The Pak–Postnikov and Naruse skew hook length formulas: a new proof

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The *Young diagram* of this partition $\lambda$ is a left-aligned table with $\lambda_i$ cells in row $i$ (indexed from the top). We call it $\text{Y}(\lambda)$. Formally:

$$\text{Y}(\lambda) = \{(i,j) \mid i > 0 \text{ and } 0 < j \leq \lambda_i\}.$$
A partition is a weakly decreasing sequence
\( \lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots) \) of nonnegative integers with only finitely many positive entries.

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\[
Y(\lambda) = \{(i,j) \mid i > 0 \text{ and } 0 < j \leq \lambda_i\}.
\]

Example: If \( \lambda = (4,2,2,0,0,0,\ldots) = (4,2,2) \) (we omit zeroes), then

\[
Y(\lambda) = \begin{array}{cccc}
\text{\ } & \text{\ } & \text{\ } & \text{\ } \\
\text{\ } & \text{\ } & \text{\ } & \text{\ } \\
\text{\ } & \text{\ } & \text{\ } & \text{\ } \\
\end{array}.
\]
A **partition** is a weakly decreasing sequence
\[ \lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots) \]
of nonnegative integers with only finitely many positive entries.

The **Young diagram** of this partition \( \lambda \) is a left-aligned table
with \( \lambda_i \) cells in row \( i \) (indexed from the top). We call it
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\[ Y(\lambda) = \{(i,j) \mid i > 0 \text{ and } 0 < j \leq \lambda_i \} . \]

Two partitions \( \mu \) and \( \lambda \) satisfy \( \mu \subseteq \lambda \) if \( Y(\mu) \subseteq Y(\lambda) \). In this case, the **skew diagram** \( Y(\lambda/\mu) \) is defined to be \( Y(\lambda) \setminus Y(\mu) \).
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**Example:** If \( \lambda = (5, 2, 2) \) and \( \mu = (1, 1) \), then

\[
Y(\lambda/\mu) = \begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]
A \textit{partition} is a weakly decreasing sequence \( \lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots) \) of nonnegative integers with only finitely many positive entries.

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More generally, any set of (square) cells is called a \textit{diagram}.

\textbf{Example:}

\[
\begin{array}{ccccccccc}
\chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi \\
\chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi \\
\chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi \\
\chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi \\
\chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi \\
\chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi \\
\chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi & \chi \\
\end{array}
\]
Given a diagram $D$, we can fill it with the numbers $1, 2, \ldots, n$. Such a filling is called a *standard tableau* (of shape $D$) if

- each of the numbers $1, 2, \ldots, n$ appears exactly once;
- the numbers increase along each row;
- the numbers increase down each column.

Example: If $\lambda = (5, 4, 3, 3)$ and $\mu = (2, 1, 1)$, then $1, 3, 9, 2, 4, 10, 5, 6, 7, 8, 11 \in \text{SYT}(\lambda/\mu)$.

Question: Given a diagram $D$, how many standard tableaux of shape $D$ exist?
Given a diagram $D$, we can fill it with the numbers $1, 2, \ldots, n$. Such a filling is called a standard tableau (of shape $D$) if
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If $D = Y(\lambda)$, we let $\text{SYT}(\lambda)$ be the set of all standard tableaux of shape $Y(\lambda)$. 

Standard tableaux

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- If $D = Y(\lambda)$, we let $\text{SYT}(\lambda)$ be the set of all standard tableaux of shape $Y(\lambda)$.
- Likewise $\text{SYT}(\lambda/\mu)$.
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Likewise $\text{SYT}(\lambda/\mu)$.

**Example:** If $\lambda = (5, 4, 3, 3)$ and $\mu = (2, 1, 1)$, then

\[
\begin{array}{ccc}
1 & 3 & 9 \\
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5 & 6 \\
7 & 8 & 11 \\
\end{array}
\]

$\in \text{SYT}(\lambda/\mu)$.  

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5 & 6 \\
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\end{array}
\in \text{SYT}(\lambda/\mu).
$$

**Question:** Given a diagram $D$, how many standard tableaux of shape $D$ exist?
For $D = Y(\lambda)$, the classical hook length formula of Frame, Robinson and Thrall (1953) gives a beautiful answer in terms of the hooks of $\lambda$. 

Example: If $\lambda = (4, 3, 3, 2)$, then $H_\lambda(2, 2) = \bullet \bullet \bullet \bullet \bullet \bullet$ and $h_\lambda(2, 2) = 4$. 


