Knots and Reidemeister theorem

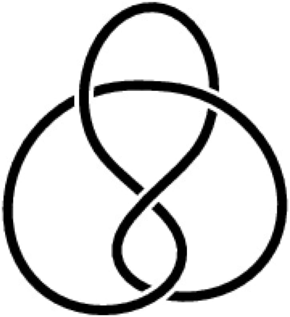
Maksym Torianyk and Ivan Balashov
Mentor: Roman Krutowski

May 2023

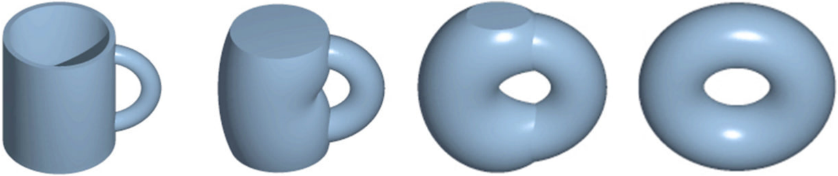
Table of Contents

Informal Definition (Knot)

Knot is a closed curve in topological $\mathbb{R}^3$ *space.*



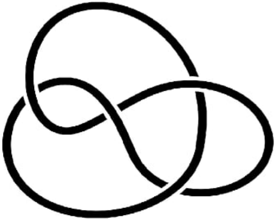
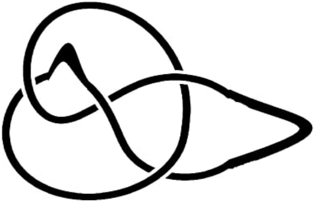
Topological deformations

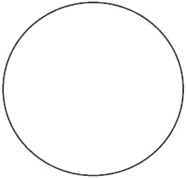
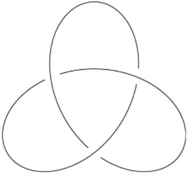
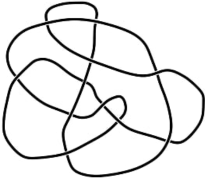
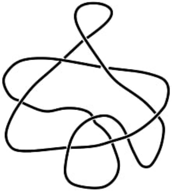


As we can see, in topology cup and torus are the same things.

R^2 plane is equal to point.

Knots

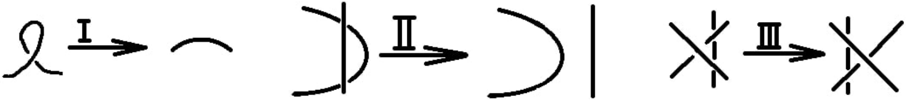



Reidemeister moves

Informal Definition

Reidemeister moves are 3 kinds of not obvious topological deformations that can be used to knots.



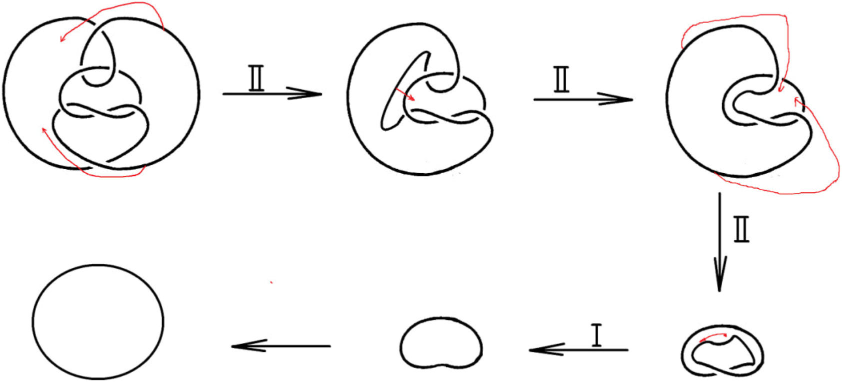
Where I, II, III are first, second and third Reidemeister moves respectively

The main tasks of Knot Theory, science that researches knots, are to answer the question "is this knot an unknot", and to distinguish one knot from another.

Theorem

Two knots are equivalent if and only if, they result from each other by a finite number of Reidemeister moves.

