therefore,

$$a_n = 2^n - 1$$

□

In the previous example we were fortunate enough to have been given the recursive formula in the problem. However, sometimes we have to derive the formula ourselves based on the information given like in the following example.

Example 3.3. We have invested 1000 dollars into a savings account that pays five percent interest at the end of each year. At the beginning of each year, we deposit another 500 dollars into this account. How much money will be in this account after n years?

Solution. First, we have to find a recurrence relation based on the information given to us in the problem.

If a_n = the amount of money in the account after n years, then $a_0 = 1000$ and thus,

$$a_{n+1} = 1.05 \cdot a_n + 500$$

.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then, multiply both sides of the recurrence relation by x^{n+1} and take the sum across all natural numbers n:

$$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} = \sum_{n=0}^{\infty} 1.05 a_n x^{n+1} + \sum_{n=0}^{\infty} 500 x^{n+1}$$

Substitute in $G(x)$:

$$G(x) - a_o = 1.05xG(x) + \frac{500x}{1-x}$$

$$G(x) - 1.05xG(x) = a_o + \frac{500x}{1-x}$$

$$G(x)(1 - 1.05x) = 1000 + \frac{500x}{1-x}$$

$$G(x) = \frac{1000}{1-1.05x} + \frac{500x}{(1-x)\cdot(1-1.05x)}$$

Given that...

$$\frac{1000}{1-1.05x} = 1000 \cdot \sum_{n=0}^{\infty} 1.05^n x^n$$

$$\frac{500x}{(1-x)(1-1,05x)} = 500x \cdot (\sum_{n=0}^{\infty} x^n)(\sum_{n=0}^{\infty} 1.05^2 x^n)$$

We can substitute and further simplify this expression:

$$\sum_{n=0}^{\infty} a_n x^n = 1000 \cdot \sum_{n=0}^{\infty} (1.05)^n x^n + 500x \cdot (\sum_{n=0}^{\infty} x^n)(\sum_{n=0}^{\infty} 1.05^2 x^n)$$

Therefore,

$$a_n = 1000 \cdot 1.05^n + 10000 \cdot (1.05^n - 1) = 1.05^n \cdot 11000 - 10000$$

□

3.4 A More Complex Example

Now that we have walked through a couple of simple, basic, and classic generating function problems, what happens when the approach is not as straight forward. An intuitive problem can also be solved with generating functions, but like complex induction problems, we have to create and set up the problem so that generating functions works.

Example 3.4. A child wants to walk up a stairway. At each step, she moves up either one or two stairs. Let $f(n)$ be the number of ways she can reach the n-th stair. Find a closed explicit formula for $f(n)$.

Solution. We are given that $f(0) = 1$ (1 way to take zero steps) and $f(1) = 1$ (1 way to take 1 step. To start this problem, we will first find a recursive formula:

$$f(n) = f(n-1) + f(n-2) \tag{1}$$

$$f(n+2) = f(n+1) + f(n) \tag{2}$$

First we will multiple Equation 2 by

$$x^{n+2}$$

and take the sum of all terms from $n = 0$ to n=$\infty.Thiswillyieldthefollowingequation:$

$$\sum_{n=0}^{\infty} f(n+2)x^{n+2} = \sum_{n=0}^{\infty} f(n+1)x^{n+2} + \sum_{n=0}^{\infty} f(n)x^{n+2} \tag{3}$$

We will then simplify 3 to become:

$$\sum_{n=0}^{\infty} f(n+2)x^{n}x^{2} = \sum_{n=0}^{\infty} f(n+1)x^{n}x^{2} + \sum_{n=0}^{\infty} f(n)x^{n}x^{2} \tag{4}$$

At this point, it is necessary for us to define what our generating function will be:

$$G(x) = \sum_{n=0}^{\infty} f(n)x^{n} \tag{5}$$

We will now substitute G(x) into our derived equation:

$$G(x) - f(0) - f(1)x = x(G(x) - f(0)) + G(x)x^2 \tag{6}$$

Simplifying this equation, substituting f(0) and f(1), and solving for G(x):

$$G(x) = \frac{1}{1-x-x^2} \tag{7}$$

Now we can use partial fractions as shown in 3.1,we can decompose this fraction.

$$G(x) = \frac{1}{1-x-x^2} = \frac{A}{x+\frac{1-\sqrt{5}}{2}} + \frac{B}{x-\frac{1+\sqrt{5}}{2}} \tag{8}$$

Solving for A and B using partial fractions, $A = \frac{1-\sqrt{5}}{3}$ and $B = \frac{1-\sqrt{5}}{6}$
So the final equation is:

$$G(x) = \frac{\frac{1-\sqrt{5}}{5}}{x+\frac{1-\sqrt{5}}{2}} + \frac{\frac{1-\sqrt{5}}{6}}{x-\frac{1+\sqrt{5}}{2}} \tag{9}$$

This equation looks very ugly but with some rearranging, this equation becomes:

$$G(x) = \frac{2-2\sqrt{5}}{6x+3-3\sqrt{5}} + \frac{2-2\sqrt{5}}{12x-6-6\sqrt{5}} \tag{10}$$

Still looks ugly, but there seems to be a common factor of

$$1-\sqrt{5}$$

in the fractions. Let us set

$$c = 1-\sqrt{5} and d = 1+\sqrt{5}$$

. Then the equation is:

$$G(x) = \frac{2c}{6x+3c} + \frac{2c}{12x-6d} \tag{11}$$

With this equation we can do some manipulating so that we can use 2. Let's divide the first term by 6x and the second term by 12x and 2:

$$G(x) = \frac{2}{3}\frac{1}{1-\frac{-2}{c}} + \frac{c}{3d}\frac{1}{\frac{-2x}{d}+1} \tag{12}$$

Now we can see that 2 can be used to an extent. With more algebra and re-arranging, we finally get the following explicit formula:

$$\frac{2(1+\sqrt{5})^n + c^n\frac{\sqrt{5}-3}{2}}{3*2^n} \tag{13}$$

Although not a very nice explicit formula, it is still an explicit formula to describes the original explicit formula. In fact, if you had seen this problem before, the answer and process is the Fibonacci sequence, which is pretty much what we received. □

4 Conclusion

As shown, Generating Functions and Induction are incredibly powerful mathematical tools. They can be used to solve similar types of problems: recurrences. When given an explicit formula, induction is a simple approach to prove this formula for all natural numbers. However, when a formula is not given or is not clear, generating functions would be the better approach because it creates a closed form for a more general situation. Given this, Generating Functions and Induction are extremely useful tools in the field of mathematics and make problems much similar than they may seem.

References

[1] Miklos Bona. *A walk through combinatorics: an introduction to enumeration and graph theory.* World Scientific, 2002.