Matching of frames in open Jacobi diagrams and chord diagrams spaces

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Definition (Vector space)

Vector space - is a set whose elements, often called vectors, may be added together and multiplied scalars.

Example

$\mathbb{R}^n$ - is a vector space over real numbers.

Definition (Equivalent relation)

Equivalent relation - is the relation that satisfy:
1. reflexivity: $x \sim x$;
2. symmetry: $x \sim y$ if , only if $y \sim x$;
3. transitivity: if $x \sim y$ and $y \sim z$ then $x \sim z$.

Let V be a vector space over a field K, and let N be a subspace of V. We define an equivalence relation $\sim$ on V by stating that $x \sim y$ if $x - y \in N$.

Definition (Equivalent classes)

For $x \in V$ equivalent class is

$$[x] = \{x + n : n \in N\} \tag{1}$$

Definition (Quotient space)

The quotient space is set of equivalent classes of V on N and denoted V/N.

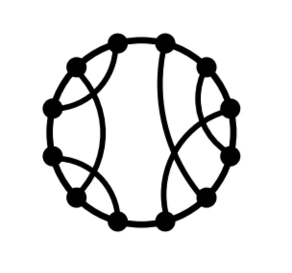
Example

Let $X = \mathbb{R}^2$ and Y is line through the origin in X. Then X/Y is space of lines that parallel to Y.

Definition (Chord diagram)

A chord diagram is a graph with an external circle and chords that lie on that circle.

Example



Definition (Vector spaces of chord diagrams)

Let $A_m = span_{\mathbb{R}}(CD_m)/(all\ 4T\ relations)$ where CD_m set of all chord diagrams with m chords ,4T relation is given by the alteration of two arbitrary chords in a diagram (Fig. 1).

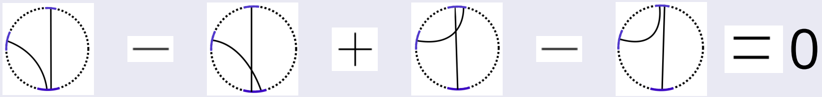


Figure: 4T relation.

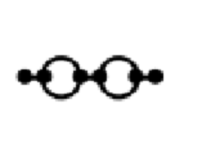
The full space of chord diagrams:

$$A = \bigoplus_{k=0}^{\infty} A_k. \tag{2}$$

Definition (Open Jacobi diagram)

An open Jacobi diagram is a connected graph with 1- and 3-valent vertices, cyclic order of half-edges at every 3-valent vertex.

Example



Relations between Jacobi diagrams

Definition (AS relation)

Definition (IHX relation)

Definition (Vector space of open Jacobi diagrams with k trivalent and h univalent vertices)

$$Jac_{k,h} = span_{\mathbb{R}}(JD_{k,h})/(AS \text{ and } IHX \text{ relations}) \tag{3}$$

where $JD_{k,h}$ is a set of open Jacobi diagrams with k trivalent and h univalent vertices.

Definition (Vector space of open Jacobi diagrams)

$$Jac = \bigoplus_{k,h=0}^{\infty} Jac_{k,h}. \tag{4}$$

Frame of a Jacobi diagram

Frame of a Jacobi diagram $F_{Jac} : Jac \longrightarrow$ "trivalent graphs" is an operation of deleting all univalent vertices and trivalent vertices connected to them from given Jacobi diagram

Example

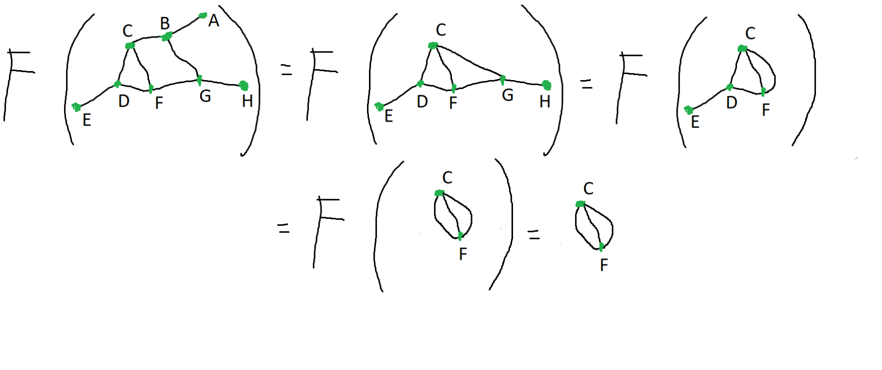


Figure: A non-trivial example of a frame

Frame of a Chord diagram

Frame of a Chord diagram $F_{CD} : CD \longrightarrow$ "trivalent graphs" is an operation that forgets a cycle on a Chord diagram and transforms it to a trivalent graph