• If \( c = (i, j) \) is a cell of a Young diagram \( Y(\lambda) \), we let the
  \textit{hook} \( H_\lambda(c) \) be

\[
\{ \text{all cells of } Y(\lambda) \text{ lying due east of } c \} \\
\cup \{ \text{all cells of } Y(\lambda) \text{ lying due south of } c \} \cup \{ c \}
\]

\[= \{(i, k) \in Y(\lambda) \mid k \geq j\} \cup \{(k, j) \in Y(\lambda) \mid k \geq i\}.\]
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The hook length \( h_\lambda(c) \) is defined to be \( |H_\lambda(c)| \), that is, the number of all cells in the hook of \( c \).
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The **hook length** \( h_\lambda(c) \) is defined to be \( |H_\lambda(c)| \), that is, the number of all cells in the hook of \( c \).

**Example:** If \( \lambda = (4, 3, 3, 2) \), then

\[
H_\lambda(2, 2) = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\]

and \( h_\lambda(2, 2) = 4 \).
The original *hook length formula* says that

$$|\text{SYT} (\lambda)| = \frac{n!}{\prod_{c \in Y(\lambda)} h_\lambda (c)},$$

where $n$ is the number of cells in $Y (\lambda)$. 

Example: If $\lambda = (3, 2)$, then

$$|\text{SYT} (\lambda)| = 5! \cdot 1 \cdot 3 \cdot 4 \cdot 1 \cdot 2 = 5.$$
The hook length formula

The original *hook length formula* says that

\[ |\text{SYT}(\lambda)| = \frac{n!}{\prod_{c \in Y(\lambda)} h_{\lambda}(c)}, \]

where \( n \) is the number of cells in \( Y(\lambda) \).

**Example:** If \( \lambda = (3, 2) \), then

\[ |\text{SYT}(\lambda)| = \frac{5!}{1 \cdot 3 \cdot 4 \cdot 1 \cdot 2} = 5. \]

Here are the hooks of all five cells:
The original *hook length formula* says that

$$|\text{SYT} (\lambda)| = \frac{n!}{\prod_{c \in Y(\lambda)} h_\lambda(c)},$$

where $n$ is the number of cells in $Y(\lambda)$.

**Example:** If $\lambda = (3, 2)$, then

$$|\text{SYT} (\lambda)| = \frac{5!}{1 \cdot 3 \cdot 4 \cdot 1 \cdot 2} = 5.$$

Here is SYT($\lambda$):

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5
\end{array}, \quad
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5
\end{array}, \quad
\begin{array}{ccc}
1 & 2 & 5 \\
3 & 4
\end{array}
\]

\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 5
\end{array}, \quad
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 4
\end{array}
\]
**Example:** The number of Dyck paths from $(0, 0)$ to $(2n, 0)$ is the $n$-th *Catalan number* $C_n = \frac{(2n)!}{n!(n+1)!}$.

This follows from the hook length formula, applied to $\lambda = (n, n)$, and a simple bijection $\{\text{Dyck paths}\} \rightarrow \text{SYT}(\lambda)$:

\[
\begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & 6 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 6 \\
\end{array}
\]
Naruse's skew hook length formula (Naruse, 2014) expresses $|\text{SYT} (\lambda/\mu)|$ in terms of excitations.
An *excited move* for a cell $c = (i, j) \in D$ means moving this cell from $(i, j)$ to $(i + 1, j + 1)$. This is allowed only if the three cells marked $\times$ (that is, $(i + 1, j)$, $(i, j + 1)$, $(i + 1, j + 1)$) are not in $D$.

![Diagram](image-url)
Excited moves

- An *excited move* for a cell \( c = (i, j) \in D \) means moving this cell from \((i, j)\) to \((i + 1, j + 1)\).
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- **Example:**

An *excited move* for a cell \( c = (i, j) \in D \) means moving this cell from \((i, j)\) to \((i + 1, j + 1)\). This is allowed only if the three cells marked \( \times \) (that is, \((i + 1, j), (i, j + 1), (i + 1, j + 1)\)) are not in \( D \).

**However,**

\[
\begin{array}{c}
  \boxed{c} \\
  \times \\
  \times \\
\end{array} \rightarrow \begin{array}{c}
  \times \\
  \times \\
  \boxed{c}
\end{array}
\]

\[
\begin{array}{c}
  \boxed{c} \\
  \end{array} \rightarrow \begin{array}{c}
  \end{array}
\]

\[
\begin{array}{c}
  \boxed{c} \\
  \end{array} \rightarrow \begin{array}{c}
  \boxed{c}
\end{array}
\]
An *excited move* for a cell \( c = (i, j) \in D \) means moving this cell from \((i, j)\) to \((i + 1, j + 1)\). This is allowed only if the three cells marked \( \times \) (that is, \((i + 1, j)\), \((i, j + 1)\), \((i + 1, j + 1)\)) are not in \( D \).

