Partial orderings of minors in the positive Grassmannian

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The Positive Grassmanian

Definition

- Let k be an integer such that $0 \leq k \leq n$. The *Grassmannian* $Gr(k,n)$ is the space of all $k-dimensional$ subspaces of $\mathbb{R}^n$.
- For $A \in Gr(k,n)$ let A_I, for $I \in \binom{[n]}{k}$, denote its $k \times k$ submatrix with column set I and $\Delta_I = det(A_I)$.
- The Δ_I are called *Plücker coordinates* and give an embedding of $Gr(k,n)$ into $\mathbb{P}^{\binom{n}{k}-1}$, the $\binom{n}{k}$-dimensional projective space.
- We will be mostly interested in *positive Grassmannian*, $Gr^+(k,n)$. It is the subset of $G(k,n)$ such that $det(A_I) \geq 0$ holds for all $A \in Gr^+(k,n)$ and $I \in \binom{[n]}{k}$.

The Positive Grassmanian(Example)

$A = \begin{pmatrix} 1 & 3 & 2 & \frac{1}{4} \\ 1 & 4 & 3 & \frac{1}{2} \end{pmatrix} \in Gr^{+}(2,4)$ has six Plücker coordinates: $A_{\{1,2\}}$, $A_{\{1,3\}}$, $A_{\{1,4\}}$, $A_{\{2,3\}}$, $A_{\{2,4\}}$, $A_{\{3,4\}}$. Which satisfies $\Delta_{\{1,2\}} = \Delta_{\{1,3\}} = \Delta_{\{2,3\}} = 1$, $\Delta_{\{1,4\}} = \Delta_{\{3,4\}} = \frac{1}{4}, \Delta_{\{2,4\}} = \frac{1}{2}$

Partition of the Positive Grassmanian

Definition

Let $\mathbb{U} = (\mathbb{U}_0, \mathbb{U}_1, \cdots \mathbb{U}_l)$ be an ordered set-partition of $\binom{[n]}{k}$ according to *Plucker* coordinates in $A \in Gr^+(k,n)$ such that:
(1) $\Delta_I = 0$ for $I \in \mathbb{U}_0$
(2) $\Delta_I = \Delta_J$ if $I, J \in \mathbb{U}_i$
(3) $\Delta_I < \Delta_J$ if $I \in \mathbb{U}_i$ $J \in \mathbb{U}_j$ with $i < j$
An arrangement of minors is an ordered set-partition $\mathbb{U}$.

Definition

We say that A_I, $I \in \binom{[n]}{k}$ is a t-largest minor in $A \in G^+(k,n)$ if for ordered set-partition of $G(k,n)$ there exist such i that $\mathbb{U}_i = \mathbb{U}_{l-t}$ and $I \in \mathbb{U}_i$.

Sort1 and Sort2 function

Definition

For a multiset S of elements from [n], let $Sort(S)$ be the non-decreasing sequence obtained by ordering the elements of S. Let $I,J \subset \binom{[n]}{k}$ and let $Sort(I \cup J){=}(a_1;a_2;\cdots;a_{2k})$. Define:

$$Sort_1(I,J) := \{a_1;a_3;\cdots;a_{2k-1}\}, \quad Sort_2(I,J) := \{a_2;a_4;\cdots;a_{2k}\}.$$

A pair I;J is called sorted if $Sort_1(I,J) = I$ and $Sort_2(I,J) = J$, or vice versa. Consequently $\mathbb{U} \subset \binom{[n]}{k}$ is called sorted if every two sets $I,J \in \mathbb{U}$ are sorted.

Example

Let $A = \begin{pmatrix} 1 & 3 & 2 & \frac{1}{4} \\ 1 & 4 & 3 & \frac{1}{2} \end{pmatrix} \in Gr^+(2,4)$ then sets $\{1,2\}$ and $\{3,4\}$ are not sorted where $Sort_1(\{1,2\},\{3,4\}){=}\{1,3\}$, $Sort_2(\{1,2\},\{3,4\}){=}\{2,4\}$

Scanderra inequality

Theorem

Let $I, J \in \binom{[n]}{k}$ *be a pair which is not sorted. Then for all* $A \in Gr^+(k,n)$*, it holds that* $\Delta_{sort_1(I,J)}\Delta_{sort_2(I,J)} > \Delta_I\Delta_J$*.*

Example

Hence from Scanderra inequality $\Delta_{\{1,3\}}\Delta_{\{2,4\}}>\Delta_{\{1,2\}}\Delta_{\{3,4\}}$.
substitution: $\Delta_{\{1,3\}}=1,\Delta_{\{2,4\}}=\frac{1}{2},\Delta_{\{1,2\}}=1,\Delta_{\{3,4\}}=\frac{1}{4}$ and $\frac{1}{2} > \frac{1}{4}$.

Maximal sorted sets

Theorem

The number of elements in the maximal sorted sets of $\binom{[n]}{k}$ is always n.

Theorem (Farber-Postnikov)

The U_l is always a sorted set.

Circuit graph

Definition

Let G be the directed graph where vertices are vectors of $\{0,1\}^n$, where exactly k ones, and vertex is connected to other if the second vertex can be obtained by shifting one 1 in right. Let's call the **circuit** the cycle of length n in the graph G.