Example

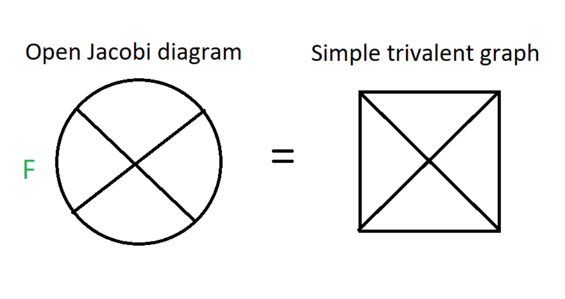


Figure: Where we have a Chord diagram on the left and a trivalent graph on the right

Important observation: We couldn't define frame operation as $F : Jac \longrightarrow CD$ because if a Jacobi diagram has 2 different Hamiltonian cycles, then its frame can be interpreted as 2 different Chord diagrams

Example

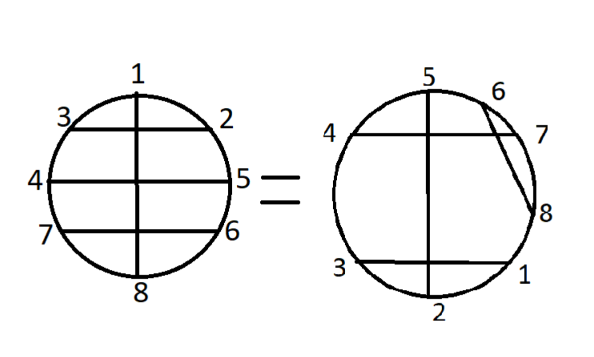
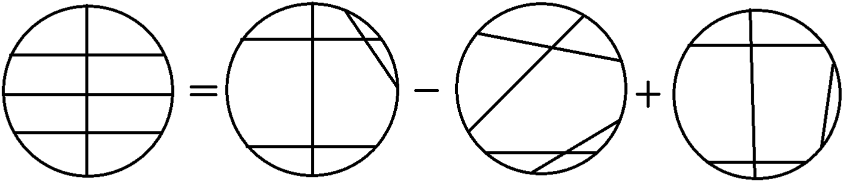


Figure: Jac diagrams with 2 different Hamiltonian paths

These Jac diagrams are equal by AS

Their frames are not equal as Chord diagrams by 4T relation



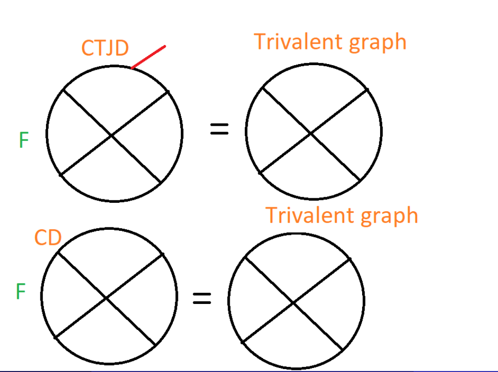
But those frames are equal as trivalent graphs. That's why we define two separate frame operations:
$F_{Jac} : Jac \longrightarrow$ "trivalent graphs" and $F_{CD} : CD \longrightarrow$ "trivalent graphs"

Chord type Jacobi diagrams

We call given Jacobi diagram a Chord type Jacobi diagram (CTJD) if it has at least 1 Hamiltonian cycle through all of its trivalent vertices.

Property: $\forall\ a \in CTJD\ \exists\ b \in CD_j : F_{Jac}(a) = F_{CD}(b)$

Example



Jacobi diagram decomposition theorem

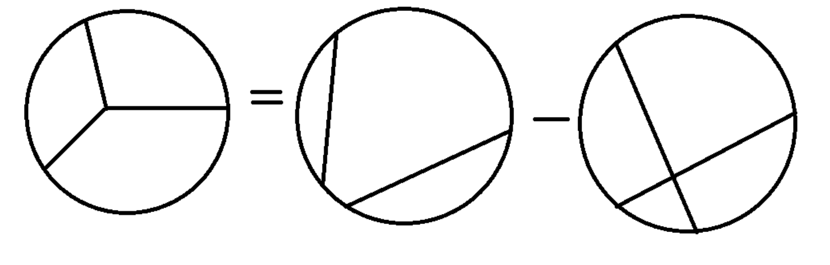
For every Jac diagram J:

$$J = \sum_k c_k J_K \tag{5}$$

$$\tag{6}$$

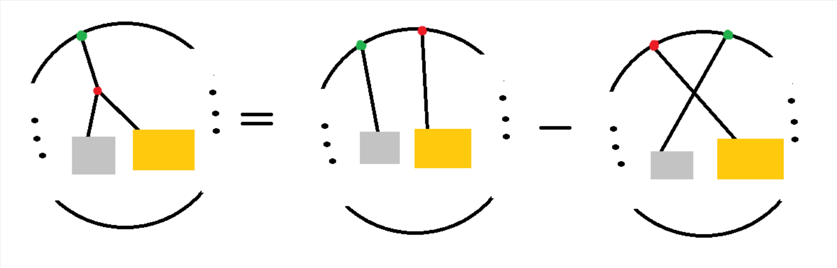
Where J_k is CTJD and $c_k \in \mathbb{Z}$

Example



Proof.

1) As long as there is a 3-valent vertice not on the circle, there is a 3-valent vertice not on the circle connected to the 3-valent vertice on the circle



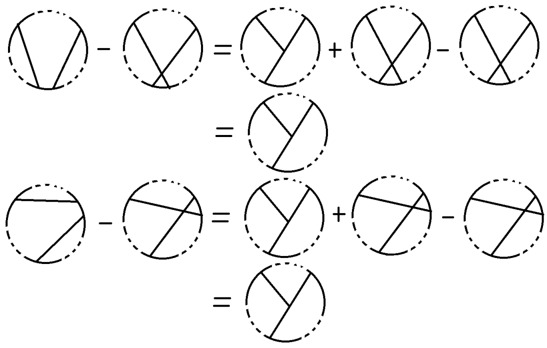
2) Induction on the number of trivalent vertices not on the circle □

Comparing relations between CTJD and ones between chord diagrams

Remark

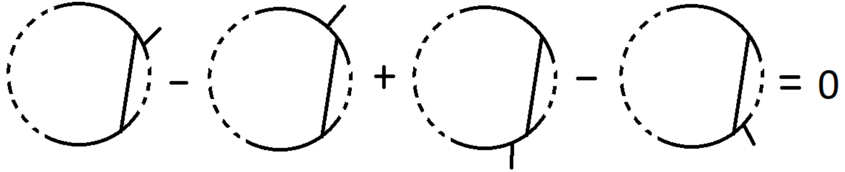
CTJD with no hairs have the same 4T relation as chord diagrams, which are made by applying IHX twice.

Example



However, not all relations between CTJD have analogous relations in A

Example (4T' relation)



QUESTION: Is the list of relations between CTJD exhaustive by $4T$, $4T'$, and internal symmetries?

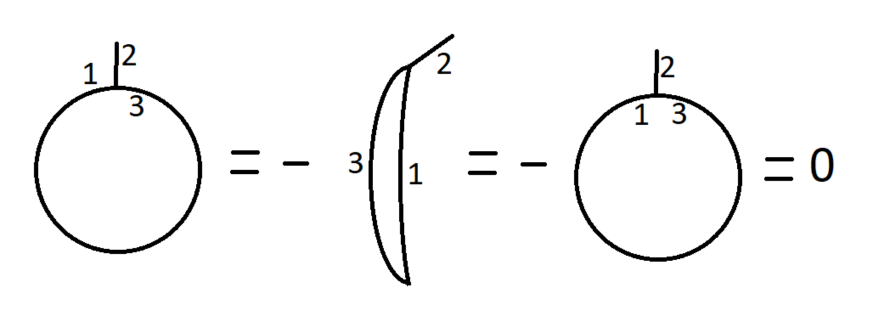
Birman's Conjecture

Conjecture

Every open Jacobi diagram with odd number of univalent vertices is equal to zero in the factor space of open Jacobi diagrams.

Example

for $k = 1, h = 1$ we have:



Simplification of Birman’s conjecture

Conjecture

Every CTJD with odd number of univalent vertices is equal to zero in the factor space of open Jacobi diagrams.

Proving this simplified conjecture is sufficient to proving the Birman’s conjecture according to Jacobi diagram decomposition theorem.

