Riemann integral in higher dimensions

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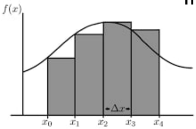
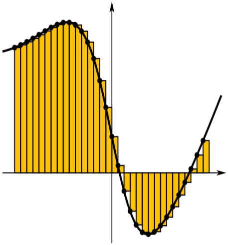
Let f be a *"good enough"* function. Informally, the integral of f is defined as an area under its graph.





Important properties and calculation techniques.

- The integral is linear.
- Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = \Big[F'(x) = f(x)\Big] = F(x)\Big|_a^b$$

- Change of variable.

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \Big[u = g(x)\Big] = \int_{g(a)}^{g(b)} f(u)\,du$$

- Integration by parts.

$$\int_a^b u\cdot v'\,dx = uv\Big|_a^b - \int_a^b u'\cdot v\,dx$$

Integration in dimensions greater than 1

Let A be a closed rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$, and $f : A \subset \mathbb{R}^N \to \mathbb{R}^1$ a bounded function. Let P be a partition of A into rectangles with sides belonging to the collection $(P_1, \cdots, P_n)$, where $P_i = (a_i = t_0 < t_1 < \cdots < t_k = b_i)$ is a partition of the corresponding side of A.
For each subrectangle $S = [a'_1, b'_1] \times \cdots \times [a'_n, b'_n]$ of P define its **volume** and **bounds** as

$$v(S) = \prod_{i=1}^{n} (b'_i - a'_i)$$

$$m_S(f) = \inf \Big\{ f(x) : x \in S \Big\}$$

$$M_S(f) = \sup \Big\{ f(x) : x \in S \Big\}.$$

Define **Lower and Upper Riemann sums** to be

$$L(f, P) = \sum_{S \in P} m_S(f) \cdot v(S)$$

$$U(f, P) = \sum_{S \in P} M_S(f) \cdot v(S).$$

Riemann integral of f on A:

$$\int_A f(x)\, dx = \sup_P L(f, P) =$$
$$= \inf_P U(f, P),$$

in case supremum and infimum are equal [3]; then $f \in R$, where R is a set of integrable functions.

Definition 1

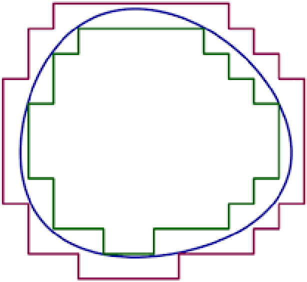
A set $A \subset \mathbb{R}^n$ has **measure 0** if

$$\forall \epsilon > 0 : \exists \{U_1, U_2, U_3, \cdots\} \text{ – closed rectangles} : \quad A \subseteq \bigcup_1^\infty U_i \text{ and } \sum_1^\infty v(U_i) < \epsilon.$$

If these collections could be made finite, we say that A has **content 0**.

Definition 2

A bounded set C whose boundary has measure 0 is called **Jordan-measurable**.



Definition 3

Let $C \subset \mathbb{R}^n$. The **characteristic function** $\mathcal{X}_C$ of the set C is defined by

$$\mathcal{X}_C(x) = \begin{cases} 0 & x \notin C, \\ 1 & x \in C. \end{cases}$$

Definition 4

Let $C \subseteq A \subset \mathbb{R}^n$ for some closed rectangle A. If $f \cdot \mathcal{X}_C \in R$, we define

$$\int_C f \; dx = \int_A f \cdot \mathcal{X}_C \; dx$$

Remark: The characteristic function $\mathcal{X}_C \in R$ if and only if C is Jordan-measurable.

Now the (n-dimensional) **volume** of a Jordan-measurable set $C \subset \mathbb{R}^n$ can be defined as

$$v(C) = \int_C 1 \; dx$$

THEOREM 5

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function. Let $B = \Big\{x : f \text{ is not continuous at } x\Big\}$. Then f is integrable if and only if B is a set of measure 0. In particular, continuity of f implies its integrability.

In particular, the indicator function of the set of rational numbers between 0 and 1 is not integrable!

