Generating Functions

Exponential Generating Functions, Catalan Numbers, and the Snake Oil Method

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Exponential Generating Functions

Definition, Product Formula and Application on Stirling Numbers of the Second Kind

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Exponential Generating Functions

Definition

- Let $f_n \ {}_{n \geq 0}$ be a sequence of real numbers, then its exponential generating function takes the form $F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$.

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  When $f_n = 1$, $F(x) = \sum_{n\geq 0} \frac{x^n}{n!} = e^x$.

Exponential Generating Functions

Product Formula

- Let a_i and b_k be two sequences with exponential generating functions $A(x) = \sum_{i \geq 0} a_i \frac{x^i}{i!}$ and $B(x) = \sum_{k \geq 0} b_k \frac{x^k}{k!}$, respectively.

Exponential Generating Functions

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  Then the terms of $A(x)B(x)$ take the form

  $$a_i \frac{x^i}{i!} \cdot b_j \frac{x^j}{j!}$$

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  $$a_i \frac{x^i}{i!} \cdot b_j \frac{x^j}{j!} = a_i b_j \frac{x^{i+j}}{i!j!}$$

Exponential Generating Functions

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$$a_i \frac{x^i}{i!} \cdot b_j \frac{x^j}{j!} = a_i b_j \frac{x^{i+j}}{i!j!} = a_i b_j \cdot \frac{x^{i+j}}{i!j!} \cdot \frac{(i+j)!}{(i+j)!} = a_i b_j \cdot \frac{x^{i+j}}{(i+j)!} \cdot \binom{i+j}{i}$$

Exponential Generating Functions

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  and are of degree n when $i + j = n$, where

$$a_i b_j \cdot \frac{x^{i+j}}{(i+j)!} \cdot \binom{i+j}{i} = \binom{n}{i} a_i b_{n-i} \frac{x^n}{n!}.$$

Exponential Generating Functions

Product Formula

- Therefore, the coefficients of $\frac{x^n}{n!}$ in $A(x)B(x)$ terms are $\binom{n}{i} a_i b_{n-i}$ for all real i, $0 \leq i \leq n$, so we obtain the coefficient

$$\sum_{i \geq 0}^{n} \binom{n}{i} a_i b_{n-i}.$$

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$$\sum_{i\geq 0}^{n}\binom{n}{i}a_ib_{n-i}.$$

Let C(x) be the exponential generating function of $c_n = \sum_{i\geq 0}^{n}\binom{n}{i}a_ib_{n-i}$. Then

$$A(x)B(x) = C(x).$$

Exponential Generating Functions

Product Formula

- Let a_n be the numbers of ways to build a certain structure on an n-element set, and b_n be the number of ways to build another structure on an n-element set.

Exponential Generating Functions

Product Formula

- Let a_n be the numbers of ways to build a certain structure on an n-element set, and b_n be the number of ways to build another structure on an n-element set.
  Let c_n be the number of ways to separate a set of n elements into disjoint subsets S and T, and then build a structure of the first kind on S and the second kind on T.

Exponential Generating Functions

Product Formula

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  Then there are $\binom{n}{i}$ ways to choose S, a_i ways to build the first structure on S and b_{n-i} ways to build the second structure on T, for all $0 \leq i \leq n$.

Exponential Generating Functions

Product Formula

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  Then there are $\binom{n}{i}$ ways to choose S, a_i ways to build the first structure on S and b_{n-i} ways to build the second structure on T, for all $0 \leq i \leq n$. Therefore

$$c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}.$$

Exponential Generating Functions

Product Formula

- Let A(x), B(x), and C(x) be the exponential generating functions of the sequences a_n, b_n and c_n, respectively. Then it follows from previously that

$$A(x)B(x) = C(x).$$

- Stirling numbers of the second kind, $S(n, k)$ describes the number of partitions of n elements into k nonempty, non-ordered subsets, where k is a fixed positive integer.

Exponential Generating Functions

Application on Stirling Numbers of the Second Kind

- Let $S_k(x) = \sum_{n \geq k} S(n,k) \frac{x^n}{n!}$ be the exponential generating function of $S(n,k)$.

