Bipartite Graphs and Matchings

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23.6.18

König's Theorem

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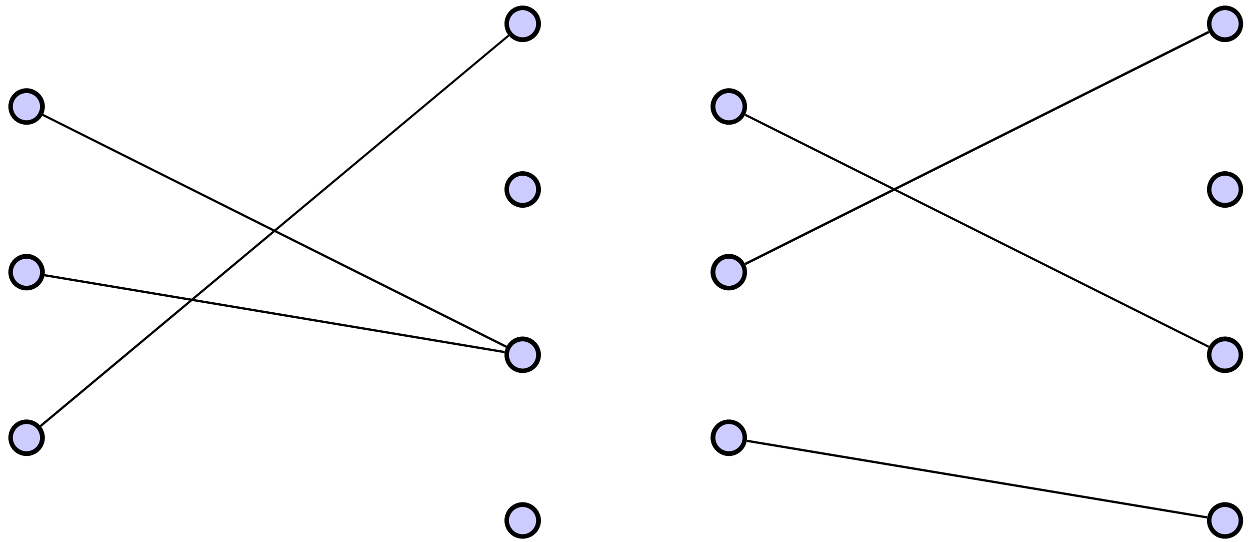
KEN 5th year Highschool

23.6.18

Introduction

- Bipartite Graph: Graph whose vertices can be partitioned into two different independent sets U and V such that no edges are between any two vertices in either set.
- Perfect matching of U into V: Set of edges without common vertices. Where every vertex in U is connected to an edge which goes to a distinct vertex in V



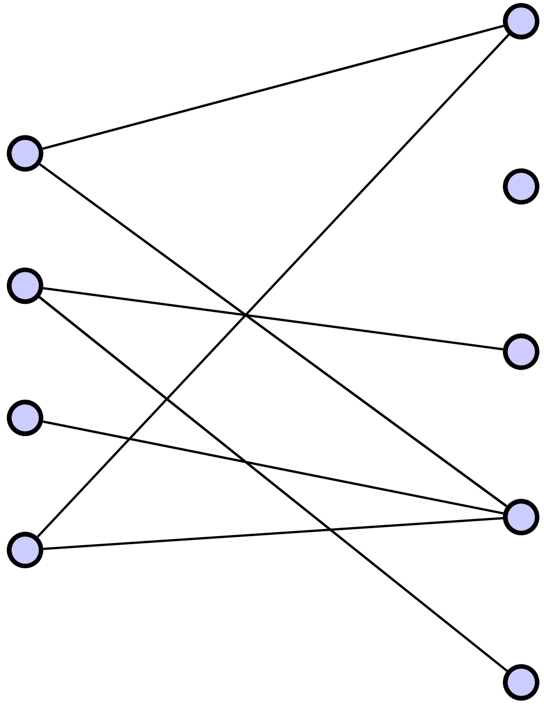
Problem

$$\begin{matrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{matrix}$$

a = minimum # of lines that cover all 1's
b = maximum # of independent 1's
Prove that a = b.

Solution

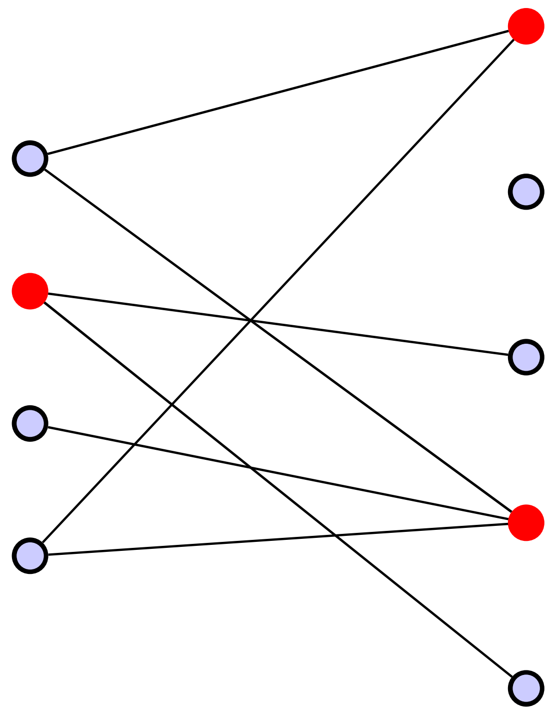
$$\begin{matrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{matrix}$$



To prove this we use a bipartite graph and translate the 1's and 0's to edges and the rows and columns to vertices.

Solution

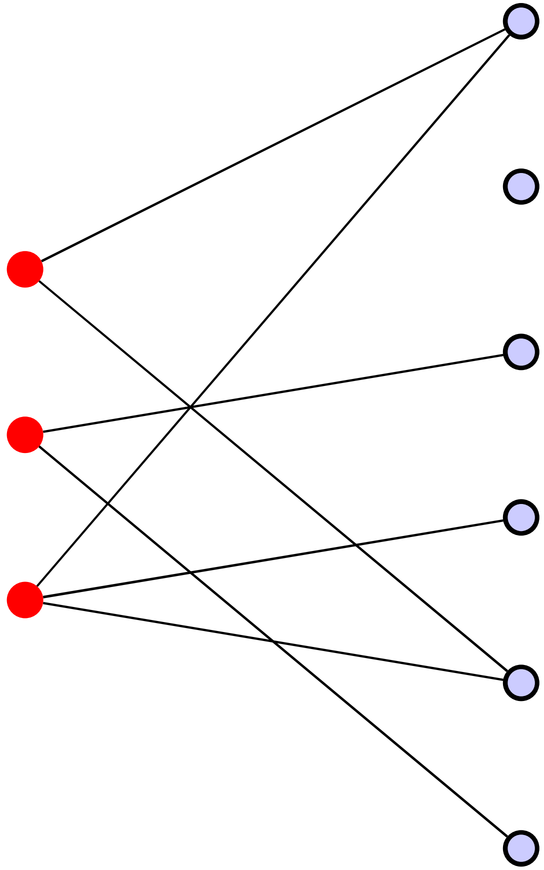
1	0	0	1	0
0	0	1	0	1
0	0	0	1	0
1	0	0	1	0



We color the minimum number of lines/vertices and take all of them to the left side and all others to the right.