\[
\begin{array}{c|c|c}
| & | & | \\
| \times & \times & \times \\
|\hline | & | & |
\end{array} \quad \rightarrow \quad \begin{array}{c|c|c}
| & | & | \\
| \times & \times & \times \\
|\hline | & | & |
\end{array}
\]

However,
An *excited move* for a cell $c = (i, j) \in D$ means moving this cell from $(i, j)$ to $(i + 1, j + 1)$. This is allowed only if the three cells marked $\times$ (that is, $(i + 1, j)$, $(i, j + 1)$, $(i + 1, j + 1)$) are not in $D$.

\[
\begin{array}{c}
\begin{array}{c}
\times \\
\times \\
\times \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\times \\
\times \\
\times \\
\end{array}
\end{array}
\]

However,
An *excitation* of a diagram $D$ is a diagram obtained from $D$ by a sequence of excited moves.
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**Example:** Original diagram $D$:

```
+---+---+---+
|   |   |   |
+---+---+---+
|   |   |   |
+---+---+
|   |
+---+
```

Now, for two partitions $\lambda$ and $\mu$, we define $E(\lambda/\mu)$ to be the set of all excitations $E$ of $Y(\mu)$ that satisfy $E \subseteq Y(\lambda)$. 
An *excitation* of a diagram $D$ is a diagram obtained from $D$ by a sequence of excited moves.

**Example:** After a single excited move:

```
  +---+
  |   |
  +---+   +---+
  |   |   |   |
  +---+   +---+
  |   +---+
  | +---+   +---+   +---+
  |   |   |   |   |   |
  +---+   +---+   +---+   +---+
```

Now, for two partitions $\lambda$ and $\mu$, we define $E(\lambda/\mu)$ to be the set of all excitations $E$ of $Y(\mu)$ that satisfy $E \subseteq Y(\lambda)$. 
An *excitation* of a diagram $D$ is a diagram obtained from $D$ by a sequence of excited moves.

**Example:** After two excited moves:
An *excitation* of a diagram $D$ is a diagram obtained from $D$ by a sequence of excited moves.

**Example:** After three excited moves:

![Diagram showing excited moves](image)
An *excitation* of a diagram $D$ is a diagram obtained from $D$ by a sequence of excited moves.