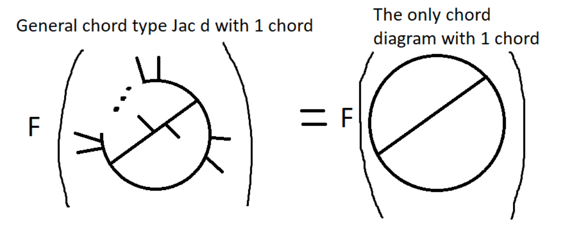
Vasiliy's Conjecture

Conjecture

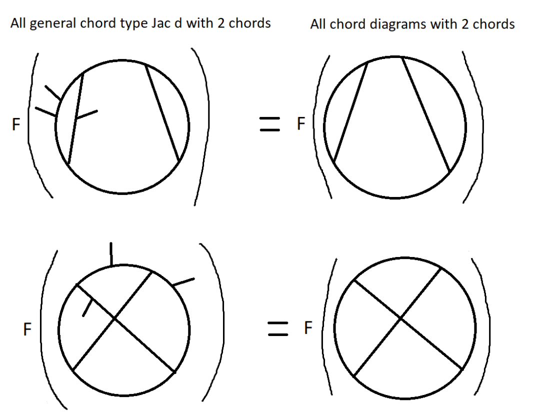
$$\forall\ m,n\ \exists\ B = <c_1, c_2, c_3, \ldots>, \text{ a basis of } A = \bigoplus_i A_i, c_i \in CD : \ \forall j \in J_{m,n}$$

$$j = \sum_k t_k j_k \ : j_k \in J_{m,n}\ ,\ F_{Jac}(j_k) \in \ \{F_{CD}(c_1), F_{CD}(c_2), F_{CD}(c_3), \ldots\} \quad (7)$$

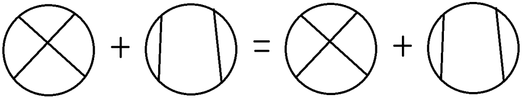
Example (Chord type Jacobi diagrams with 1 chord)



Example (Chord type Jacobi diagrams with 2 chords)

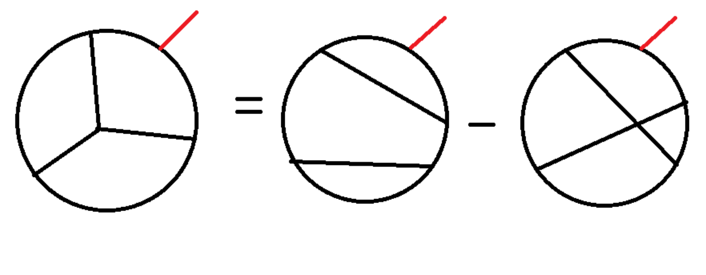


The only 4T relation for chord diagrams with 2 chords is trivial



For Jacobi diagrams that are not CTJD we need to use the Jacobi diagram decomposition theorem first when working on Vasiliy's Conjecture

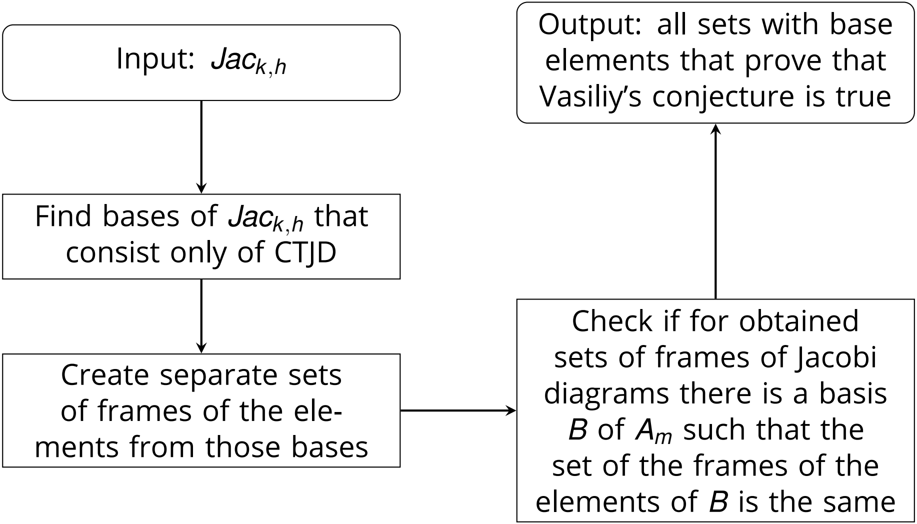
Example



Computer experimentation

- Generation and creation of A_m. Acknowledgment goes to Vasiliy Dolgushev.
- Generation of $JD_{k,h}$. Acknowledgment goes to Vincent Delecroix.
- Generation of IHX, AS; computation of $Jac_{k,h}$.
- Computation of frame functions.
- Finding all unique Hamiltonian paths.
- Functions required for verification of whether Vasiliy's conjecture is valid in $Jac_{k,h}$.

The algorithm for verifying Vasiliy's conjecture



The End

Questions? Comments?