Exponential Generating Functions

Application on Stirling Numbers of the Second Kind

- Let $S_k(x) = \sum_{n \geq k} S(n,k) \frac{x^n}{n!}$ be the exponential generating function of $S(n,k)$.

  $S(n,k)$ requires a partition of n elements into k nonempty disjoint subsets, and nothing is to be done on each subset.

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  $S(n,k)$ requires a partition of n elements into k nonempty disjoint subsets, and nothing is to be done on each subset.

  As there is one way to do nothing on a nonempty subset, each task has exponential generating function

$$A(x) = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1.$$

- Using the product formula, we obtain the generating function $(A(x))^k$ for the combined task of partitioning n elements into k ordered subsets.

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  But as the order does not matter, our task has generating function

$$S_k(x) = \frac{1}{k!}(A(x))^k = \frac{1}{k!}(e^x - 1)^k.$$

- To obtain an explicit formula for $S(n,k)$, we need to find the coefficient of $\frac{x^n}{n!}$ in $S_k(x)$.

Exponential Generating Functions

Application on Stirling Numbers of the Second Kind

$$S_k(x) = \frac{1}{k!}(e^x - 1)^k$$

Exponential Generating Functions

Application on Stirling Numbers of the Second Kind

$$S_k(x) = \frac{1}{k!}(e^x - 1)^k = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}e^{(k-i)x}$$

Exponential Generating Functions

Application on Stirling Numbers of the Second Kind

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Exponential Generating Functions

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Exponential Generating Functions

Application on Stirling Numbers of the Second Kind

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Exponential Generating Functions

Application on Stirling Numbers of the Second Kind

$$S_k(x) = \sum_{n \geq k} S(n,k)\frac{x^n}{n!} = \sum_{n \geq 0}\sum_{i=0}^{k}(-1)^i \frac{(k-i)^n}{i!(k-i)!}\frac{x^n}{n!}$$

Exponential Generating Functions

Application on Stirling Numbers of the Second Kind

$$S_k(x) = \sum_{n \geq k} S(n,k) \frac{x^n}{n!} = \sum_{n \geq 0} \sum_{i=0}^{k} (-1)^i \frac{(k-i)^n}{i!(k-i)!} \frac{x^n}{n!}$$

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Exponential Generating Functions

Definition, Product Formula and Application on Stirling Numbers of the Second Kind

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The Catalan Numbers

Definition, Example, and the Generating Function of the Catalan Numbers

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- Catalan numbers form a sequence of natural numbers that occur in various counting problems.

The Catalan Numbers

Example

- A student moves into a new room and upon his arrival, he puts an empty jar on his kitchen counter. From then on, every day he either puts a dollar coin in the jar, or takes a dollar coin out of the jar. After 2n days, the jar is empty again. In how many different ways could this happen?

The Catalan Numbers

Example

- Let p = put in a coin = +1, t = take out a coin = -1
- The sequence must start with a p and end with a t, and the sum of the terms from day 0 to day j, $0 \leq j \leq 2n$ must never be negative.

The Catalan Numbers

Example

- $n = 0$......... $2n = 0$ .... Ways: 1
- $n = 1$......... $2n = 2$ .... Ways: 1 (pt)
- $n = 2$......... $2n = 4$ .... Ways: 2 (ptpt, pptt)
- $n = 3$......... $2n = 6$ .... Ways: 5 (ptptpt, ppttpt, ptpptt, pptptt, pppttt)
- ...
- Let c_n be the number of ways to put in or take out coins for $2n$ days, then we have
  $c_0 = 1$, $c_1 = 1$, $c_2 = 2$, $c_3 = 5$

- A sequence of length $2n$:

  $p......1......t_a......2......$

  Let t_a be on the $(2i+2)^{th}$ day that the jar is empty again for the first time,

  1: a sequence of length $2i$, $0 \leq i \leq n-1$
  2: a sequence of length $2n-2i-2$

The Catalan Numbers

Example

$p......1......t_a......2......$

Sequences 1 and 2 both have to start with a p and end with a t, and are never negative,

1: a sequence of length $2i$, $0 \leq i \leq n-1$, c_i ways
2: a sequence of length $2n-2i-2$, c_{n-i-1} ways, for $0 \leq i \leq n-1$

Then we have $c_n = \sum_{i=0}^{n-1} c_i c_{n-i-1}$.