**Example:** After four excited moves:

```
  
```

```
Excitations

- An *excitation* of a diagram $D$ is a diagram obtained from $D$ by a sequence of excited moves.

- Now, for two partitions $\lambda$ and $\mu$, we define $\mathcal{E}(\lambda/\mu)$ to be the set of all excitations $E$ of $Y(\mu)$ that satisfy $E \subseteq Y(\lambda)$. 
Naruse’s skew hook length formula says that

\[ |\text{SYT}(\lambda/\mu)| = n! \sum_{E \in E(\lambda/\mu)} \prod_{c \in Y(\lambda) \setminus E} \frac{1}{h_\lambda(c)} \]

if \( \lambda \) and \( \mu \) are two partitions with \( \mu \subseteq \lambda \) with \( |Y(\lambda/\mu)| = n \).
Naruse’s skew hook length formula says that

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if \( \lambda \) and \( \mu \) are two partitions with \( \mu \subseteq \lambda \) with \( |Y(\lambda/\mu)| = n \).

\textbf{Example:} If \( \lambda = (2, 2, 2) \) and \( \mu = (1, 1) \), then

\[ \mathcal{E} (\lambda/\mu) = \left\{ \begin{array}{c|c|c}
\ast & & \\
\ast & & \\
\ast & & \\
\hline & & \\
& & \\
& & \\
\hline & & \\
& & \\
& & \\
\end{array}, \quad \begin{array}{c|c|c}
\ast & & \\
& & \\
& & \\
\hline & & \\
& & \\
& & \\
\hline & & \\
& & \\
& & \\
\end{array}, \quad \begin{array}{c|c|c}
& & \\
& & \\
& & \\
\hline & & \\
& & \\
& & \\
\hline & & \\
& & \\
& & \\
\end{array} \right\} . \]

Thus,

\[ |\text{SYT} (\lambda/\mu)| = 4! \cdot \left( \frac{1}{3 \cdot 2 \cdot 1 \cdot 2} + \frac{1}{3 \cdot 2 \cdot 3 \cdot 2} + \frac{1}{3 \cdot 2 \cdot 3 \cdot 4} \right) = 3 \]
Naruse’s skew hook length formula says that

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if $\lambda$ and $\mu$ are two partitions with $\mu \subseteq \lambda$ with $|Y(\lambda/\mu)| = n$.

**Example:** If $\lambda = (2, 2, 2)$ and $\mu = (1, 1)$, then

$\text{SYT}(\lambda/\mu) = \left\{ \begin{array}{ccc} 1 & & 2 \\ 3 & 2 & 3 \\ 2 & 4 & 1 \end{array} \right\}$. 
Naruse’s skew hook length formula says that

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\[
\text{SYT}(\lambda/\mu) = \left\{ \begin{array}{c}
1 \\
3 \\
2 \\
4 \\
\end{array}, \quad \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}, \quad \begin{array}{c}
2 \\
3 \\
1 \\
4 \\
\end{array} \right\}.
\]

Known proofs use algebraic geometry (Naruse) or complicated combinatorics (Morales/Pak/Panova and Konvalinka).
In 2001, Pak and Postnikov generalized the classical hook length formula in a different direction.
The Pak–Postnikov generalization

- If \( T \) is a standard tableau (of any shape), and if \( k \) is a positive integer, then \( c_T(k) \) shall denote the difference \( j - i \), where \((i, j)\) is the cell of \( T \) that contains the entry \( k \).

**Example:** If \( T = \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 \\
\end{array} \), then

\[
\begin{align*}
  c_T(1) &= 1 - 1 = 0, \\
  c_T(2) &= 1 - 2 = -1, \\
  c_T(3) &= 2 - 1 = 1, \\
  c_T(4) &= 3 - 1 = 2, \\
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\end{align*}
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- If $T$ is a standard tableau (of any shape), and if $k$ is a positive integer, then $c_T(k)$ shall denote the difference $j - i$, where $(i, j)$ is the cell of $T$ that contains the entry $k$.
- Let $\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots$ be commuting indeterminates.
If $T$ is a standard tableau (of any shape), and if $k$ is a positive integer, then $c_T(k)$ shall denote the difference $j - i$, where $(i,j)$ is the cell of $T$ that contains the entry $k$.

Let $\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots$ be commuting indeterminates.

For any cell $c = (i,j)$ of $Y(\lambda)$, we define the \textit{algebraic hook length} $h_\lambda(c; z)$ by

$$h_\lambda(c; z) := \sum_{(i,j) \in H_\lambda(c)} z_{j-i}.$$
The Pak–Postnikov generalization

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\[
h_\lambda(c; z) := \sum_{(i, j) \in H_\lambda(c)} z_{j-i}.
\]

- For any standard tableau \( T \) with \( n \) cells, we define the fraction