Solution

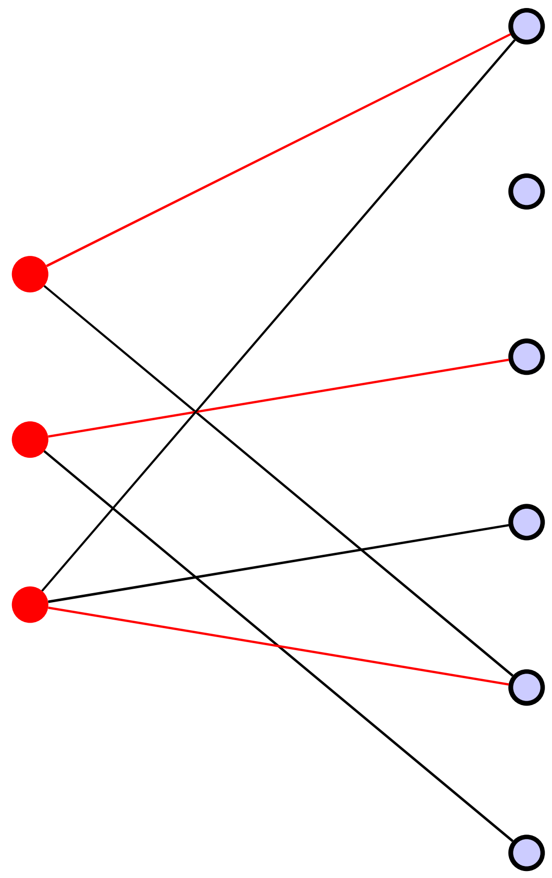
$$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}$$



Now we get a new bipartite graph in witch we have to find a matching from U into V. This will represent a set of independent 1's.

Solution

1	0	0	1	0
0	0	1	0	1
0	0	0	1	0
1	0	0	1	0



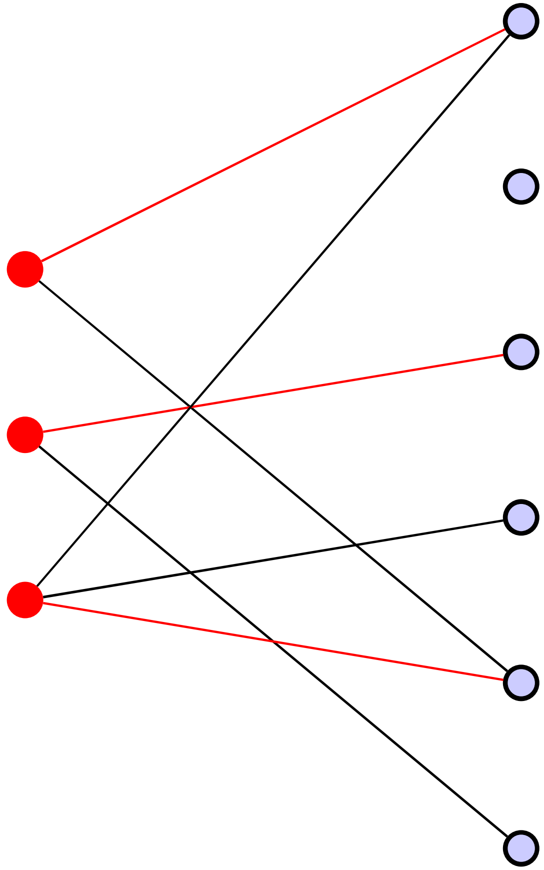
We need to prove that a matching exists. For this we use Phillip Hall's Theorem.

Philip Hall's Theorem

Let G be a bipartite graph with bipartite sets X and Y. For a set W of vertices in X, let $N_G(W)$ denote the neighborhood of W in G, i.e. the set of all vertices in Y adjacent to some element of W. The theorem in this formulation states that there is a matching that entirely covers X if and only if for every subset W of X: $|W| \leq |N_G(W)|$.

Solution

1	0	0	1	0
0	0	1	0	1
0	0	0	1	0
1	0	0	1	0



Proof by contradiction: Suppose there exists a subset W where $|W| > |N_G(W)|$.Then we could just switch W with $N_G(W)$ and get a smaller set of lines! From this follows that b $=$ a! Q.E.D.

Stable Matchings

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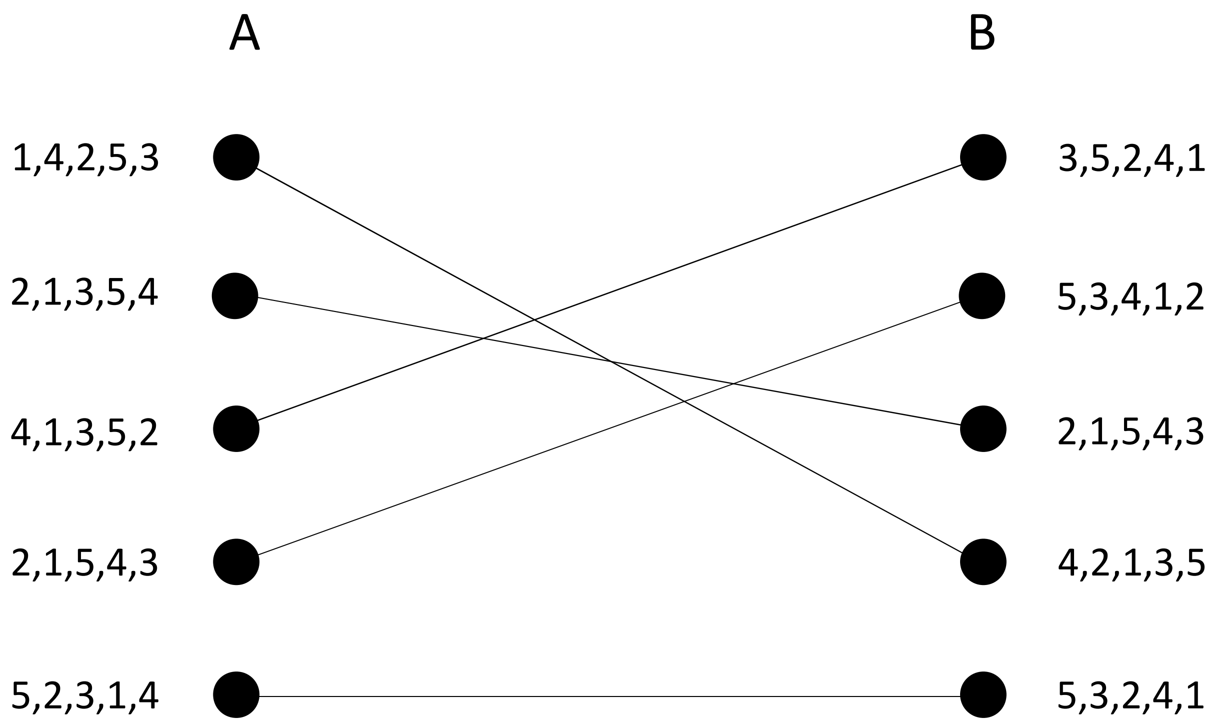
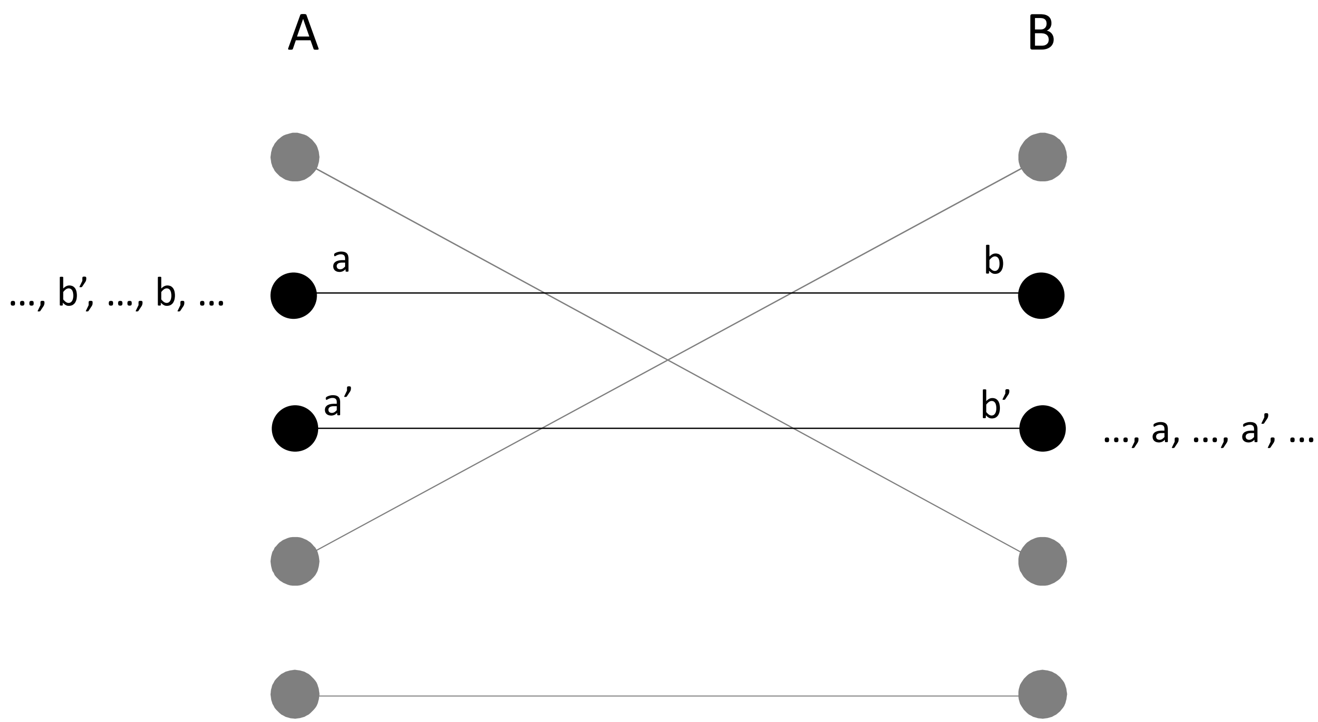
Requirements for stable matchings