The Catalan Numbers

Example

$c_n = \sum_{i=0}^{n-1} c_i c_{n-i-1}$

For $n = 4$, $2n = 8$ days, we obtain $c_4 = \sum_{i=0}^{3} c_i c_{3-i} = 14.$

We can check this by listing the ways in order of i, where sequence 1 corresponding to c_i is **bold** and sequence 2 corresponding to c_{n-i-1} is $\underline{\text{underlined}}$:

$i = 0$: $\mathrm{pt}_a\underline{\mathrm{pppttt}}$, $\mathrm{pt}_a\underline{\mathrm{pptptt}}$, $\mathrm{pt}_a\underline{\mathrm{ppttpt}}$, $\mathrm{pt}_a\underline{\mathrm{ptpptt}}$, $\mathrm{pt}_a\underline{\mathrm{ptptpt}}$
$i = 1$: $\mathrm{p}\mathbf{pt}\mathrm{t}_a\underline{\mathrm{pptt}}$, $\mathrm{p}\mathbf{pt}\mathrm{t}_a\underline{\mathrm{ptpt}}$
$i = 2$: $\mathrm{p}\mathbf{pptt}\mathrm{t}_a\underline{\mathrm{pt}}$, $\mathrm{p}\mathbf{ptpt}\mathrm{t}_a\underline{\mathrm{pt}}$
$i = 3$: $\mathrm{p}\mathbf{pppttt}\mathrm{t}_a$, $\mathrm{p}\mathbf{ptpptt}\mathrm{t}_a$, $\mathrm{p}\mathbf{pptptt}\mathrm{t}_a$, $\mathrm{p}\mathbf{ptptpt}\mathrm{t}_a$, $\mathrm{p}\mathbf{ppttpt}\mathrm{t}_a$

The Catalan Numbers

Finding the Generating Function

- Let c_n be the Catalan numbers which are defined by the previous terms.

$$
\begin{aligned}
c_n &= c_0 c_{n-1} + c_1 c_{n-2} + c_2 c_{n-3} + \ldots + c_{n-1} c_0 \\
&= \sum_{i=0}^{n-1} c_i c_{n-i-1}
\end{aligned}
$$

The Catalan Numbers

Finding the Generating Function

Let $C(x) = \sum_{n \geq 0} c_n x^n$ be the ordinary generating function of the sequence of the numbers c_n.

according to

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-i-1}$$

$$\begin{aligned} C(x) &= c_0 + c_1 x + c_2 x^2 + \ldots \\ &= 1 + 1x + 2x^2 + 5x^3 + \ldots \\ &= \sum_{n \geq 0} c_n x^n \end{aligned}$$

The Catalan Numbers

Finding the Generating Function

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$$\begin{aligned} C(x)^2 &= c_0^2 + (c_0 c_1 + c_1 c_0)x + (c_0 c_2 + c_1 c_1 + c_0 c_2)x^2 + \ldots \\ &= c_1 + c_2 x + c_3 x^2 + \ldots \\ &= \sum_{n \geq 0} c_{n+1} x^n \end{aligned}$$

The Catalan Numbers

Finding the Generating Function

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\begin{aligned}
C(x)^2 &= c_0^2 + (c_0c_1 + c_1c_0)x + (c_0c_2 + c_1c_1 + c_0c_2)x^2 + ... \\
&= c_1 + c_2x + c_3x^2 + ... \\
&= \sum_{n \geq 0} c_{n+1}x^n
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$$