\[
z_T := \frac{1}{\prod_{k=1}^{n} \left( z_{c_T(k)} + z_{c_T(k+1)} + \cdots + z_{c_T(n)} \right)}.
\]
The Pak–Postnikov generalization

- If \( T \) is a standard tableau (of any shape), and if \( k \) is a positive integer, then \( c_T(k) \) shall denote the difference \( j - i \), where \((i, j)\) is the cell of \( T \) that contains the entry \( k \).
- Let \( \ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots \) be commuting indeterminates.
- For any cell \( c = (i, j) \) of \( Y(\lambda) \), we define the **algebraic hook length** \( h_\lambda(c; z) \) by
  \[
  h_\lambda(c; z) := \sum_{(i, j) \in H_\lambda(c)} z_{j-i}.
  \]
- For any standard tableau \( T \) with \( n \) cells, we define the fraction
  \[
  z_T := \frac{1}{\prod_{k=1}^{n} \left( z_{c_T(k)} + z_{c_T(k+1)} + \cdots + z_{c_T(n)} \right)}
  \]
- The **Pak-Postnikov generalization of the hook length formula** states that
  \[
  \sum_{T \in SYT(\lambda)} z_T = \prod_{c \in Y(\lambda)} \frac{1}{h_\lambda(c; z)}.
  \]
Example: For $\lambda = (2, 1)$, we have

$$\text{SYT}(\lambda) = \begin{cases} 1 & 2 \\ 3 & 2 \end{cases}, \quad \begin{cases} 1 & 3 \\ 2 & \end{cases},$$

so the formula becomes

$$\frac{1}{z_1 (z_1 + z_{-1}) (z_{-1} + z_1 + z_0)} + \frac{1}{z_1 (z_1 + z_{-1}) (z_1 + z_{-1} + z_0)}$$

$$= \frac{1}{(z_1 + z_{-1} + z_0) z_1 z_{-1}}.$$
The Pak–Postnikov generalization: example

**Example:** For \( \lambda = (2, 1) \), we have

\[
\text{SYT}(\lambda) = \begin{cases} 
1 & 2 \\
3 & 
\end{cases}, \quad \begin{cases} 
1 & 3 \\
2 & 
\end{cases}
\]

so the formula becomes

\[
\frac{1}{z_1 (z_1 + z_1)(z_1 + z_1 + z_0)} + \frac{1}{z_1 (z_1 + z_1)(z_1 + z_1 + z_0)}
\]

\[
= \frac{1}{(z_1 + z_{-1} + z_0) z_1 z_{-1}}.
\]

Known proofs involve polytopes (Pak/Postnikov) or P-partitions and tropical RSK (Hopkins).
We propose a generalization of the Pak–Postnikov formula to skew diagrams, thus extending Naruse’s hook length formula as well.
Main theorem. Let $\lambda$ and $\mu$ be two partitions with $\mu \subseteq \lambda$ such that the skew diagram $Y(\lambda/\mu)$ has $n$ cells.
Define $z_T$ for $T \in \text{SYT}(\lambda/\mu)$ as before.
Define $h_\lambda(c; z)$ for $c \in Y(\lambda)$ as before (this does not depend on $\mu$!).

Example: For $\lambda = (2, 2)$ and $\mu = (1)$, we have $\text{SYT}(\lambda/\mu) = \{12, 21\}$ and $E(\lambda/\mu) = \{\ast, \ast\}$, so the formula becomes

$$1 \cdot z_0 \cdot (z_0 + z_1) \cdot (z_0 + z_1 + z_0) + 1 \cdot z_0 \cdot (z_0 + z_1) \cdot (z_0 + z_1 + z_0) = 1 \cdot (z_1 + z_0) \cdot z_0 \cdot (z_0 + z_1) + 1 \cdot (z_1 + z_0) \cdot (z_0 + z_1 + z_0) \cdot (z_0 + z_1).$$

This was first observed by Grinberg. An intricate combinatorial proof was sketched by Konvalinka in 2019.
Main theorem. Let $\lambda$ and $\mu$ be two partitions with $\mu \subseteq \lambda$ such that the skew diagram $Y(\lambda/\mu)$ has $n$ cells. Then,

$$\sum_{T \in \text{SYT}(\lambda/\mu)} z^T = \sum_{E \in \mathcal{E}(\lambda/\mu)} \prod_{c \in Y(\lambda) \setminus E} \frac{1}{h_\lambda(c; z)}.$$
Main theorem. Let $\lambda$ and $\mu$ be two partitions with $\mu \subseteq \lambda$ such that the skew diagram $Y(\lambda/\mu)$ has $n$ cells. Then,

$$
\sum_{T \in \text{SYT}(\lambda/\mu)} z_T = \sum_{E \in \mathcal{E}(\lambda/\mu)} \prod_{c \in Y(\lambda) \setminus E} \frac{1}{h_\lambda(c; z)}.
$$

Example: For $\lambda = (2, 2)$ and $\mu = (1)$, we have

$$
\text{SYT}(\lambda/\mu) = \left\{ \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad \begin{array}{c}
2 \\
1 \\
3
\end{array} \right\} \quad \text{and} \quad \mathcal{E}(\lambda/\mu) = \left\{ \begin{array}{c}
\ast
\end{array}, \quad \begin{array}{c}
\ast
\end{array} \right\},
$$

so the formula becomes

$$
\frac{1}{z_0 \cdot (z_0 + z_{-1}) \cdot (z_0 + z_{-1} + z_1)} + \frac{1}{z_0 \cdot (z_0 + z_1) \cdot (z_0 + z_1 + z_{-1})} = \\
\frac{1}{(z_1 + z_0) \cdot z_0 \cdot (z_{-1} + z_0)} + \frac{1}{(z_1 + z_0) \cdot (z_0 + z_{-1} + z_1) \cdot (z_{-1} + z_0)}.