- Bipartite Graph
- |A| = |B|
- Preferences

Definition of a stable matching



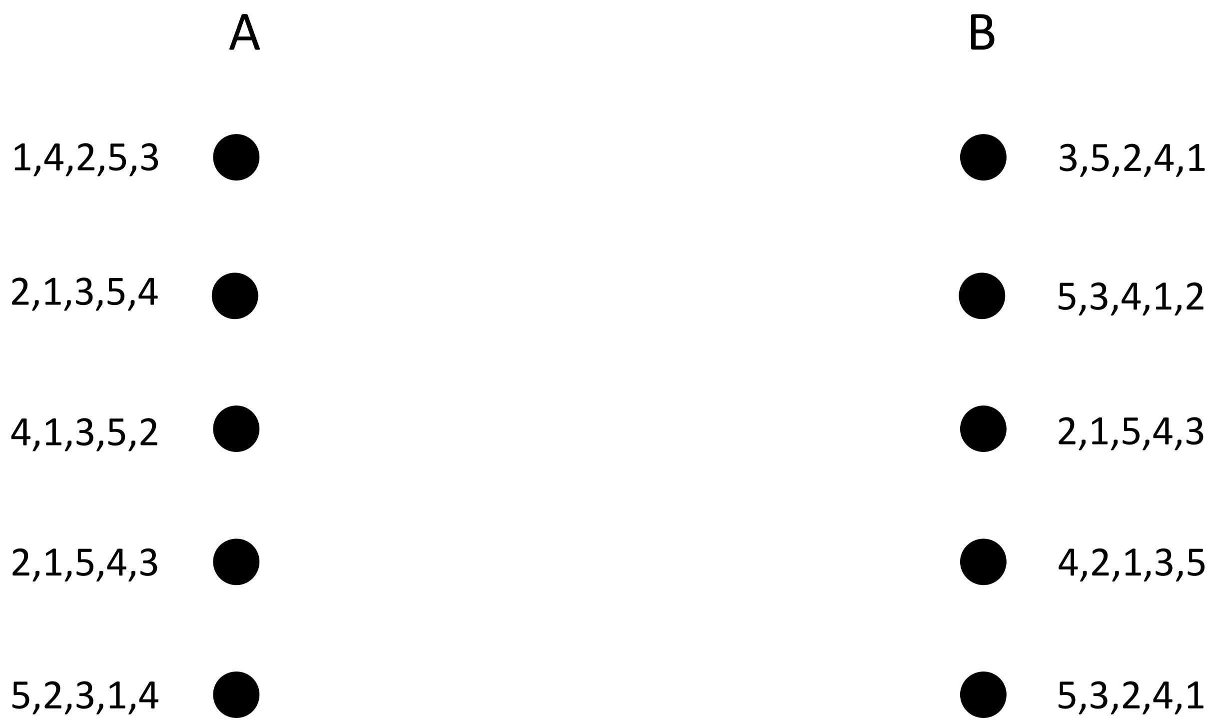
A perfect matching is not stable if ∃ vertices a, b, a' and b' such that:

- There is an edge ab but a prefers b'