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xC(x)^2 &= c_0^2x + (c_0c_1 + c_1c_0)x^2 + (c_0c_2 + c_1c_1 + c_0c_2)x^3 + ... \\
&= c_1x + c_2x^2 + c_3x^3 + ... \\
&= \sum_{n \geq 0} c_{n+1}x^{n+1}
\end{aligned}
$$

The Catalan Numbers

Finding the Generating Function

$$C(x) = 1 + c_1 x + c_2 x^2 + ...$$
$$xC(x)^2 = c_1 x + c_2 x^2 + c_3 x^3 + ...$$

The Catalan Numbers

Finding the Generating Function

$$xC(x)^2 = C(x) - 1$$
$$xC(x)^2 - C(x) + 1 = 0$$

The Catalan Numbers

Finding the Generating Function

$$C(x)_1 = \frac{1 + \sqrt{1 - 4x}}{2x}$$

$$C(x)_2 = \frac{1 - \sqrt{1 - 4x}}{2x}$$

The Catalan Numbers

Finding the Generating Function

Recall:

$\sqrt{1-4x}$ is the unique power series whose square is $1-4x$ and whose constant coefficient is 1.

By using the binomial thereom,

$$(1+x)^m = \sum_{n\geq 0} \binom{m}{n} x^n,$$

then

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4x)^n.$$

The Catalan Numbers

Finding the Generating Function

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4x)^n$$

To simplify this expression,

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdots\frac{-2n+3}{2}}{n!} = (-1)^{n-1}\frac{(2n-3)!!}{2^n\cdot n!},$$

then we have

$$\sqrt{1-4x} = 1 - 2x - \sum_{n\geq 2} \frac{2^n(2n-3)!!}{2^n\cdot n!}x^n.$$

$$\sqrt{1-4x} = 1 - 2x - \sum_{n\geq 2} \frac{2^n(2n-3)!!}{2^n \cdot n!} x^n,$$

and

$$\frac{2^n(2n-3)!!}{n!} = \frac{2^n(2n-3)!!}{n!} \cdot \frac{(n-1)!}{(n-1)!} = 2\frac{(2n-2)!}{n!(n-1)!},$$

then

$$\sqrt{1-4x} = 1 - 2x - 2\sum_{n\geq 2} \frac{1}{n}\binom{2n-2}{n-1} x^n.$$

Going back to the two solutions of the quadratic equation, substituting

$$\sqrt{1-4x} = 1 - 2x - 2\sum_{n\geq 2} \frac{1}{n}\binom{2n-2}{n-1}x^n$$

into them, we obtain

$$C(x)_1 = \frac{1 + 1 - 2x - 2\sum_{n\geq 2} \frac{1}{n}\binom{2n-2}{n-1}x^n}{2x}$$

and

$$C(x)_2 = \frac{1 - 1 + 2x + 2\sum_{n\geq 2} \frac{1}{n}\binom{2n-2}{n-1}x^n}{2x}.$$

The Catalan Numbers

Finding the Generating Function

Taking the valid solution $C(x)_2$, we have

$$
\begin{aligned}
C(x) &= \frac{2x + 2\sum_{n\geq 2} \frac{1}{n}\binom{2n-2}{n-1} x^n}{2x} \\
&= 1 + \sum_{n\geq 2} \frac{1}{n}\binom{2n-2}{n-1} x^{n-1} \\
&= 1 + \sum_{n\geq 1} \frac{1}{n+1}\binom{2n}{n} x^n \\
&= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n.
\end{aligned}
$$

As

$$C(x) = \sum_{n \geq 0} c_n x^n = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n,$$

therefore

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$

The Snake Oil Method

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The Snake Oil Method

Background

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

The Snake Oil Method

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- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
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The Snake Oil Method

Background

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
- $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$
- $\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$

The Snake Oil Method

Background

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
- $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$
- $\sum_{n=0}^{\infty} (n+1) x^n = \frac{1}{(1-x)^2}$
- $\sum_{n=0}^{\infty} (n+1)(n+2) x^n = \frac{2}{(1-x)^3}$

