Exploration of Grothendieck-Teichmüller (GT) shadows for the dihedral poset

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Background

Definition (Braid group $B_3$)

The Artin braid group on 3 strands is

$$B_3 := \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$
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Let us denote by \( \rho \) a homomorphism that is defined as

\[ \rho : B_3 \rightarrow S_3, \]

where \( S_3 \) is a symmetric group on a set of 3 elements and

\[ \rho(\sigma_1) := (12), \quad \rho(\sigma_2) := (23). \]
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**Definition (Pure braid group $PB_3$)**

Pure braid group $PB_3$ is the kernel of $\rho$:

$$PB_3 := \ker(\rho).$$
For $N \trianglelefteq_{f.i.} G$ the sets $\text{NFI}(G)$ and $\text{NFI}_N(G)$ are defined as

$$\text{NFI}(G) := \{ H \mid H \trianglelefteq_{f.i.} G \}, \quad \text{NFI}_N(G) := \{ H \mid H \subset N, H \trianglelefteq_{f.i.} G \}.$$
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Let us denote \(x_{12} := \sigma_1^2, x_{23} := \sigma_2^2, c := (\sigma_1 \sigma_2)^3\).

It can be shown that \(\langle x_{12}, x_{23} \rangle \cong F_2\), so we will identify \(F_2\) with \(\langle x_{12}, x_{23} \rangle\). It is known that \(PB_3 \cong F_2 \times \langle c \rangle\).
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**Definition ($N_{ord}$ and $N_{F_2}$)**

Let $N \in \text{NFI}_{\text{PB}_3}(B_3)$ and let

$$N_{ord} := \text{lcm}(\text{ord}(x_{12}N), \text{ord}(x_{23}N), \text{ord}(cN)),$$

$$N_{F_2} := N \cap F_2.$$
Definition \((GT\text{-pair with the target } N)\)

A GT-pair with the target \(N\) is a pair

\[
(m + N_{\text{ord}}\mathbb{Z}, fN_{F_2}) \in \mathbb{Z}/N_{\text{ord}}\mathbb{Z} \times F_2/N_{F_2}
\]

satisfying hexagon relations:

\[
\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} fN = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m N
\]

and

\[
f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1} N = \sigma_2 \sigma_1 x_{23}^{-m} c^m fN.
\]

By writing \([m, f]\), we will mean the GT-pair represented by \((m, f)\).

We will motivate the hexagon relations by the fact that if they are satisfied for a GT-pair \([m, f]\), then \(T_{m,f}\), the map that is described on the next slide, is a homomorphism.
For every GT-pair with the target N, \([m, f]\), the formulas

\[
T_{m,f}(\sigma_1) = \sigma_1^{2m+1}N, \quad T_{m,f}(\sigma_2) = f^{-1}\sigma_2^{2m+1}fN
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define a group homomorphism from \(B_3\) to \(B_3/N\).
Proposition

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Definition \((GT\text{-shadow with the target } N)\)

A GT-pair \([m, f]\) is called a \textbf{GT-shadow with the target }\(N\) if it satisfies two technical conditions and if the homomorphism \( T_{m,f} \) is surjective. The set of GT-shadows with the target \( N \) is denoted by \( \text{GT}(N) \).
Proposition (Groupoid $\text{GTSh}$)

GT-shadows form a groupoid $\text{GTSh}$ with $\text{Ob}(\text{GTSh}) := N\text{FI}_{\text{PB}_3}(B_3)$ and $\text{GTSh}(K, N) := \{[m, f] \in \text{GT}(N) | \ker(T_{m,f}) = K\}$. 

Let $E_{m,f}: F_2 \to F_2$, $E_{m,f}(x_12) := x_2^m + x_1^2$, $E_{m,f}(x_23) := f - x_2^m + x_2^2f$. Then composition of morphisms in $\text{GTSh}$ is defined like this: $[m_1, f_1] \circ [m_2, f_2] := [2m_1m_2 + m_1 + m_2, f_1E_{m_1,f_1}(f_2)]$. 

Ivan B. and Vadym P. Dihedral Poset and GT-Shadows
Proposition \textit{(Groupoid GTSh)}

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Let $E_{m,f} : F_2 \rightarrow F_2$,

$E_{m,f}(x_{12}) := x_{12}^{2m+1}$, \quad $E_{m,f}(x_{23}) := f^{-1}x_{23}^{2m+1}f$. 