$$
**Main theorem.** Let $\lambda$ and $\mu$ be two partitions with $\mu \subseteq \lambda$ such that the skew diagram $Y(\lambda/\mu)$ has $n$ cells. Then,

$$
\sum_{T \in \text{SYT}(\lambda/\mu)} z_T = \sum_{E \in \mathcal{E}(\lambda/\mu)} \prod_{c \in Y(\lambda) \setminus E} \frac{1}{h_\lambda(c; z)}.
$$

**Example:** For $\lambda = (2, 2)$ and $\mu = (1)$, we have

\[
\text{SYT}(\lambda/\mu) = \left\{ \begin{array}{c} 1 \\ 2 & 3 \\ 
2 & 3 \\ 1 \\ 2 & 3 \end{array} \right\} \quad \text{and} \quad \mathcal{E}(\lambda/\mu) = \left\{ * \quad, \quad \begin{array}{c} * \\ \end{array} \right\},
\]

so the formula becomes

$$
\frac{1}{z_0 \cdot (z_0 + z_{-1}) \cdot (z_0 + z_{-1} + z_1)} + \frac{1}{z_0 \cdot (z_0 + z_1) \cdot (z_0 + z_{1} + z_{-1})} =
\frac{1}{z_0 \cdot (z_0 + z_{-1}) \cdot (z_0 + z_{-1} + z_1) \cdot (z_{-1} + z_0)} + \frac{1}{z_1 + z_0) \cdot z_0 \cdot (z_{-1} + z_0)}.
$$

This was first observed by Grinberg. An intricate combinatorial proof was sketched by Konvalinka in 2019.
We propose a new, elementary proof of this generalized formula.
We propose a new, elementary proof of this generalized formula.

Induct on $|Y(\lambda/\mu)|$, increasing $\mu$ by one cell in the induction step.
Proof idea: the Konvalinka recursion, 1

Let \( f(\lambda/\mu) = \sum_{T \in \text{SYT}(\lambda)} z_T \).

Example: Let \( \lambda = (3, 3, 2) \) and \( \mu = (2, 1) \). If \( T \) is ... then \( T' \) is ...
Let $f(\lambda/\mu) = \sum_{T \in \text{SYT}(\lambda)} z_T$.

We easily obtain the recurrence

$$z_T = \frac{1}{\sum_{(i,j) \in Y(\lambda/\mu)} z_{j-i}} \cdot z_{T'},$$

where $T'$ is the same tableau as $T$, with the entry 1 removed and all other entries decreased by 1.
Proof idea: the Konvalinka recursion, 1

- Let \( f(\lambda/\mu) = \sum_{T \in \text{SYT}(\lambda)} z_T \).
- We easily obtain the recurrence

\[
z_T = \frac{1}{\sum_{(i,j) \in Y(\lambda/\mu)} z_{j-i}} \cdot z_{T'},
\]

where \( T' \) is the same tableau as \( T \), with the entry 1 removed and all other entries decreased by 1.

**Example:** Let \( \lambda = (3, 3, 2) \) and \( \mu = (2, 1) \).

If \( T \) is ...

\[
\begin{array}{ccc}
2 & & \\
1 & 3 & \\
4 & 5 & \\
\end{array}
\]

\( \in \text{SYT}(\lambda/\mu) \)

then \( T' \) is ...

\[
\begin{array}{ccc}
1 & & \\
2 & & \\
3 & 4 & \\
\end{array}
\]

\( \in \text{SYT}(\lambda/\nu) \)

for \( \nu = (2, 2) \). Thus, \( z_T = \frac{1}{z_{-1} + z_{-2} + z_1 + z_2 + z_0} \cdot z_{T'} \).
Proof idea: the Konvalinka recursion, 2

- Thus we get a recurrence for $f(\lambda/\mu)$:

  $$f(\lambda/\mu) = \frac{1}{\sum_{(i,j) \in Y(\lambda/\mu)} z_{j-i}} \cdot \sum_{\mu < \nu \subseteq \lambda} f(\lambda/\nu).$$

- Here, $\mu \lessdot \nu$ means that the partition $\nu$ is obtained by adding 1 to some entry of $\mu$. 
Proof idea: the Konvalinka recursion, 2

- Thus we get a recurrence for $f(\lambda/\mu)$:

$$f(\lambda/\mu) = \frac{1}{\sum_{(i,j) \in Y(\lambda/\mu)} z_{j-i}} \cdot \sum_{\mu < \nu \subseteq \lambda} f(\lambda/\nu).$$

- Here, $\mu < \nu$ means that the partition $\nu$ is obtained by adding 1 to some entry of $\mu$.

- The induction step thus reduces to the following claim:

**Konvalinka recursion.** Let $\lambda/\mu$ be any skew partition, and let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be two infinite families of commuting indeterminates. Then,

$$\left( \sum_{\# i: \lambda_k - k = \mu_i - i} x_k + \sum_{\# j: \lambda^t_p - p = \mu^t_j - j} y_p \right) \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (x_i + y_j)$$

$$= \sum_{\mu < \nu \subseteq \lambda} \sum_{D \in \mathcal{E}(\lambda/\nu)} \prod_{(i,j) \in D} (x_i + y_j).$$
Proof ingredient 1: Flagged SSYTs, 1

- Let $D$ be a diagram. A *semistandard tableau* (of shape $D$) means a filling of the cells of $D$ with positive integers such that
  - the numbers *weakly* increase along each row,
  - the numbers *strictly* increase down each column.

- **Example:** Here is a semistandard tableau for $\mu = (4, 3, 1)$:

```
  1 1 1 2
  2 3 3
  4
```
Proof ingredient 1: Flagged SSYTs, 1

- Let $D$ be a diagram. A *semistandard tableau* (of shape $D$) means a filling of the cells of $D$ with positive integers such that
  - the numbers *weakly* increase along each row,
  - the numbers *strictly* increase down each column.
- A *flagging* means a sequence $b := (b_1, b_2, b_3, \ldots)$ of positive integers.
Let $D$ be a diagram. A \textit{semistandard tableau} (of shape $D$) means a filling of the cells of $D$ with positive integers such that

- the numbers \textit{weakly} increase along each row,
- the numbers \textit{strictly} increase down each column.

A \textit{flagging} means a sequence $b := (b_1, b_2, b_3, \ldots)$ of positive integers.

A \textit{flagged semistandard tableau} of shape $(\mu, b)$ is a semistandard tableau of shape $\nu(\mu)$ in which all entries in row $i$ are $\leq b_i$.

\[
\begin{array}{c}
& \leq b_1 \\
\leq b_2 \\
\leq b_3 \\
\end{array}
\]
Let $D$ be a diagram. A \textit{semistandard tableau} (of shape $D$) means a filling of the cells of $D$ with positive integers such that
- the numbers \textbf{weakly} increase along each row,
- the numbers \textbf{strictly} increase down each column.

A \textit{flagging} means a sequence $b := (b_1, b_2, b_3, \ldots)$ of positive integers.

A \textit{flagged semistandard tableau} of shape $(\mu, b)$ is a semistandard tableau of shape $Y(\mu)$ in which all entries in row $i$ are $\leq b_i$.

For two partitions $\lambda$ and $\mu$, we define $\mathcal{F}(\lambda/\mu)$ to be the set of flagged semistandard tableaux of shape $(\mu, b)$, where $b := (b_1, b_2, b_3, \ldots)$ with

$$b_i := \max \{k \geq i \mid \lambda_k - k \geq \mu_i - i\} \quad \text{for all } i \geq 1.$$
Now, there is a bijection from $\mathcal{E}(\lambda/\mu)$ to $\mathcal{F}(\lambda/\mu)$, defined as follows: Each excitation $D \in \mathcal{E}(\lambda/\mu)$ is sent to the flagged semistandard tableau $T$ of shape $(\mu, b)$, where

$$T(i,j) = i + (\# \text{ of excited moves that cell } (i,j) \text{ makes in } D) .$$

Here $T(i,j)$ means the entry of $T$ in cell $(i,j)$.

**Example:** For $\lambda = (3, 3, 3, 1)$ and $\mu = (2, 1)$:
Proof ingredient 1: Flagged SSYTs, 2

Now, there is a bijection from $\mathcal{E}(\lambda/\mu)$ to $\mathcal{F}(\lambda/\mu)$, defined as follows: Each excitation $D \in \mathcal{E}(\lambda/\mu)$ is sent to the flagged semistandard tableau $T$ of shape $(\mu, b)$, where

$$T(i, j) = i + (\text{# of excited moves that cell } (i, j) \text{ makes in } D).$$

Here $T(i, j)$ means the entry of $T$ in cell $(i, j)$.

**Example:** For $\lambda = (3, 3, 3, 1)$ and $\mu = (2, 1)$:

Thus, we can work with flagged SSYTs instead of excited diagrams.
**Theorem (generalized Jacobi–Trudi formula).** Let 
\[ \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) \] and 
\[ \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k) \] be two partitions. Let \[ a_1 \leq a_2 \leq \cdots \leq a_k \] and 
\[ b_1 \leq b_2 \leq \cdots \leq b_k \] be positive integers. Let \( u_{i,j} \) be a variable for each pair \((i, j) \in \mathbb{Z}^2\). Then,

\[
\sum_{T \text{ is a semistandard tableau}} \prod_{(i,j) \in \mathcal{Y}(\lambda/\mu)} u_{j-i, \ T(i,j)}
\]

\[
= \det \left( \sum_{a_i \leq t_{\mu_i+1} \leq t_{\mu_i+2} \leq \cdots \leq t_{\lambda_j} \leq b_j} \prod_{c=\mu_i+1}^{\lambda_j} u_{c-i, \ t_c} \right)_{i,j \in [k]}.
\]
Theorem (generalized Jacobi–Trudi formula). Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) \) and \( \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k) \) be two partitions. Let \( a_1 \leq a_2 \leq \cdots \leq a_k \) and \( b_1 \leq b_2 \leq \cdots \leq b_k \) be positive integers. Let \( u_{i,j} \) be a variable for each pair \( (i, j) \in \mathbb{Z}^2 \).

