The Generalized Stokes Theorem and its applications

Dmytro Kulish Yaroslav Ibragimov Dang Minh Cong

Mentor: Kyrylo Muliarchyk

May 15, 2023

Contents

Generalized Stokes Theorem

Theorem (Generalized Stokes Theorem)

$$\int_{\partial M} \omega = \int_{M} d\omega$$

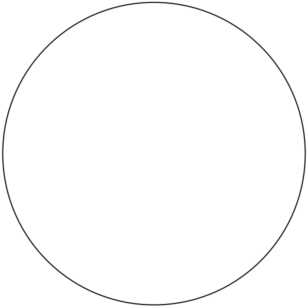
where M is a k-manifold, the border of M, ∂M is an oriented $k-1$-manifold, ω is a differential $k-1$-form.

k-manifolds

$M \subset \mathbb{R}^n$, that locally looks like $\mathbb{R}^k$, i.e. any point inside M has a neighborhood that is a deformed k-dimensional hypersphere, for a point on the border – half of the hypersphere.

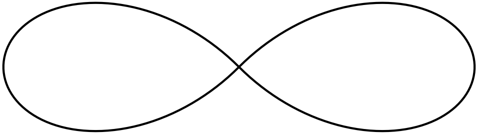
Examples of manifolds

A circle is a 1-manifold.



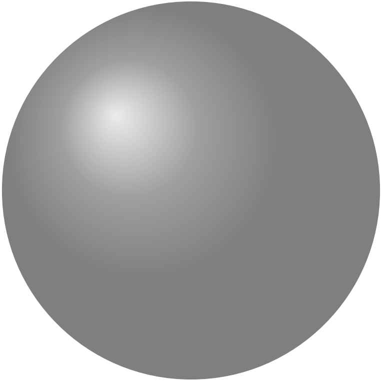
Examples of manifolds

A lemniscate is not a manifold.



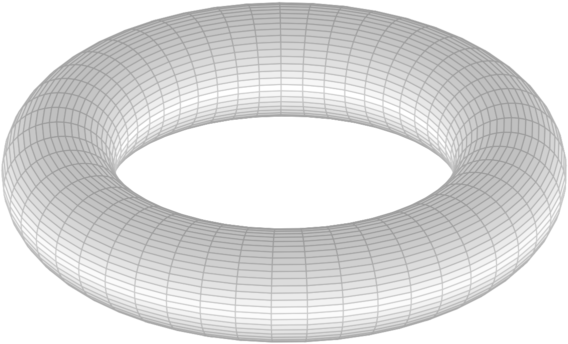
Examples of manifolds

A ball is a 3-manifold whose boundary (a sphere) is a 2-manifold.



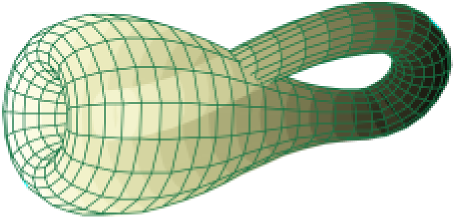
Examples of manifolds

A torus is a 2-manifold.



Examples of manifolds

Klein bottle is a non-oriented 2-manifold.



Differential form

Definition

$$\omega = \sum f(x_1, \dots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

is called a k-differential form.

The operation $\wedge$ is called the wedge product and satisfies the following properties:

- $dx_i \wedge dx_j = -dx_j \wedge dx_i = \pm dx_i dx_j,\ i \neq j,$
- $dx_i \wedge dx_i = 0.$

Differential forms examples

Examples

In $\mathbb{R}^3$ 1-form is the following expression:

$$\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz.$$

2-form is the expression

$$\omega = P(x, y, z)dx \wedge dy + Q(x, y, z)dy \wedge dz + R(x, y, z)dz \wedge dx,$$

whereas 3-form is the expression

$$\omega = f(x, y, z)dx \wedge dy \wedge dz.$$

Derivative of ω

If ω is a k-form, we shall denote the derivative of ω by $d\omega$.

- If f is a 0-form, then

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j.$$

- If ω_1 and ω_2 are k-forms, then

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2.$$

Therefore,

$$\begin{aligned} d\omega &= \sum df \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \\ &= \sum \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

Integration on manifolds

- $\int_{[a,b]} \mathrm{f} dx$- a simple Riemann integral,
- $\int_P dxdy$-the area of the projection of P on the xy plane where P is a region on a plane in the space $\mathbb{R}^3$.

Corollaries

- FTC(Fundamental theorem of calculus)
- Green's theorem
- Kelvin-Stokes' theorem
- Divergence theorem

Fundamental theorem of calculus

Theorem

$$\int_a^b f^{'}(x)dx = f(a) - f(b)$$

Stokes theorem $\Rightarrow$ FTC

Let $\omega = f(x)$, 0-differential form and $M = [a, b]$, then $\partial M = \{a-, b+\}$.