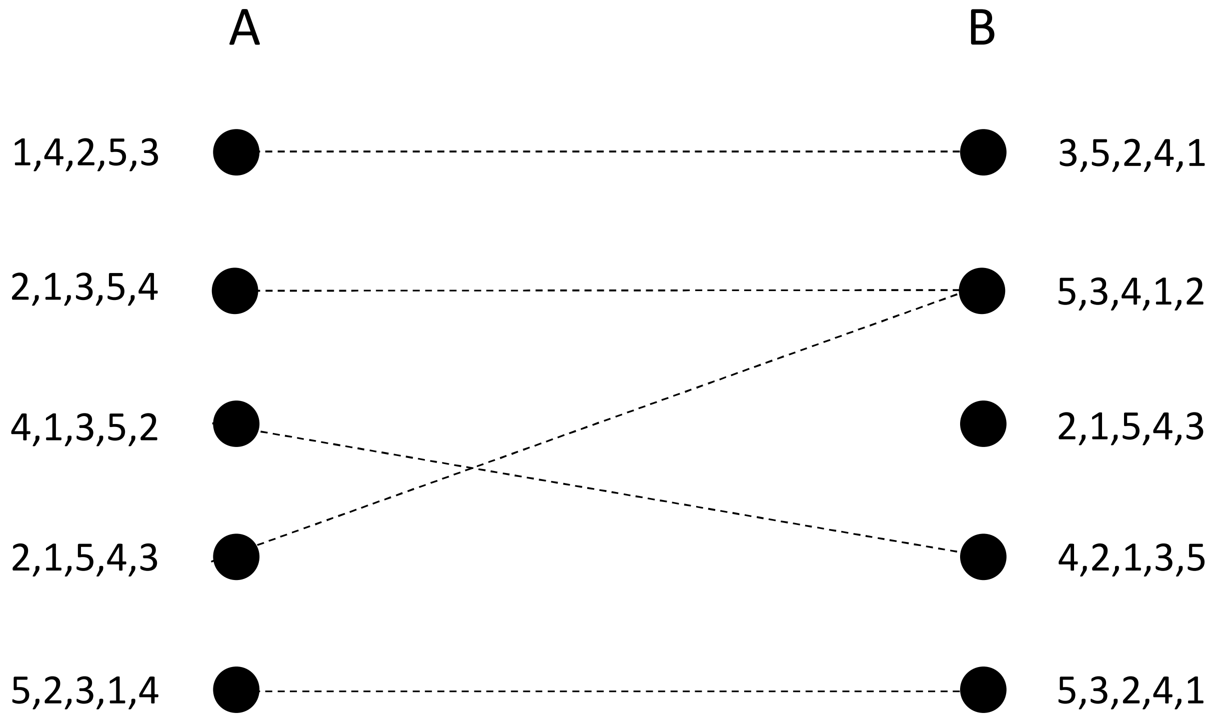
and

- There is an edge a'b' but b' prefers a

Gale-Shapley Algorithm

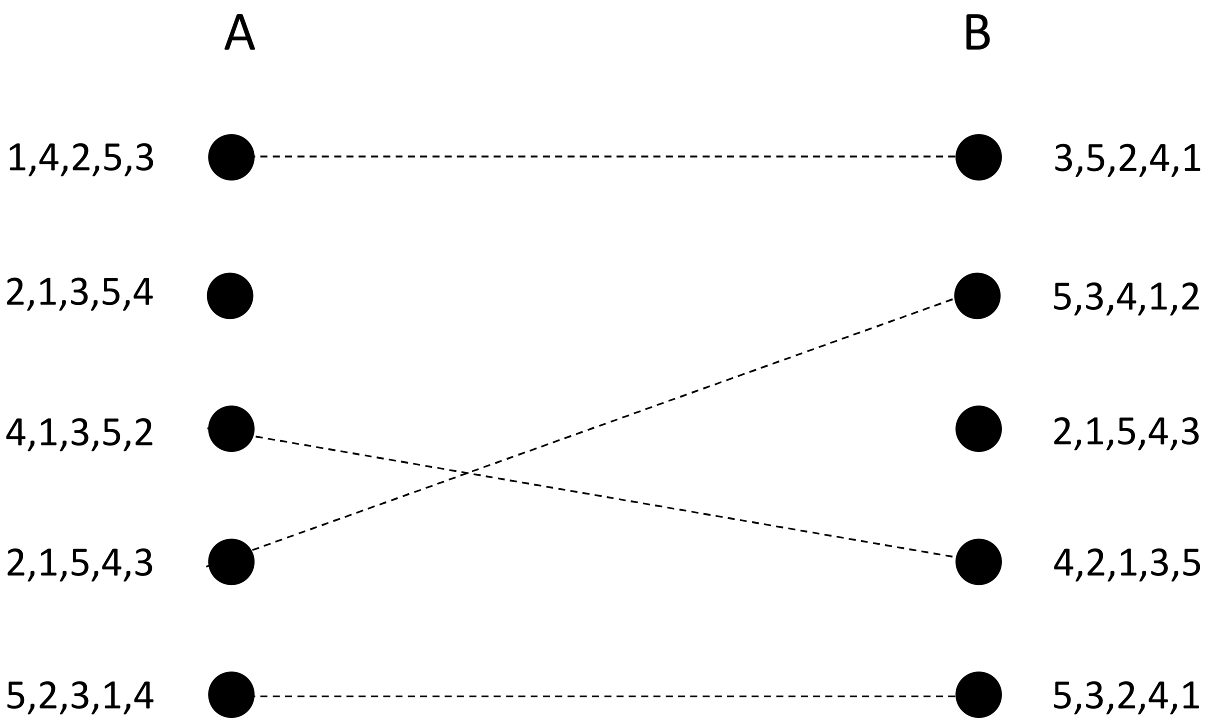


Gale-Shapley Algorithm



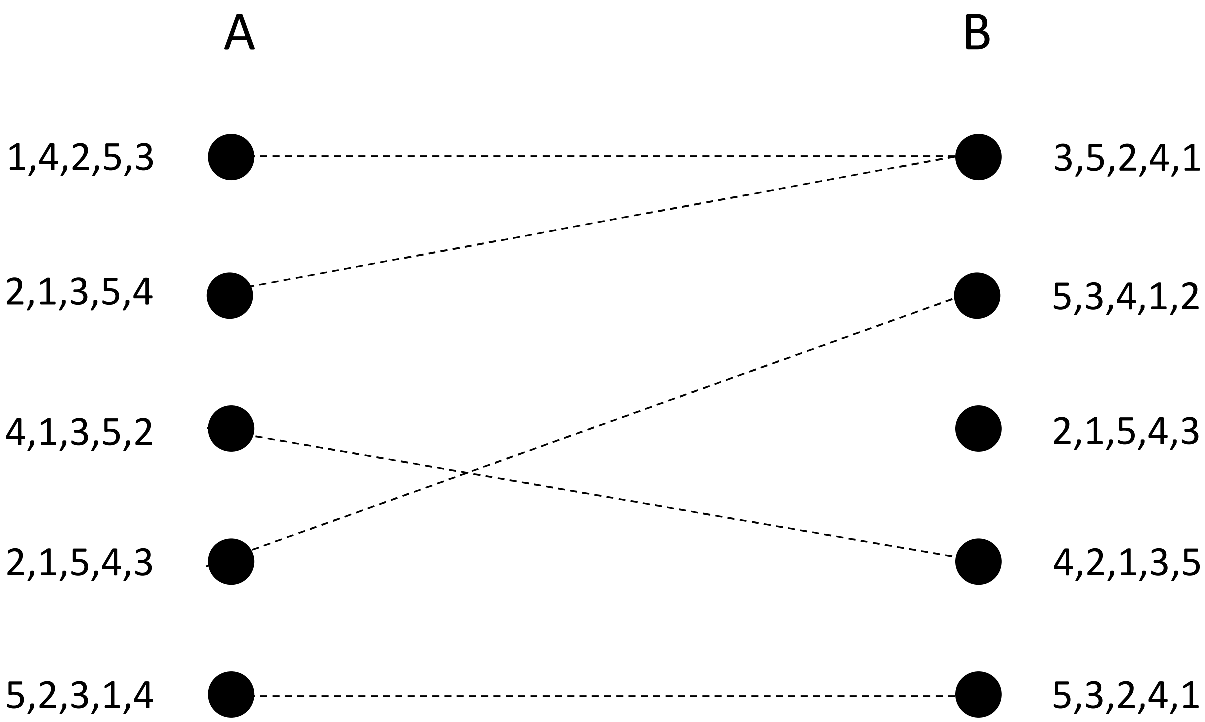
1. Applicants apply for favorite job.