Then,

\[
\sum_{T \text{ is a semistandard tableau} \atop \text{of shape } \gamma(\lambda/\mu)} \prod_{(i,j) \in \gamma(\lambda/\mu)} u_{j-i, \ T(i,j)}
\]

\[
= \det \left( \sum_{a_i \leq t_{\mu_i+1} \leq t_{\mu_i+2} \leq \cdots \leq t_{\lambda_j} \leq b_j} \prod_{c=\mu_i+1}^{\lambda_j} u_{c-i, \ t_c} \right)_{i,j \in [k]}.
\]

This is implicit in a preprint of Gessel and Viennot 1989.
If $\mu = (0, 0, \ldots, 0)$ and all $a_i$ are 0 as well, and if $u_{i,j} = x_j + y_{i+j}$, and if we rename $\lambda$ as $\mu$, then the left hand side here becomes

$$\sum_{T \in \text{FSSYT}(\mu, b)} \prod_{(i,j) \in Y(\mu)} (x_{T(i,j)} + y_{T(i,j)+j-i}),$$
If \( \mu = (0,0,\ldots,0) \) and all \( a_i \) are 0 as well, and if
\( u_{i,j} = x_j + y_{i+j} \), and if we rename \( \lambda \) as \( \mu \), then the left hand side here becomes

\[
\sum_{T \in \text{FSSYT}(\mu,b)} \prod_{(i,j) \in Y(\mu)} \left( x_{T(i,j)} + y_{T(i,j)+j-i} \right),
\]

which equals the

\[
\sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (x_i + y_j)
\]

in the Konvalinka recursion.
Proof ingredient 3: a determinantal identity

- Jacobi–Trudi transforms both sides of the Konvalinka recursion into sums of determinants.
- After some nontrivial work, it becomes an easy determinantal identity:

\[
\begin{align*}
\text{det} \begin{pmatrix}
A & B & C \\
a & b & c \\
a' & b' & c'
\end{pmatrix} + \\
\text{det} \begin{pmatrix}
a & b & c \\
A' & B' & C' \\
a' & b' & c'
\end{pmatrix} + \\
\text{det} \begin{pmatrix}
a & B & C \\
a & b & c \\
a' & b' & c'
\end{pmatrix} = \\
\text{det} \begin{pmatrix}
A & b & c \\
A' & b' & c' \\
a & b & c
\end{pmatrix} + \\
\text{det} \begin{pmatrix}
a & B & c \\
a & b & c' \\
a' & b' & c'
\end{pmatrix} + \\
\text{det} \begin{pmatrix}
a & B & C \\
a & b & c \\
a' & b' & c'
\end{pmatrix}.
\end{align*}
\]
Proof ingredient 3: a determinantal identity

**Theorem.** Let $M$ and $N$ be two $n \times n$-matrices. Then,

$$\sum_{k=1}^{n} \det(M \text{ with its } k\text{-th row replaced by the } k\text{-th row of } N) = \sum_{k=1}^{n} \det(M \text{ with its } k\text{-th column replaced by the } k\text{-th column of } N).$$

**Example:**

$$\begin{align*}
\det \begin{pmatrix} A & B & C \\
 a' & b' & c' \\
 a'' & b'' & c'' \end{pmatrix} + \det \begin{pmatrix} a & b & c \\
 A' & B' & C' \\
 a'' & b'' & c'' \end{pmatrix} + \det \begin{pmatrix} a & b & c \\
 a' & b' & c' \\
 A'' & B'' & C'' \end{pmatrix} = \\
\det \begin{pmatrix} A & b & c \\
 a' & B' & c' \\
 a'' & B'' & c'' \end{pmatrix} + \det \begin{pmatrix} a & B & c \\
 A' & b' & c' \\
 a'' & b'' & c'' \end{pmatrix} + \det \begin{pmatrix} a & b & C \\
 A' & b' & C' \\
 a'' & b'' & C'' \end{pmatrix}.
\end{align*}$$
Proof ingredient 4: Combinatorial lemmas

- Two of the lemmas used along the way:

  **Lemma 1.** Let $\lambda$ be a partition. Let $\lambda^t$ be its conjugate (i.e., Young diagram flipped across the main diagonal). Then, the sets

  $$\{ \lambda_i - i \mid i \in \mathbb{N} \} \quad \text{and} \quad \{ j - \lambda^t_j - 1 \mid j \in \mathbb{N} \}$$

  are disjoint and their union is $\mathbb{Z}$.

  **Lemma 2.** Let $b = (b_1, b_2, b_3, \ldots)$ be the flagging of $\lambda/\mu$. Let $\mu^+_i$ be the partition obtained from $\mu$ by increasing the $i$-th entry by 1. Let $b^*_i = (b^*_1, b^*_2, b^*_3, \ldots)$ be the flagging induced by $\lambda/\mu^+_i$. Then:

  $$-1 \leq b^*_i - b_i \leq 0, \quad \text{and} \quad b^*_k = b_k \text{ for all } k \neq i.$$


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