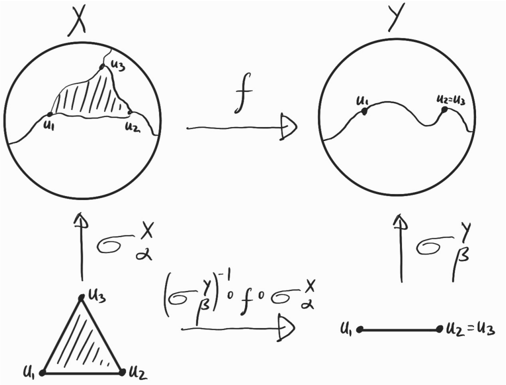
Example



Definition (p.l.-embedding)

Let X and Y be Hausdorff spaces and $f\colon X \to Y$ be an embedding. Then we call f *piecewise linear* iff there exist delta complexes on X with embeddings σ^X_α for $\alpha \in I_X$ and Y with embeddings σ^Y_β for $\beta \in I_Y$ such that:

- for every $\alpha' \in I_X$ we have $\beta' \in I_Y$ such that $f \circ \sigma^X_{\alpha'}\left\{\Delta^{n(\alpha')}\right\} = \sigma^Y_{\beta'}\left\{\Delta^{n(\beta')}\right\}$.
- function $\left(\sigma^Y_{\beta'}\right)^{-1} \circ f \circ \sigma^X_{\alpha'}$ is linear.

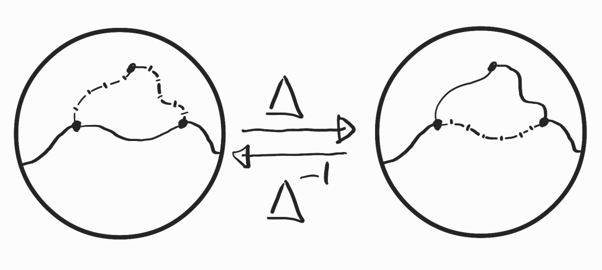


Definition (Tame knots)

A *tame knot* (or simply a knot) K is a p.l.-embedding of S^1 into S^3.

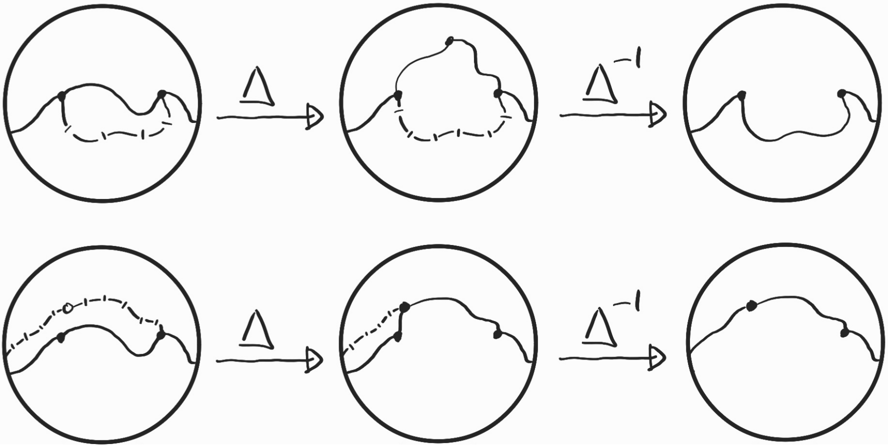
Definition (Δ-move)

Let K and K' be two knots. We say K' *results from* K *by a* Δ*-move* (or equivalently K *results from* K' *by a* Δ^{-1}*-move*) iff the following is true. We can define these p.l.-embeddings for equal delta complexes on S^3. There is a 2-simplex D in them with 1-faces u, v, w. $D \cap K = u$ and $K' = (K \setminus u) \cup v \cup w$.



Definition (Knots Equivalence)

We say that K *and* K' *are equivalent* iff K' results from K by a finite sequence of $\Delta^{\pm 1}$-moves.



Definition (Regular Projection)

A projection p of a knot K is called regular iff

- there are only finitely many points that have more than one preimage point in K, and for all of them the number of them is two. These are called *crossings*.
- no vertex of K is mapped onto a crossing.

Definition (Knot Diagram)

A diagram of a knot is its regular projection with the additional information of which of two segments is higher and which one is lower.