Gale-Shapley Algorithm



1. Applicants apply for favorite job.
2. Preferred applicants are hired temporarily, rest rejected.

Gale-Shapley Algorithm



1. Applicants apply for favorite job.
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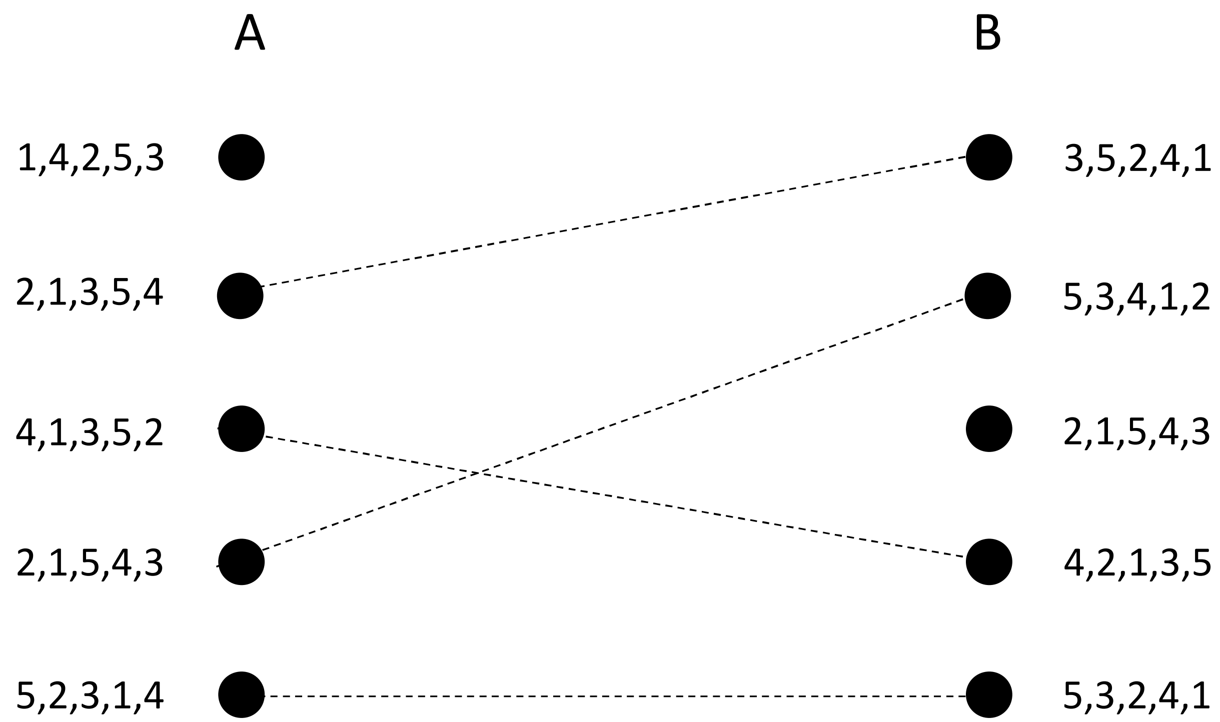
3. Applicants that aren’t hired temporarily apply for next choice.

Gale-Shapley Algorithm



1. Applicants apply for favorite job.
2. Preferred applicants are hired temporarily, rest rejected.
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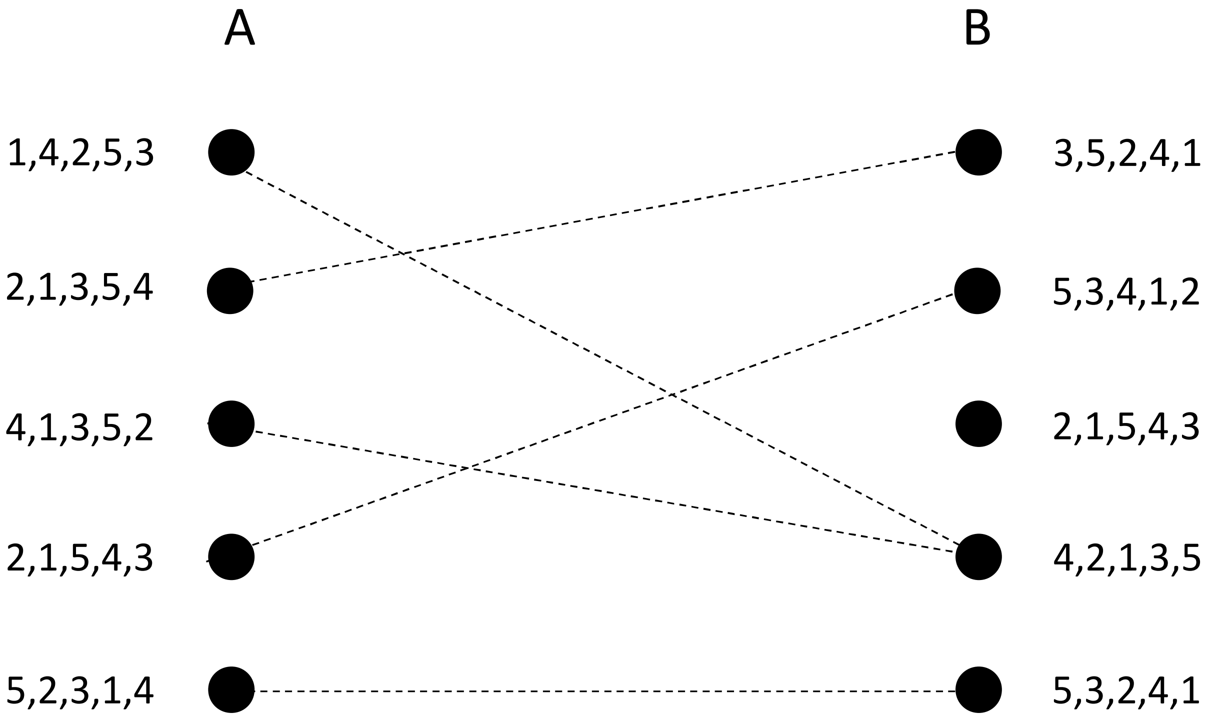
4. Repeat 2 and 3 until everyone is hired.

Gale-Shapley Algorithm



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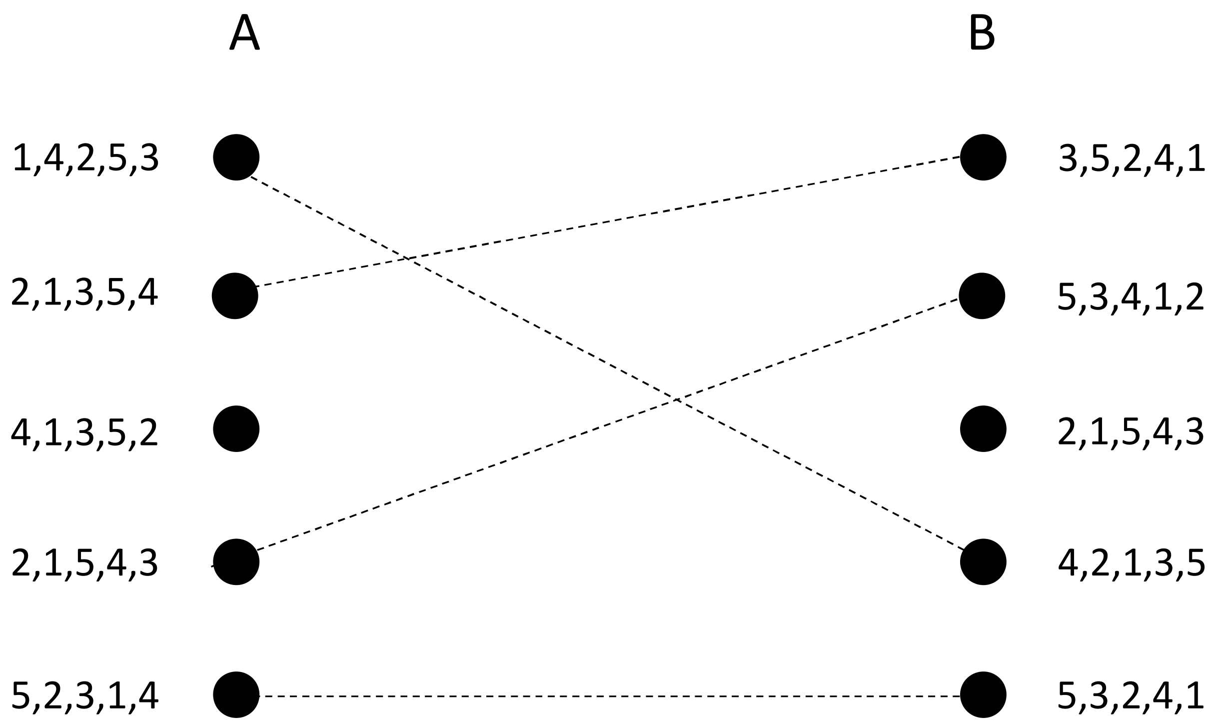