The Snake Oil Method

Background

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- $\sum_{n=0}^{\infty} (n+1)(n+2)x^n = \frac{2}{(1-x)^3}$
- $\sum_{n=0}^{\infty} \binom{n+2}{2} x^n = \frac{1}{(1-x)^3}$

The Snake Oil Method

Background

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
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- $\sum_{n=0}^{\infty} \binom{n+3}{3} x^n = \frac{1}{(1-x)^4}$

The Snake Oil Method

Background

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- $\sum_{n=0}^{\infty} \binom{n+2}{2}x^n = \frac{1}{(1-x)^3}$
- $\sum_{n=0}^{\infty} \binom{n+3}{3}x^n = \frac{1}{(1-x)^4}$
- $\sum_{n=0}^{\infty} \binom{n+k}{k}x^n = \frac{1}{(1-x)^{k+1}}$

The Snake Oil Method

The Question

- $a_n = \sum_{k \geq 0} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$

The Snake Oil Method

The Solution Part 1

- $a_n = \sum_{k \geq 0} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$
- $a_n x^n = \sum_{k \geq 0} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} x^n$

The Snake Oil Method

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- $\sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} \sum_{k \geq 0} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} x^n$

The Snake Oil Method

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- $\sum_{n \geq 0} a_n x^n = \sum_{k \geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \sum_{n \geq 0} \binom{n+k}{m+2k} x^n$

The Snake Oil Method

The Solution Part 2

- $\sum_{n\geq 0} a_n x^n = \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \sum_{n\geq 0} \binom{n+k}{m+2k} x^n$

The Snake Oil Method

The Solution Part 2

- $\sum_{n\geq 0} a_n x^n = \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \sum_{n\geq 0} \binom{n+k}{m+2k} x^n$
- $\sum_{n\geq 0} \binom{n+k}{m+2k} x^n$

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The Solution Part 2

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The Solution Part 2

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- $\sum_{n\geq m+k} \binom{n+k}{m+2k} x^n$
- $\sum_{n\geq 0} \binom{n+m+2k}{m+2k} x^{n+m+k}$

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The Solution Part 2

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- $x^{m+k} \sum_{n\geq 0} \binom{n+m+2k}{m+2k} x^n$

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The Solution Part 2

- $\sum_{n\geq 0} a_n x^n = \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \sum_{n\geq 0} \binom{n+k}{m+2k} x^n$
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- $\sum_{n\geq 0} \binom{n+m+2k}{m+2k} x^{n+m+k}$
- $x^{m+k} \sum_{n\geq 0} \binom{n+m+2k}{m+2k} x^n$
- Recall: $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}$

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- $\sum_{n\geq 0} a_n x^n = \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \sum_{n\geq 0} \binom{n+k}{m+2k} x^n$
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- $\sum_{n\geq 0} \binom{n+m+2k}{m+2k} x^{n+m+k}$
- $x^{m+k} \sum_{n\geq 0} \binom{n+m+2k}{m+2k} x^n$
- Recall: $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}$
- $\frac{x^{m+k}}{(1-x)^{m+2k+1}}$

The Snake Oil Method

The Solution Part 3

- $\sum_{n\geq 0} a_n x^n = \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \frac{x^{m+k}}{(1-x)^{m+2k+1}}$

The Snake Oil Method

The Solution Part 3

- $\sum_{n\geq 0} a_n x^n = \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \frac{x^{m+k}}{(1-x)^{m+2k+1}}$
- $\sum_{n\geq 0} a_n x^n = \frac{x^m}{(1-x)^{m+1}} \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \frac{x^k}{(1-x)^{2k}}$

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The Solution Part 3

- $\sum_{n\geq 0} a_n x^n = \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \frac{x^{m+k}}{(1-x)^{m+2k+1}}$
- $\sum_{n\geq 0} a_n x^n = \frac{x^m}{(1-x)^{m+1}} \sum_{k\geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \frac{x^k}{(1-x)^{2k}}$
- $\sum_{n\geq 0} a_n x^n = \frac{x^m}{(1-x)^{m+1}} \sum_{k\geq 0} \binom{2k}{k} \frac{1}{k+1} (\frac{-x}{(1-x)^2})^k$