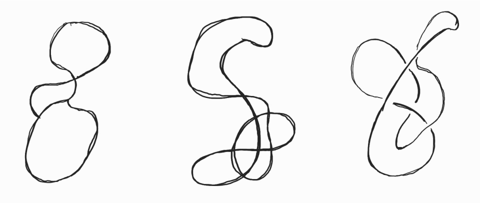
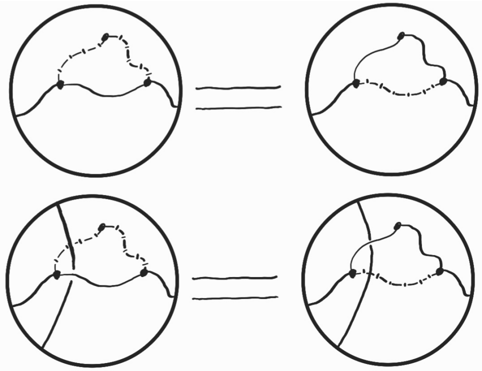


Figure: Knot (a) projection, (b) regular projection, (c) Diagram

Definition (Diagram Isotopy)

We say that two knot diagrams of K and K' are *isotopic* iff K' results from K by a finite sequence of $\Delta^{\pm 1}$-moves. Moreover, each of these moves are required to:

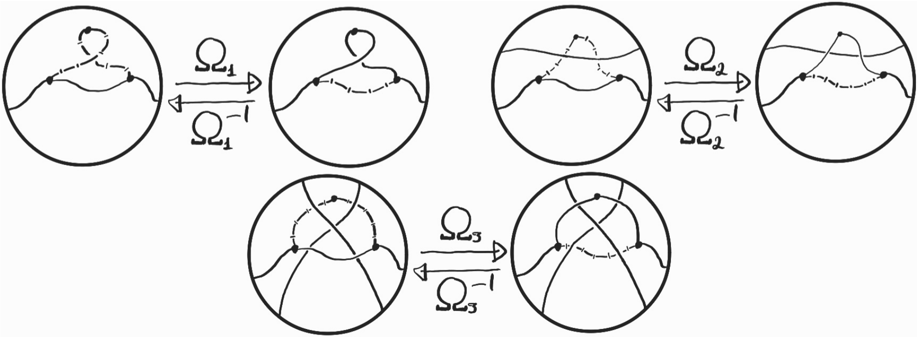
- preserve diagram regularity.
- preserve all crossings or replace one by a crossing on the same segment and of the same type.



Definition (Reidemeister Moves)

Let K and K' be two knots with regular diagrams. The phrase K' *results from* K *by a Reidemeister move* Ω_i, $i \in \{1, 2, 3\}$ means:

1. Ω_1 is equivalent to a Δ which add one crossing generated by two new segments.
2. Ω_2 is equivalent to a Δ which add two crossings of the same type generated by two new segments and an unchanged segment.
3. Ω_3 is equivalent to a Δ which replace two crossings on two unchanged segments by equivalent on them.



Theorem (Reidemeister Theorem)

Two knots are equivalent iff they result from each other by a finite number of Reidemeister moves and diagram isotopy.

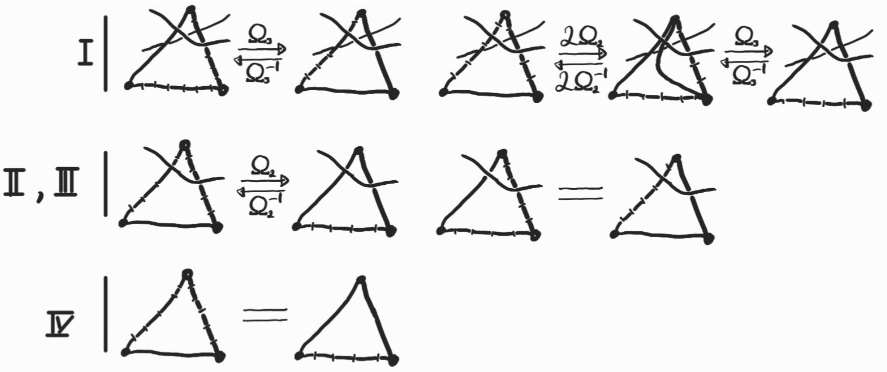
Proof.

