Exploration of Grothendieck-Teichmüller (GT) shadows for the dihedral poset

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Background

Definition (*Braid group B_3*)

The Artin braid group on 3 strands is

$$B_3 := \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

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Let us denote by ρ a homomorphism that is defined as

$$\rho : B_3 \to S_3,$$

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Definition (*Pure braid group PB_3*)

Pure braid group PB_3 is the kernel of ρ:

$$PB_3 := ker(\rho).$$

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Definition (*NFI*)

For $N \trianglelefteq_{f.i.} G$ the sets $NFI(G)$ and $NFI_N(G)$ are defined as

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Let us denote $x_{12} := \sigma_1^2$, $x_{23} := \sigma_2^2$, $c := (\sigma_1\sigma_2)^3$.
It can be shown that $< x_{12}, x_{23} > \cong F_2$, so we will identify F_2 with $< x_{12}, x_{23} >$. It is known that $PB_3 \cong F_2 \times \langle c \rangle$.

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Definition (N_{ord} *and* N_{F_2})

Let $\mathrm{N} \in NFI_{PB_3}(B_3)$ and let

$$\mathrm{N}_{ord} := \mathrm{lcm}(\mathrm{ord}(x_{12}N), \mathrm{ord}(x_{23}N), \mathrm{ord}(cN)),$$

$$\mathrm{N}_{\mathrm{F}_2} := \mathrm{N} \cap \mathrm{F}_2.$$

Definition (*GT-pair with the target N*)

A GT-pair with the target N is a pair

$$(m + N_{ord}\mathbb{Z}, fN_{F_2}) \in \mathbb{Z}/N_{ord}\mathbb{Z} \times \mathrm{F}_2/\mathrm{N}_{\mathrm{F}_2}$$

satisfying **hexagon relations**:

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f\mathrm{N} = f^{-1}\sigma_1\sigma_2 x_{12}^{-m} c^m \mathrm{N}$$

and

$$f^{-1}\sigma_2^{2m+1} f \sigma_1^{2m+1}\mathrm{N} = \sigma_2\sigma_1 x_{23}^{-m} c^m f\mathrm{N}.$$

By writing $[m, f]$, we will mean the GT-pair represented by (m, f).

We will motivate the hexagon relations by the fact that if they are satisfied for a GT-pair $[m, f]$, then $T_{m,f}$, the map that is described on the next slide, is a homomorphism.

Background

Proposition

For every GT-pair with the target N, $[m, f]$, the formulas

$$T_{m,f}(\sigma_1) = \sigma_1^{2m+1}\mathrm{N}, \quad T_{m,f}(\sigma_2) = f^{-1}\sigma_2^{2m+1} f\mathrm{N}$$

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Definition (*GT-shadow with the target N*)

A GT-pair $[m, f]$ is called a **GT-shadow with the target N** if it satisfies two technical conditions and if the homomorphism $T_{m,f}$ is surjective. The set of GT-shadows with the target N is denoted by GT(N).

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Proposition (*Groupoid GTSh*)

GT-shadows form a groupoid GTSh with $Ob(\mathsf{GTSh}) := \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$ and $\mathsf{GTSh}(\mathsf{K}, \mathsf{N}) := \{[m, f] \in \mathsf{GT}(N) | \ker(T_{m,f}) = \mathsf{K}\}$.

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$$\text{Let } E_{m,f} : F_2 \to F_2,$$

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Then composition of morphisms in GTSh is defined like this:

$$[m_1, f_1] \circ [m_2, f_2] := [2m_1 m_2 + m_1 + m_2, f_1 E_{m_1, f_1}(f_2)].$$

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Let us denote $D_n := \langle r, s \mid r^n, s^2, srsr \rangle, n \in \mathbb{Z}_{\geq 3}$ the standard dihedral group and ψ_n the homomorphism $\mathrm{PB}_3 \longrightarrow D_n^3$, which is defined by formulas:

$$\psi_n(x_{12}) := (r, s, s), \quad \psi_n(x_{23}) := (rs, r, rs), \quad \psi_n(c) := (1, 1, 1).$$

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It can be shown that $\mathrm{K}^{(n)} \trianglelefteq B_3$, and thus $\mathrm{Dih} := \{\mathrm{K}^{(n)} \mid n \in \mathbb{Z}_{\geq 3}\}$ is a subposet of $\mathrm{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. We call Dih the **dihedral poset.**

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Note that $K^{(n)}_{ord} = lcm(n, 2)$.