The Snake Oil Method

Catalan numbers

- $\sum_{k\geq 0}\binom{2k}{k}\frac{1}{k+1}x^k = \frac{1-\sqrt{1-4x}}{2x}$

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The Snake Oil Method

Catalan numbers

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- $\sum_{k\geq 0} \binom{2k}{k} \frac{1}{k+1} (\frac{-x}{(1-x)^2})^k = \frac{1-\sqrt{1+\frac{4x}{(1-x)^2}}}{\frac{-2x}{(1-x)^2}}$

The Snake Oil Method

The Solution Part 4

- $\sum_{n\geq 0} a_n x^n = (\frac{x^m}{(1-x)^{m+1}})\left(\frac{1-\sqrt{1+\frac{4x}{(1-x)^2}}}{\frac{-2x}{(1-x)^2}}\right)$

The Snake Oil Method

The Solution Part 4

- $\sum_{n\geq 0} a_n x^n = (\frac{x^m}{(1-x)^{m+1}})\left(\frac{1-\sqrt{1+\frac{4x}{(1-x)^2}}}{\frac{-2x}{(1-x)^2}}\right)$
- $\sum_{n\geq 0} a_n x^n = (\frac{-x^{m-1}}{2(1-x)^{m-1}})\left(1-\sqrt{1+\frac{4x}{(1-x)^2}}\right)$

The Snake Oil Method

The Solution Part 4

- $\sum_{n\geq 0} a_n x^n = (\frac{x^m}{(1-x)^{m+1}})\left(\frac{1-\sqrt{1+\frac{4x}{(1-x)^2}}}{\frac{-2x}{(1-x)^2}}\right)$
- $\sum_{n\geq 0} a_n x^n = (\frac{-x^{m-1}}{2(1-x)^{m-1}})\left(1 - \sqrt{1 + \frac{4x}{(1-x)^2}}\right)$
- $\sum_{n\geq 0} a_n x^n = (\frac{-x^{m-1}}{2(1-x)^{m-1}})(1 - \frac{1+x}{1-x})$

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The Solution Part 4

- $\sum_{n\geq 0} a_n x^n = (\frac{x^m}{(1-x)^{m+1}})\left(\frac{1-\sqrt{1+\frac{4x}{(1-x)^2}}}{\frac{-2x}{(1-x)^2}}\right)$
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- $\sum_{n\geq 0} a_n x^n = \frac{x^m}{(1-x)^m}$

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The Solution Part 4

- $\sum_{n\geq 0} a_n x^n = (\frac{x^m}{(1-x)^{m+1}})\left(\frac{1-\sqrt{1+\frac{4x}{(1-x)^2}}}{\frac{-2x}{(1-x)^2}}\right)$
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- $\sum_{n\geq 0} a_n x^n = (\frac{-x^{m-1}}{2(1-x)^{m-1}})(1-\frac{1+x}{1-x})$
- $\sum_{n\geq 0} a_n x^n = \frac{x^m}{(1-x)^m}$
- $a_n = \binom{n-1}{m-1} = \sum_{k\geq 0} \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$

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Background On Final step

- $\sum_{n\geq 0}^{\infty} x^n = \frac{1}{1-x}$

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Background On Final step

- $\sum_{n\geq 0}^{\infty} x^n = \frac{1}{1-x}$
- $\sum_{n\geq 0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}$
- $\sum_{n\geq 0}^{\infty} \binom{n+m-1}{m-1} x^{n+m} = \frac{x^m}{(1-x)^m}$

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Background On Final step

- $\sum_{n\geq 0}^{\infty} x^n = \frac{1}{1-x}$
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- $\sum_{n\geq 0}^{\infty} \binom{n+m-1}{m-1} x^{n+m} = \frac{x^m}{(1-x)^m}$
- $\sum_{n\geq m}^{\infty} \binom{n-1}{m-1} x^n = \frac{x^m}{(1-x)^m}$