By Stokes theorem for $n = k = 1$ M is 1-manifold and ω is a 0-differential form we have:

$$\int\limits_{\partial M} f = \int\limits_{M} df \Rightarrow f(b) - f(a) = \int\limits_{a}^{b} f^{'}(x)dx.$$

Green's theorem

Theorem

$$\int_{\partial M} Pdx + Qdy = \iint_M \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy$$

Stokes theorem $\Rightarrow$ Green's theorem

$\omega = Pdx + Qdy$, where P and Q are functions on x, y.

$d\omega = dPdx + dQdy = (\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy) \wedge dx + (\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy) \wedge dy = \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$.

The Stokes theorem for $n = k = 2$, ω and 2-manifold M completes the proof.

Kelvin-Stokes' theorem

Theorem

$$\int_{\partial M} (F_x\,dx + F_y\,dy + F_z\,dz) = \iint_M \left(\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) dy\,dz + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) dz\,dx + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx\,dy \right)$$

Stokes theorem $\Rightarrow$ Kelvin-Stokes' theorem

Let $\omega = F_x dx + F_y dy + F_z dz$, where F_x, F_y, F_z are functions on x, y, z, so ω is a differential 1-form.

Kelvin-Stokes' theorem

Then the exterior derivative of ω is

$$\begin{aligned} d\omega &= dF_x dx + dF_y dy + dF_z dz \\ &= \left(\frac{\partial F_x}{\partial x}dx + \frac{\partial F_x}{\partial y}dy + \frac{\partial F_x}{\partial z}dz\right) \wedge dx \\ &\quad + \left(\frac{\partial F_y}{\partial x}dx + \frac{\partial F_y}{\partial y}dy + \frac{\partial F_y}{\partial z}dz\right) \wedge dy \\ &\quad + \left(\frac{\partial F_z}{\partial x}dx + \frac{\partial F_z}{\partial y}dy + \frac{\partial F_z}{\partial z}dz\right) \wedge dz = \end{aligned}$$

Kelvin-Stokes' theorem

$$\frac{\partial F_x}{\partial y} dy \wedge dx + \frac{\partial F_x}{\partial z} dz \wedge dx + \frac{\partial F_y}{\partial x} dx \wedge dy + \frac{\partial F_y}{\partial z} dz \wedge dy + \frac{\partial F_z}{\partial x} dx \wedge dz + \frac{\partial F_z}{\partial y} dy \wedge dz =$$

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) dy\, dz + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) dz\, dx + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx\, dy.$$

The Stokes theorem for $n = 3$, $k = 2$, ω and 2-manifold M completes the proof.

Divergence theorem

Theorem

$$\int_{\partial M}\Big(F_x\, dydz + F_y\, dzdx + F_z\, dxdy\Big)$$
$$= \int_M \left(\frac{\partial F_z}{\partial z} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) dxdydz.$$

Stokes theorem $\Rightarrow$ Divergence theorem
Let $\omega = F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy$, where F_x, F_y, F_z are functions on x, y, z, so ω is a differential 2-form. Then

$$d\omega = dF_z \wedge dx \wedge dy + dF_x \wedge dy \wedge dz + dF_y \wedge dz \wedge dx =$$

Divergence theorem

$$
\begin{aligned}
&= \left(\frac{\partial F_z}{\partial x}dx + \frac{\partial F_z}{\partial y}dy + \frac{\partial F_z}{\partial z}dz\right) \wedge dx \wedge dy \\
&\quad + \left(\frac{\partial F_x}{\partial x}dx + \frac{\partial F_x}{\partial y}dy + \frac{\partial F_x}{\partial z}dz\right) \wedge dy \wedge dz \\
&\quad\quad + \left(\frac{\partial F_y}{\partial x}dx + \frac{\partial F_y}{\partial y}dy + \frac{\partial F_y}{\partial z}dz\right) \wedge dz \wedge dx \\
&= \frac{\partial F_z}{\partial z}dz \wedge dx \wedge dy + \frac{\partial F_x}{\partial x}dx \wedge dy \wedge dz + \frac{\partial F_y}{\partial y}dy \wedge dz \wedge dx \\
&= \left(\frac{\partial F_z}{\partial z} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) dxdydz.
\end{aligned}
$$

The Stokes theorem for $n = 3$, $k = 3$, ω and 3-manifold M completes the proof.

Acknowledgments

We would like to thank:

- Our mentor Kyrylo Muliarchyk
- Our parents
- Our teachers
- Yulia's Dream organizers