Results

- We have explicitly described the set $\mathsf{GT}(\mathsf{K}^{(n)})$ of GT-shadows with the target $\mathsf{K}^{(n)}$. If

$$\mathcal{X}_n := \{m : m \in \{0, 1, \ldots, K^{(n)}_{\mathrm{ord}} - 1\} \mid \gcd(2m+1, K^{(n)}_{\mathrm{ord}}) = 1\};$$

$$\varkappa(m) := \begin{cases} m+1, & \text{if } 2 \nmid m, \\ -m, & \text{if } 2 \mid m; \end{cases}$$

then

$$\mathsf{GT}(\mathsf{K}^{(n)}) = \{(m, (r^{2k}, r^{-2k}, r^{\varkappa(m)})) \mid m \in \mathcal{X}_n, k \equiv \frac{\varkappa(m)}{2} \bmod 2\}.$$

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- We have shown that $K^{(n)}$ is an isolated object of the groupoid GTSh for every $n \in \mathbb{Z}_{\geq 3}$ (i.e., for $H \neq K^{(n)}$, $H \not\to K^{(n)}$).

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- We have shown that if $n, q \in \mathbb{Z}_{\geq 3}$ and $\mathsf{K}^{(q)} \subset \mathsf{K}^{(n)}$, then the natural reduction map

$$\mathcal{R}_{\mathsf{K}^{(q)},\mathsf{K}^{(n)}} : \mathsf{GT}(\mathsf{K}^{(q)}) \to \mathsf{GT}(\mathsf{K}^{(n)})$$

is surjective.

Future Plans

- Since $K^{(n)}$ is an isolated object of $\mathtt{GTSh}$, $\mathsf{GT}(K^{(n)})$ is a group. We would like to describe this group.

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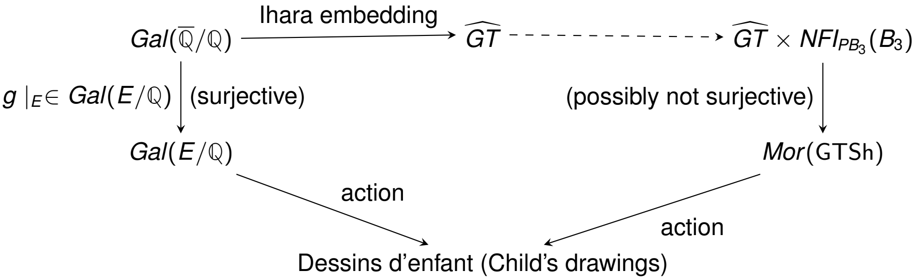
- We would like to find the limit of the functor from Dih to the category of the finite groups.
  On the level of objects, the functor operates like this:

$$K^{(n)} \to \mathrm{GT}(K^{(n)}).$$

  On the level of morphisms, the functor sends the natural morphism $K^{(q)} \to K^{(n)}$ to the reduction homomorphism $\mathcal{R}_{\mathrm{K}^{(q)},\mathrm{K}^{(n)}} : \mathrm{GT}(\mathrm{K}^{(q)}) \to \mathrm{GT}(\mathrm{K}^{(n)})$.

Motivation



$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) = Aut(\overline{\mathbb{Q}}), \qquad \widehat{GT} \subset \widehat{\mathbb{Z}} \times \widehat{F_2},$$
$$Gal(E/\mathbb{Q}) = Aut(E), \quad E/\mathbb{Q}.$$

Selected References

[1] V. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Algebra i Analiz **2**, 4 (1990) 149–181.

[2] V.A. Dolgushev and J.J. Guynee, GT-shadows for the gentle version $\widehat{\mathrm{GT}}_{gen}$ of the Grothendieck-Teichmueller group, in preparation.

[3] D. Harbater and L. Schneps, Approximating Galois orbits of dessins, *Geometric Galois actions,* 1, 205–230, London Math. Soc. Lecture Note Ser., **242**, Cambridge Univ. Press, Cambridge, 1997.

[4] V. A. Dolgushev, K. Q. Le, A. A. Lorenz "What are GT-shadows?" arXiv:2008.00066.

[5] J. Xia, GT-Shadows related to finite quotients of the full modular group, Master Thesis, 2021, Temple University, `https://scholarshare.temple.edu/handle/20.500.12613/6910`.

[6] Y. Ihara, On the embedding of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ into GT, Cambridge Univ. Press, Cambridge, 2011.

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THANK YOU!