YULIA'S DREAM 2024: ENTRANCE PROBLEM SET

Program description. Yulia's Dream is a math enrichment and research program for exceptional high school students (entering grades 10-11 in September 2024) from Ukraine. Under this program, students would meet online once a week in small groups to study advanced math topics beyond high school curriculum or work on math research projects under the guidance of academic mentors from MIT and other universities in the U.S. and Europe. The instruction will be available in Ukrainian, English and Russian.

Yulia's Dream is an initiative under PRIMES Program for Research in Mathematics, Engineering and Science for High School Students at the Massachusetts Institute of Technology. The working groups at Yulia's Dream will operate similarly to those at PRIMES-USA, the remote section of PRIMES. Yulia's Dream is dedicated to the memory of Yulia Zdanovska, a 21-year-old graduate of the National University of Kyiv, a silver medalist at the 2017 European Girls' Mathematical Olympiad, and a teacher for the Teach for Ukraine program who was killed by a Russian-fired missile in her home city of Kharkiv. We hope to help other Ukrainian boys and girls fulfill her dream.

General advice. Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps over several days. Try to solve as many problems as you can, but we encourage you to apply if you have solved at least three problems, in whole or in part.

Solutions format. You may write your solutions in Ukrainian, Russian, or English - whichever language you choose. You can type the solutions or write them up by hand and then scan them or take a picture. Please save your solutions as a file (preferably, PDF), upload it to the Web, and submit the link along with your application. Make sure the file is accessible by link and does not require special permission to access. The name of the attached file must start with your last name, for example, "lastname-solutions.pdf". Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted. PDFs produced by LATEX are preferred, but PDFs produced by Word and scans of handwritten solutions are also accepted.

Academic integrity. You are allowed to use any resources to solve these problems, except other people's help. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Contact. If you have any questions, please contact Yulia's Dream program coordinator Dmytro Matvieievskyi at yuliasdream@mit.edu

Problems

Notation. We let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and positive integers, respectively. Also, we let $\mathbb{Z}/n\mathbb{Z}$ and $(\mathbb{Z}/N\mathbb{Z})^{\times}$ denote the set of remainders and the set of non-zero remainders modulo n, respectively.

Problem 1. Hogwarts has quite peculiar habits and games.

(a) Gryffindor fans tell the truth when Gryffindor wins and lie when it loses. Fans of Hufflepuff, Ravenclaw, and Slytherin behave similarly. After two matches of quidditch with the participation of these four teams (with no draws and each team playing exactly one game), among the wizards who watched the broadcast, 500 answered positively to the question "Do you support Gryffindor?", 600 answered positively to the question "Do you support Hufflepuff?", 300 answered positively to the question "Do you support Ravenclaw?", and 200 answered positively to the question "Do you support Slytherin?". How many wizards support each of the teams?

Note: Each wizard is fan of exactly one of the teams.

(b) There is a bucket of N candies leftover from Halloween ($N \geq 2$). Two friends, Hermione Granger and Ron Weasley, take turns to disappear candies from the bucket as follows. The first turn, Hermione must disappear at least one candy and cannot disappear all of the candies. Then taking turns, each of them must disappear at least one candy and at most $9/4$ times the number of candies disappeared by her/his friend in the previous turn. The winner is the one disappearing the last candy. Assume that Hermione and Ron play optimally.

(i) For which numbers N does Hermione have a winning strategy? Justifying your answer.

(ii) Answer the previous question replacing $9/4$ by 3.

Problem 2. Suppose that each edge of a given convex hexagon has distance 1 to the origin (this means, each edge is contained in a line whose distance to the origin equals 1). What is the minimum possible area enclosed by this hexagon? Justify your answer.

Problem 3. For any positive $a, b \in \mathbb{Z}$, we define $\operatorname{pow}(a, b)$ inductively in the following way: $\operatorname{pow}(a, 1) = a$ and $\operatorname{pow}(a, b) = a^{\operatorname{pow}(a,b-1)}$ if $b \geq 2$.

(a) Prove that for any positive $k, n \in \mathbb{Z}$ with $\gcd(k, n) = 1$, there exists $c \in \mathbb{Z}$ with $0 \leq c < n$ and $M \in \mathbb{N}$ such that $\operatorname{pow}(k, m) \equiv c \pmod{n}$ for all $m \in \mathbb{Z}$ such that $m \geq M$: we denote the integer c by $f_n(k)$.

(b) Prove that for every positive integer n, the inclusion $(\mathbb{Z}/n\mathbb{Z})^{\times} \subseteq \operatorname{Im}(f_n)$ holds, where $\operatorname{Im}(f_n)$ is the image of the function $f_n \colon \mathbb{Z} \to \mathbb{Z}$.

Problem 4.

(a) Describe an algorithm, with proof, to compute all possible ways to write a given $n \in \mathbb{N}$ as the sum of squares of consecutive positive integers. For example, for $n = 25$, we can write $25 = 5^2$ and $25 = 3^2 + 4^2$. Include your code as part of your solution (feel free to use your favorite programming language).

(b) What is the time complexity of your algorithm?

(c) What is the first number that is NOT a perfect square which can be written as the sum of squares of consecutive positive integers in three different ways? Hint: it is less than 150000.

Problem 5. A nonempty set S consisting of positive real numbers is called an *additive set* if $x + y \in S$ when $x, y \in S$. Let S be an additive set. An element of S is called *indecomposable* if it is not the sum of two (not necessarily distinct) elements of S, and S is called *decomposable* if every element of S can be written as a finite sum of indecomposable elements (allowing repetitions and sums consisting of only one summand). Prove that if S is an additive set and there exists a strictly decreasing sequence $(x_n)_{n\geq 1}$ such that $\{x_n, x_n - x_{n+1} : n \in \mathbb{N}\} \subseteq S$, then there exists an additive set contained in S that is not decomposable.