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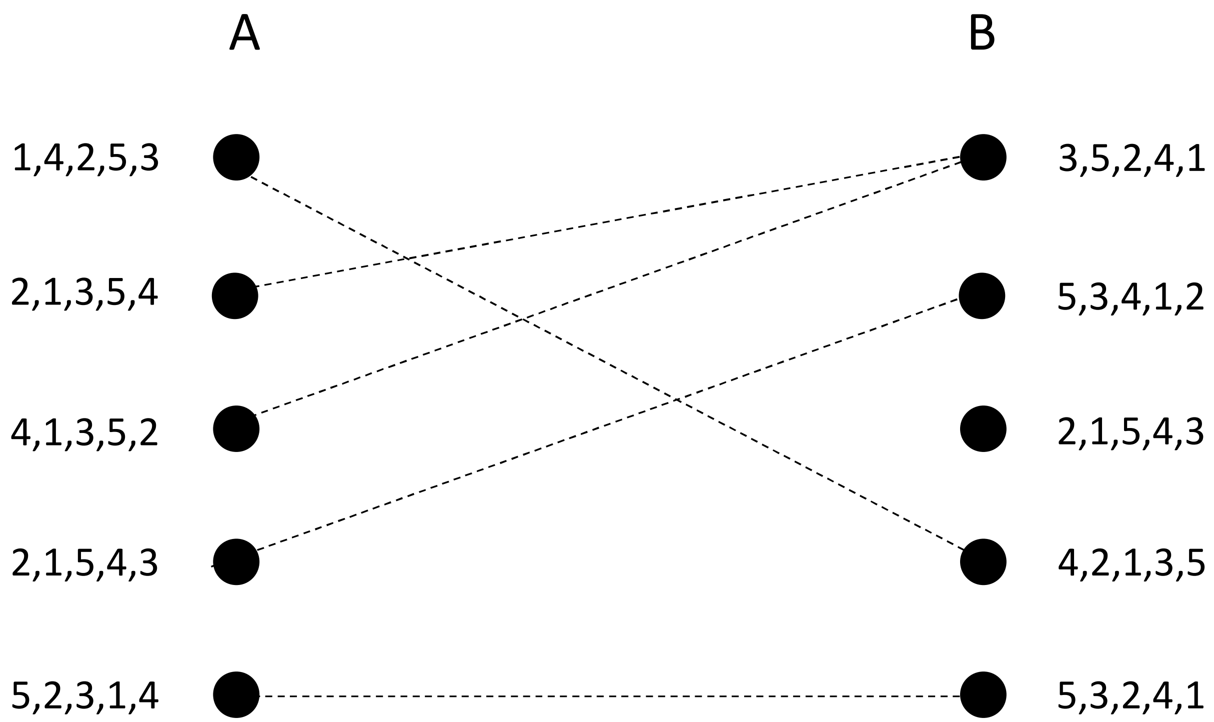

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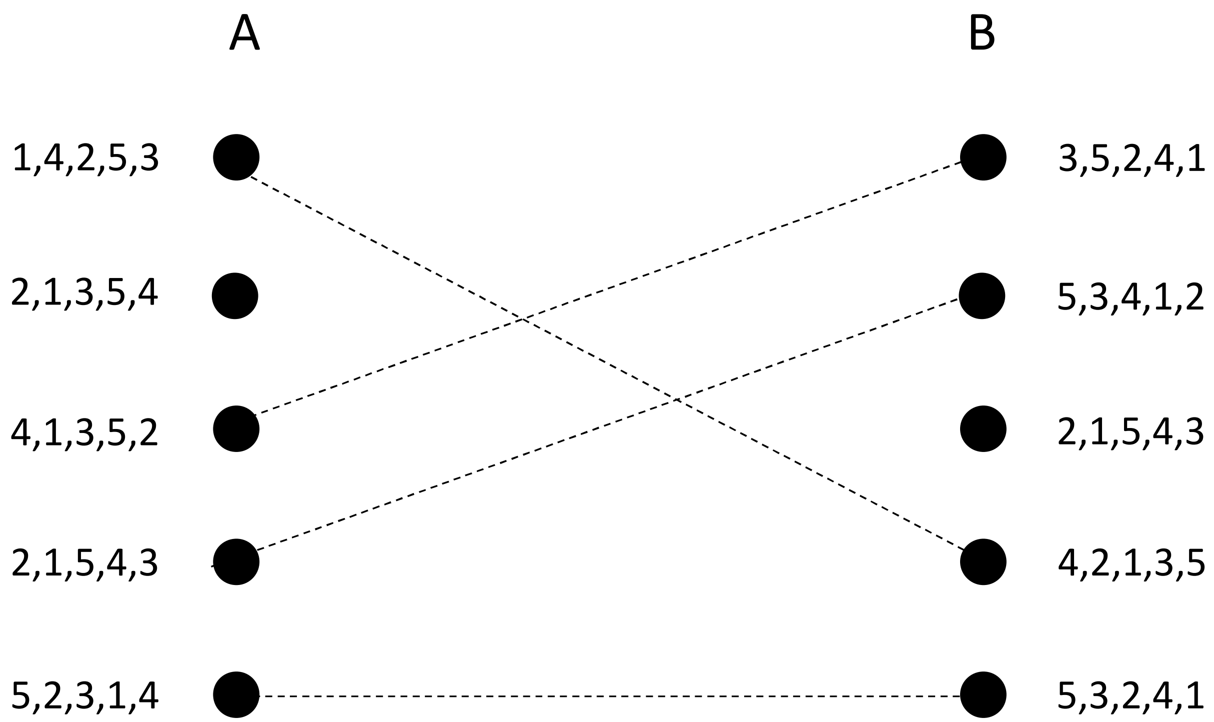