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$$[m_1, f_1] \circ [m_2, f_2] := [2m_1 m_2 + m_1 + m_2, f_1 E_{m_1,f_1}(f_2)].$$
Let us denote $D_n := \langle r, s \mid r^n, s^2, srs \rangle$, $n \in \mathbb{Z}_{\geq 3}$ the standard dihedral group and $\psi_n$ the homomorphism $\text{PB}_3 \rightarrow D_n^3$, which is defined by formulas:

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\psi_n(x_{12}) := (r, s, s), \quad \psi_n(x_{23}) := (rs, r, rs), \quad \psi_n(c) := (1, 1, 1).
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Also we define $K^{(n)} := \ker(\psi_n)$ and $K^{(n)}_{F_2} = K^{(n)} \cap F_2$. It can be shown that $K^{(n)} \leq B_3$, and thus $\text{Dih} := \{K^{(n)} \mid n \in \mathbb{Z}_{\geq 3}\}$ is a subposet of $\text{NFI}_{\text{PB}_3}(B_3)$. We call $\text{Dih}$ the \textbf{dihedral poset}. 
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Note that $K_{\text{ord}}^{(n)} = \text{lcm}(n, 2)$. 

Results

We have explicitly described the set $\text{GT}(K^{(n)})$ of GT-shadows with the target $K^{(n)}$. If

$$\mathcal{X}_n := \{ m : m \in \{0, 1, \ldots, K^{(n)}_{\text{ord}} - 1\} \mid \gcd(2m + 1, K^{(n)}_{\text{ord}}) = 1 \};$$

$$\kappa(m) := \begin{cases} 
    m + 1, & \text{if } 2 \nmid m, \\
    -m, & \text{if } 2 \mid m;
\end{cases}$$

then

$$\text{GT}(K^{(n)}) = \{ (m, (r^{2k}, r^{-2k}, r^{\kappa(m)})) \mid m \in \mathcal{X}_n, k \equiv \frac{\kappa(m)}{2} \mod 2 \}. $$
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$$\varphi(m) := \begin{cases} m + 1, & \text{if } 2 \nmid m, \\ -m, & \text{if } 2 \mid m; \end{cases}$$

then

$$\text{GT}(K^{(n)}) = \{(m, (r^{2k}, r^{-2k}, \varphi(m))) \mid m \in \mathcal{X}_n, k \equiv \frac{\varphi(m)}{2} \text{ mod } 2\}.$$ 

We have shown that $K^{(n)}$ is an isolated object of the groupoid GTSh for every $n \in \mathbb{Z}_{\geq 3}$ (i.e., for $H \neq K^{(n)}$, $H \not\rightarrow K^{(n)}$).
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\[ K^{(q)} \subsetneq K^{(n)} \iff n \mid \text{lcm}(q, 2). \]
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We have shown that if \( n, q \in \mathbb{Z}_{\geq 3} \) and \( K^{(q)} \subset K^{(n)} \), then the natural reduction map

\[ R_{K^{(q)}, K^{(n)}} : \text{GT}(K^{(q)}) \to \text{GT}(K^{(n)}) \]

is surjective.
Since $K^{(n)}$ is an isolated object of $\text{GTSh}$, $\text{GT}(K^{(n)})$ is a group. We would like to describe this group.
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$$K^{(n)} \rightarrow \text{GT}(K^{(n)}).$$

On the level of morphisms, the functor sends the natural morphism $K^{(q)} \rightarrow K^{(n)}$ to the reduction homomorphism $\mathcal{R}_{K^{(q)},K^{(n)}} : \text{GT}(K^{(q)}) \rightarrow \text{GT}(K^{(n)})$. 
Motivation

\[ \text{Ihara embedding} \]

\[ \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \widehat{\text{GT}} \rightarrow \widehat{\text{GT}} \times \text{NFI}_{PB_3}(B_3) \]

\[ g \mid_{E \in \text{Gal}(E/\mathbb{Q})} \] (surjective)

\[ \text{Gal}(E/\mathbb{Q}) \]

Dessins d’enfant (Child’s drawings)

\[ \text{Mor}(\text{GTSh}) \]

\[ \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \text{Aut}(\overline{\mathbb{Q}}), \quad \widehat{\text{GT}} \subset \hat{\mathbb{Z}} \times \hat{F}_2, \]

\[ \text{Gal}(E/\mathbb{Q}) = \text{Aut}(E), \quad E/\mathbb{Q}. \]
Selected References


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THANK YOU!