PROPOSITION

R is an algebra, i.e. $\forall f, g \in R$: (i) $f + g \in R$, (ii) $fg \in R$, (iii) $const \cdot f \in R$.

This follows from the properties of Riemann sums [2].

Fubini's theorem allows us to reduce the compuation of the integral in higher dimensions to repeated computation of integrals in dimension 1. Below we state a simplified version of this theorem for continuous functions.

Theorem 1 (Fubini's Theorem for continuous functions [3])

Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be closed rectangles and let f be continuous on $A \times B$, then $f : A \times B \to \mathbb{R}$ is integrable and

$$\int_{A\times B} f = \int_A (\int_B f(x,y)dy)dx = \int_B (\int_A f(x,y)dx)dy$$

The full version of the theorem allows to work with some non-continuous functions.
Note: the last equation from the theorem is also extremely useful by itself:

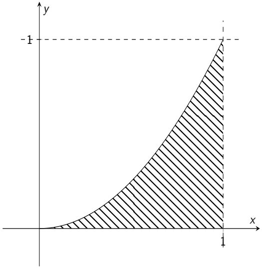
$$\int_A (\int_B f(x,y)dy)dx = \int_B (\int_A f(x,y)dx)dy$$

Problem 6 ([1])

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dxdy = ?$$

Let's understand how domain of integration looks like. From the task we know that

$$\begin{cases} 0 \le y \le 1 \\ \sqrt{y} \le x \le 1 \end{cases} \quad and \quad \begin{cases} x = \sqrt{y} \\ y \ge 0 \end{cases} \quad \Leftrightarrow \begin{cases} y = x^2 \\ x \ge 0 \end{cases}$$



$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dxdy = \int_0^1 \int_0^{x^2} e^{x^3} dydx = \int_0^1 x^2 e^{x^3} dx = \frac{1}{3}(e-1)$$

Recall, that a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point x, if it admits good linear approximations in a neighborhood of x.
More precisely, $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable (see [2]) at x if there exists a matrix A of size $m \times n$ such that

$$\lim_{|h| \to 0} \frac{|f(x+h) - f(x) - A \cdot h|}{|h|} = 0$$

Note: $h \in \mathbb{R}^n$, and the map $h \mapsto Ah$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Moreover, the entries of A are the partial derivatives

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where $f(x) = (f_1(x), \dots, f_m(x))$ with $f_i : \mathbb{R}^n \to \mathbb{R}$. Then A is denoted by $Df(x)$.

Let's consider a simple matrix A of size $n \times n$. Let $v_i = (a_{1,i}, a_{2,i}, \dots, a_{n,i}), i = 1, .., n$, be the collection of columns of A . Then determinant of this matrix will be equal to a directed volume of the n-dimensional parallelepiped defined by these vectors.
There are several formal definitions of the determinant Δ (see [4]):

1. The determinant is an n-linear antisymmetric function which takes value 1 for the standard basis on $\mathbb{R}^n$.
2. The determinant of matrix A is defined as the following sum

$$\det A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i},$$

   where S_n is the set of all permutations of $[n]$.

For a function $f : \mathbb{R}^n \to \mathbb{R}^n$ we can set $A = Df$ as in the previous slide and then calculate the **Jacobian**

$$J = \det A = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

The absolute value of the Jacobian at x_0 gives us the factor by which the function f expands or shrinks volumes near x_0.

Recall the change of variables in dimension 1:

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g'(x)dx$$

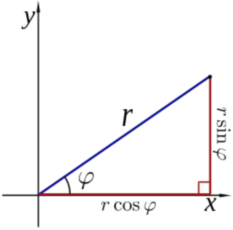
With previously mentioned constructions it can be generalized to the higher dimensions as follows.

THEOREM 2 (CHANGE OF VARIABLE[3])

Let $A \subset \mathbb{R}^n$ be an open set and let $g : A \to \mathbb{R}^n$ be a one-to-one continuously differentiable mapping such that $\forall x \in A$ $\det g'(x) \neq 0$. If $f : g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} f \; dV = \int_A f(g)|\det g'| \; dV$$

Where $\det g'(x)$ is exactly the Jacobian of g at the point x.