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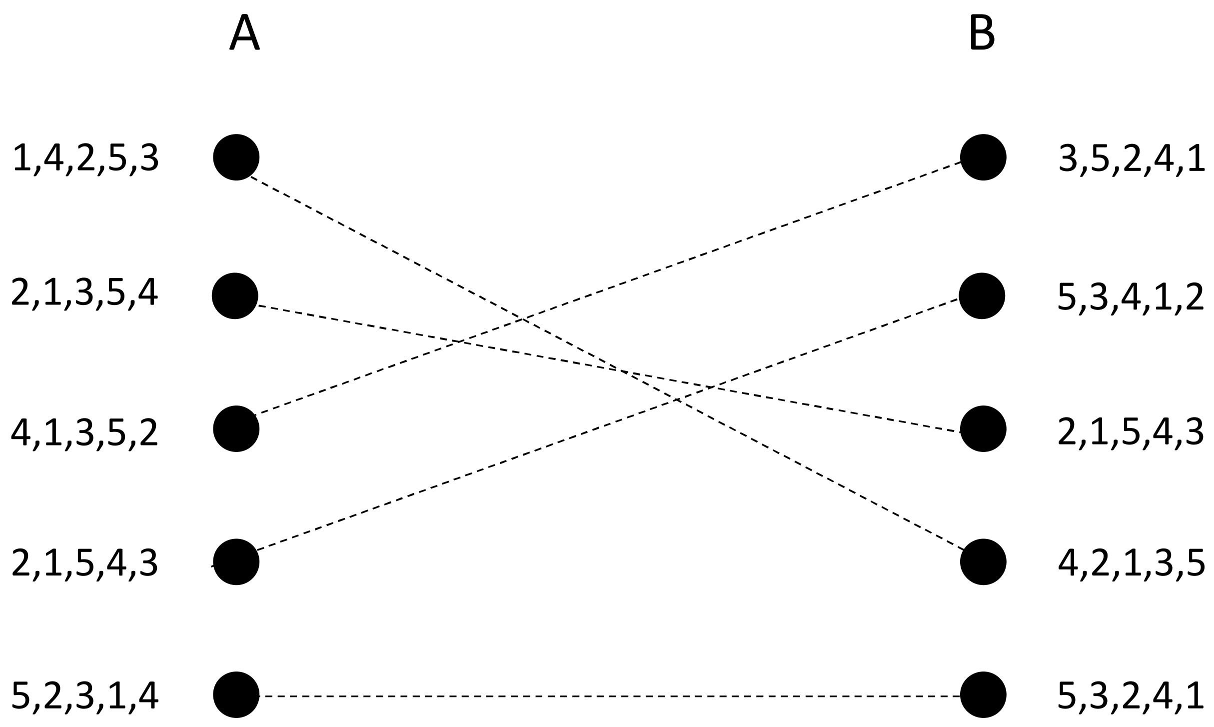

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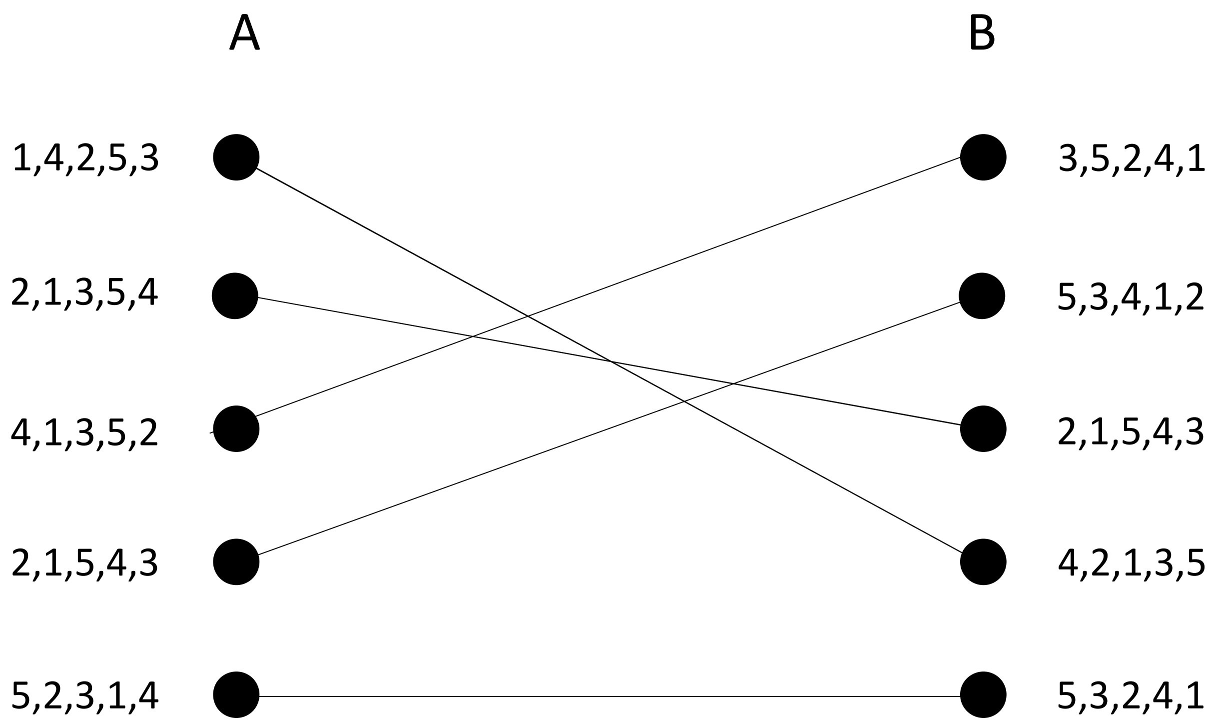
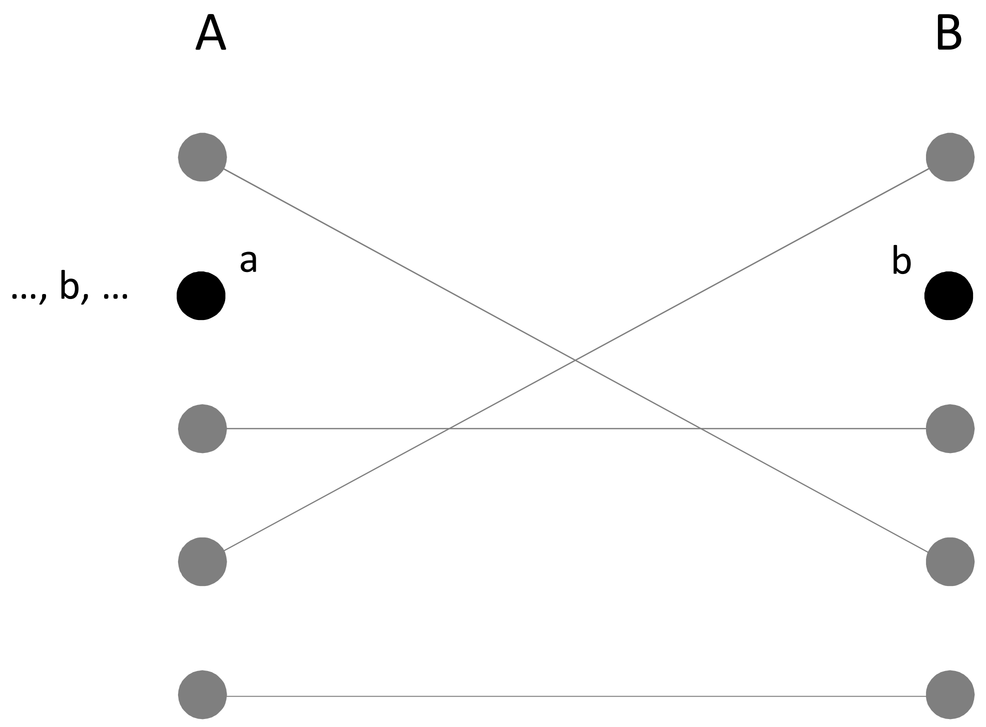
Gale-Shapley Algorithm



1. Applicants apply for favorite job.
2. Preferred applicants are hired temporarily, rest rejected.
3. Applicants that aren’t hired temporarily apply for next choice.
4. Repeat 2 and 3 until everyone is hired.
5. All applicants are hired permanently.