Theorem

Maximal sorted sets are in bijection with circuits.

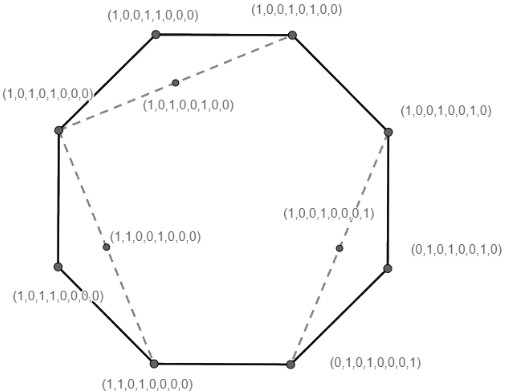
Some examples:

- In $\binom{[6]}{3}$, $\{1,2,4\}$ corresponds to $(1,1,0,1,0,0)$
- $(1,1,0,1,0,0)$ is connected to $(1,1,0,0,1,0)$ and $(1,0,1,1,0,0)$
- In $\binom{[4]}{2}$ the circuit is $\{(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,0,1)\}$

Detour of circuit

Definition

Let's look at three consecutive elements I, J, K in the circuit, if J can be replaced with some J_1, then let's call this process a *detour* of the circuit.



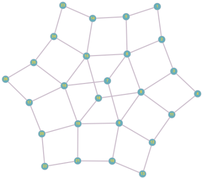
Dual Graph

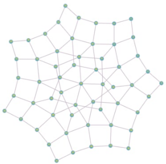
Definition

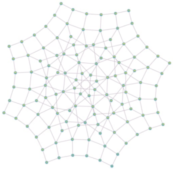
Let's build the dual Graph Γ as follows, the vertices are circuits, and two circuits are connected if they can be obtained one from the other by detour.

Theorem

Dual graphs have $A(k, n)$ vertices, where A is a number of alternating permutations.







Cubical distance

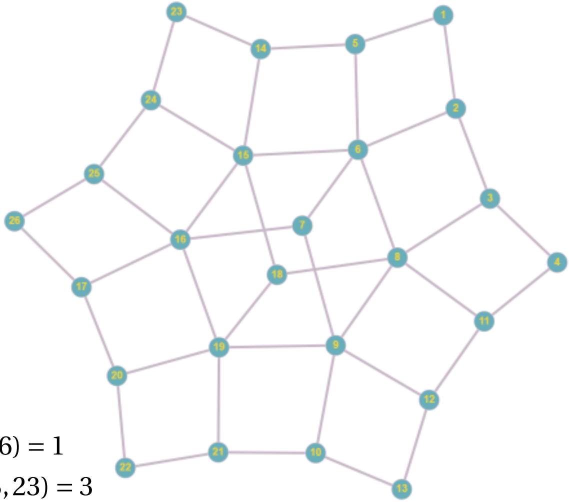
Definition

For two vertices of the dual Graph that are lying on the same cube (of any dimension), the cubical distance is one. And cubical distance between $\mathscr{J}_1$ and $\mathscr{J}_2$ is a number of cubes throw which go path minimally with regard to this property.Cubical distance from $I \in \binom{[n]}{k}$ to $\mathscr{J}$ is the minimal cubical distance from all circuits containing I to $\mathscr{J}$. It is denoted $cube(I, \mathscr{J})$.

Conjecture (Farber-Mandelshtam, 2015)

Let $\mathscr{J} \subset \binom{[n]}{k}$ be an arrangement of largest minors, then the two following statements if $cube(W, \mathscr{J}) = t$ then W is $(\geq t+1, J)$-largest minor.

Examples with the $\Gamma_{(6,3)}$



- $cube_d(1,6) = 1$
- $cube_d(13,23) = 3$

Circuit stratification of graph

Definition

Let's divide the circuit graph into vertical and horizontal levels in the following way. Vertically, we shall arrange them by the remainder of the sum of all numbers modulo n.

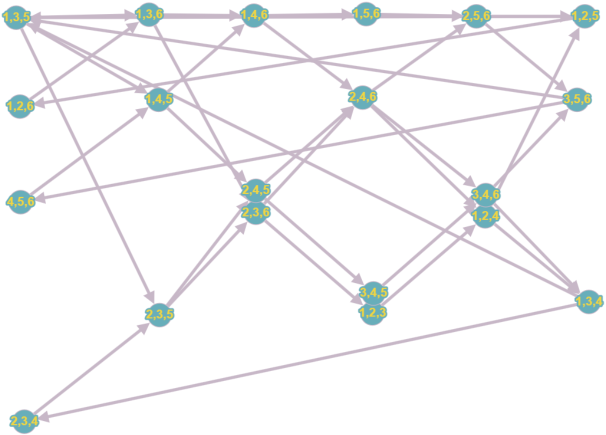
Definition

Horizontal levels are defined inductively. First are the maximal minors, and second are those that are connected with two from the first level. And as follows, the t-th level is those that have at least two connections to the union of the previous levels.

Conjecture

The stratification to horizontal levels is equivalent to the partition according to cubical distance.

Example



Conjecture

For every vertex of the stratification exist a "good quadrilateral"

The end

Thanks to everyone for making this program possible.