In polar coordinates,

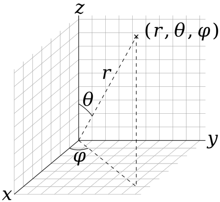
$$x = r \cdot cos\phi, y = r \cdot sin\phi,$$

where radius r varies from 0 to ∞, and angle ϕ from 0 to 2π.
Then Jacobi matrix and its determinant will look like:

$$J = \det \begin{bmatrix} \cos\phi & -r \cdot sin\phi \\ \sin\phi & r \cdot cos\phi \end{bmatrix} = r$$

So, in general, by considering a transformation from Cartesian to polar coordinates,

$$dxdy = rdrd\phi$$



Consider a spherical system of coordinates.
We can express x, y, z through our new coordinates in the next way:

$$x = r \cdot sin\,\theta \cdot cos\,\phi, y = r \cdot sin\,\theta \cdot sin\,\phi, z = r \cdot cos\,\theta,$$

where $r \in [0, \infty]$, angle $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$.
Jacobian matrix of partial derivatives will look like:

$$J = \det \begin{bmatrix} sin\,\theta cos\,\phi & r\,cos\,\theta cos\,\phi & -r\,sin\,\theta sin\,\phi \\ sin\,\theta sin\,\phi & r\,cos\,\theta sin\,\phi & r\,sin\,\theta cos\,\phi \\ cos\theta & -r\,sin\,\theta & 0 \end{bmatrix} = r^2 \cdot sin\,\theta$$

So, in general, by considering a transformation from Cartesian 3-dimensional coordinates to spherical,

$$dx\,dy\,dz = (r^2\,sin\,\theta)\,dr\,d\phi\,d\theta.$$

Now, let's move to some examples, where we may use the transformation of coordinates.

DEFINITION 7

In statistics, a **normal distribution** or **Gaussian distribution** is a type of continuous probability distribution for a real-valued random variable. The general form of its probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

(The parameter μ is the mean or expectation of the distribution, while the parameter σ is its standard deviation.)

As you know, for any distribution, the sum of all probabilities should be 1. So, what really stands for it is the Gaussian integral,

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Let's calculate it. Let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

So,

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy =$$

$$= \iint_{R^2} e^{-x^2-y^2}\, dx\, dy$$

Passing to polar coordinates,

$$I^2 = \iint_{R^2} e^{-x^2-y^2}\, dx\, dy = \iint_{R^2} r \cdot e^{-r^2}\, dr\, d\phi =$$

$$= \int_0^{2\pi} \int_0^{\infty} r \cdot e^{-r^2}\, dr\, d\phi = \int_0^{2\pi} -\frac{1}{2} e^{-r^2} \Big|_0^{\infty} d\phi =$$

$$= \int_0^{2\pi} \frac{1}{2}(e^0 - lim_{x\to\infty} e^{-x}) d\phi = \int_0^{2\pi} \frac{1}{2}\, d\phi = \frac{2\pi - 0}{2} = \pi,$$

which, after further transformation, gives us the desired result.

As we may know, the volume of a ball is just an integral of 1 over a ball:

$$\iiint_{x^2+y^2+z^2\leq R^2} 1\, dx\, dy\, dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta\, dr\, d\theta\, d\phi =$$

$$= \frac{1}{3}R^3 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = -\frac{1}{3}R^3 \int_0^{2\pi} \cos\theta\Big|_0^{\pi} d\phi =$$

$$= \frac{2}{3}R^3 \int_0^{2\pi} d\phi = \frac{2}{3}R^3 \cdot \phi\Big|_0^{2\pi} = \frac{4}{3}\pi R^3$$

Thank you for your attention!

Now just look at this chilling capybara

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[2] W. Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. McGraw-Hill, Inc., 1976. ISBN: 0-07-054235-X.

[3] Michael Spivak. *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*. The Advanced Book Program. Addison-Welsey Publishing Company, 1965. ISBN: 0-8053-9021-9.

[4] Vinberg. *A Course in Algebra*. Graduate Studies in Mathematics. American Mathematical Society, 2003. ISBN: 978-0821834138.