Obviously, (Reidemeister moves $\Rightarrow$ Knot Equivalence). Therefore, we only need to show (Δ-move $\Rightarrow$ Reidemeister moves).
Let K be a knot and K' results from K by a Δ-move on a triangle D: $u \mapsto v \cup w$.
We can guarantee that there are no segments with an endpoint in δu inside of D by applying Ω_1 to every of them.
Having this done, it is possible to split D into several triangles of four types:

1. triangle contains exactly two segments with a crossing and each of these segments crosses the same two sides of the triangle.
2. triangle contains exactly one vertex and segments that it connects. Each of these segments crosses different sides of the triangle once.
3. triangle contains only a segment which crosses two different sides of the triangle once.
4. triangle contains nothing.

Proof.

Theorem is now equivalent to the fact: using Reidemeister moves, it is possible to make every Δ-move on all four types of triangles.
For the first type works either Ω_3 or a combination of Ω_2 and Ω_3. For the second and the third — Ω_2 or simply diagram isotopy. For the fourth — diagram isotopy. □



Bracket polynomial

Definition

The Bracket polynomial $< K >$ is based on 3 principles:

1. $< \bigcirc > = 1$
2. $< \times > = t^{1/4} < \asymp > + t^{-1/4} <)(>$
   $< \times > = t^{1/4} <)(> + t^{-1/4} < \asymp >$
3. $< K \cup \bigcirc > = (-t^{1/2} - t^{-1/2}) < K >$

The only problem with this polynomial, is that by using I Reidemeister move, it would change:

$< \curlyvee > = -t^{3/4} < \frown >$

$< \curlyvee > = -t^{-3/4} < \frown >$ We see that I Reidemeister move change value of polynomial of knot in $-t^{3/4}$ times depending on orientation.

Jones polynomial

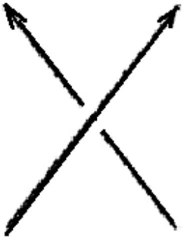
Definition

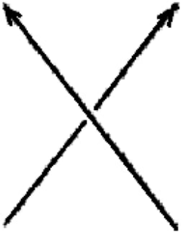
Writhe of knot $w(K)$ is a number that equal to difference of number of *up* and *under* crossings in oriented knot.

Definition

Jones polynomial $V(K)$ of a knot, is a polynomial, which is equal to product of bracket polynomial and $-t^{-3/4}$ to the power of writhe

$$V(K) = (-t^{-3/4})^{w(K)} < K >$$



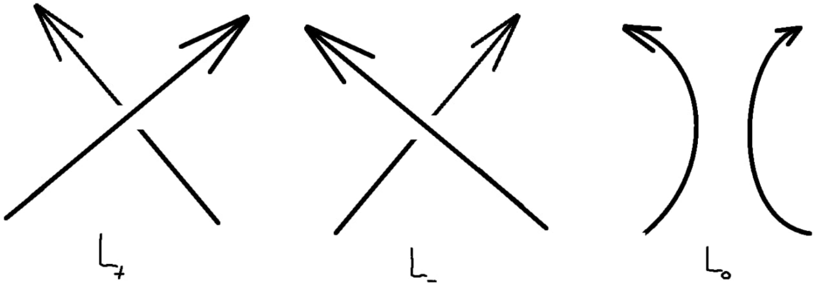


HOMFLY

Definition

HOMFLY is a polynomial with two variables that represent knots, and follow 2 rules:

1. $P(\bigcirc) = 1$
2. $lP(L_+) + l^{-1}P(L_-) = mP(L_0)$



Actually, if we substitute $l = \sqrt{-1}t^{-1}$ and $m = \sqrt{-1}(t^{-1/2} - t^{1/2})$ we will get that $P(K) = V(K)$

Bibliography

- John M. Lee (2011) *Introduction to Topological Manifolds*, GTM.
- Allen Hatcher (2000) *Algebraic topology*, Cambridge Univ. Press.
- Gerhard Burde and Heiner Zieschang (2003) *Knots*, Walter de Gruyter.
- Victor V. Prasolov and Alexey Sossinsky (1997) *Knots, links, braids and 3-manifolds*, MCCME.
- Colin C. Adams (1994) *The Knot Book*, American Mathematical Soc.

The End.

Thank you for the attention!