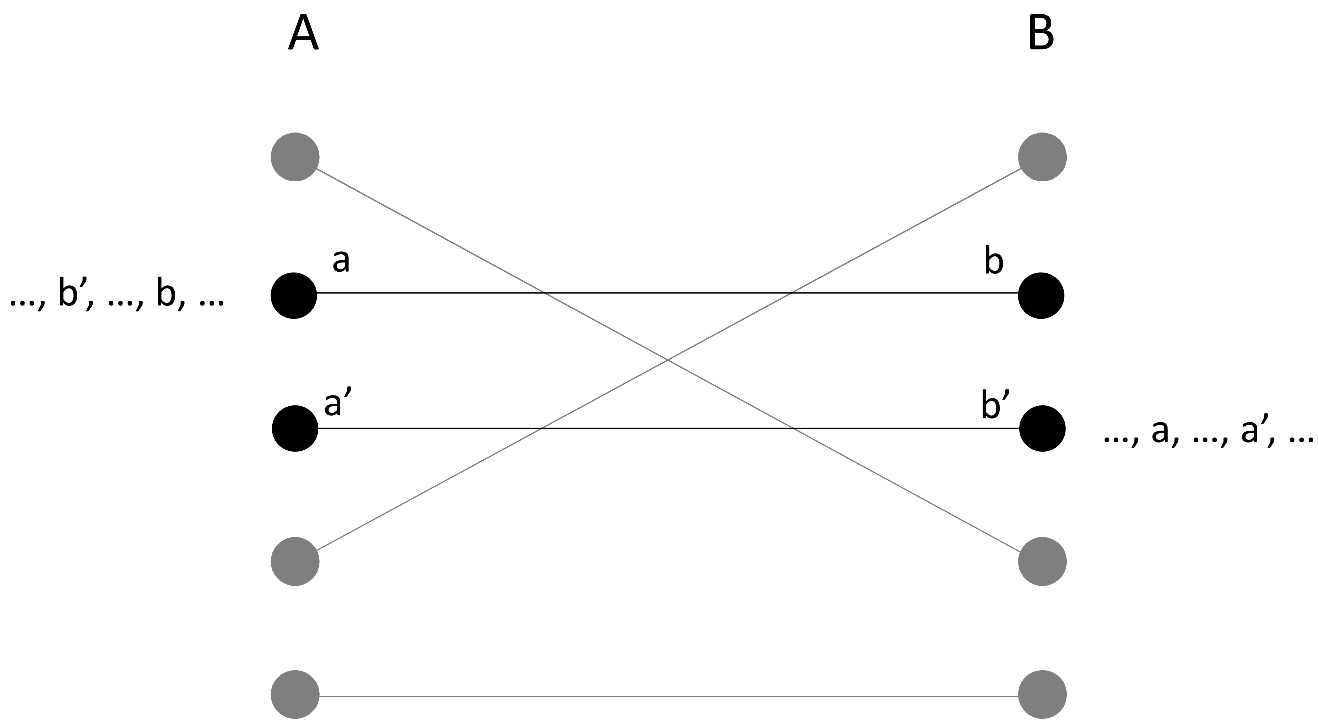
Proof that it produces a perfect matching



Assume the opposite is true:

- If the process is over a must have applied to b.
- From that point on b has to have a temporarily hired applicant.
- Contradiction

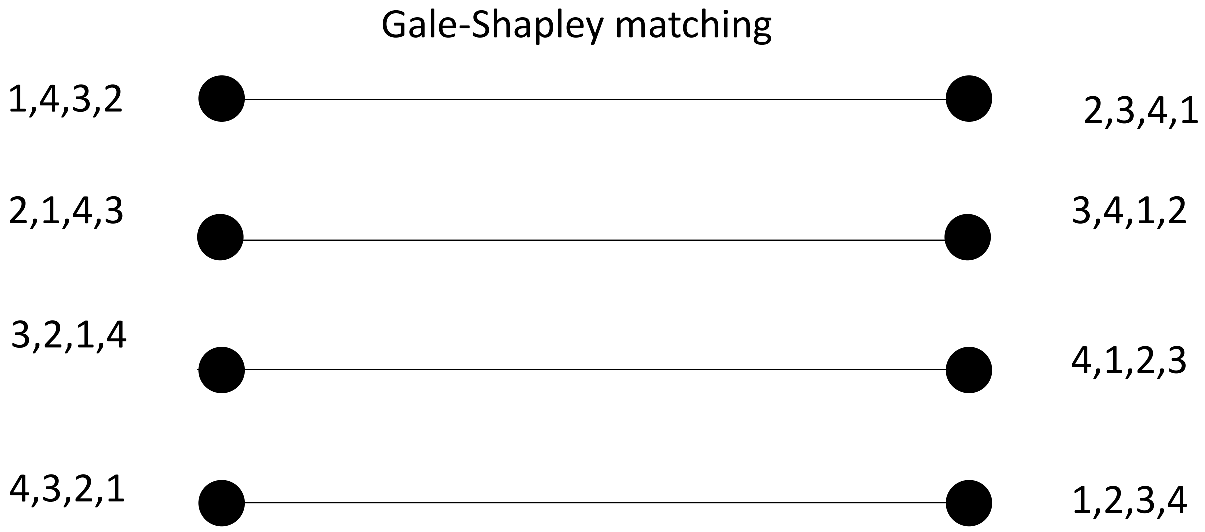
Proof of stability



Assume the opposite is true:

- a must have applied to b’ before applying to b.
- Either a was rejected directly or temporarily hired and then rejected. In both cases, b’ must have found a vetex x that he prefers to a.
- If x is not a’ then b’ even prefers a’ to x.
- Contradiction because it would imply that b’ prefers a’ to a.

Does the algorithm produce the best matching?



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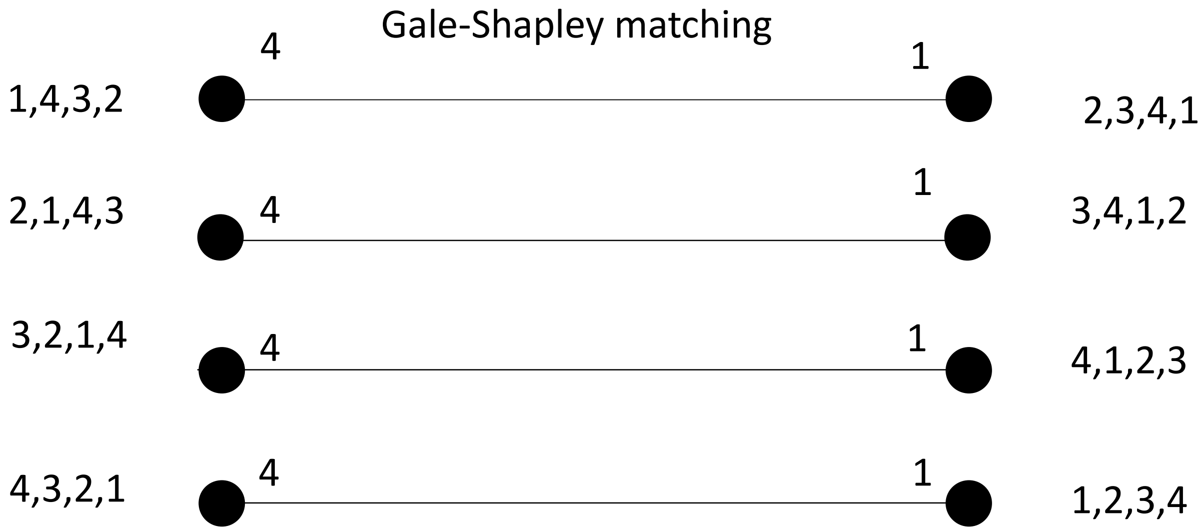

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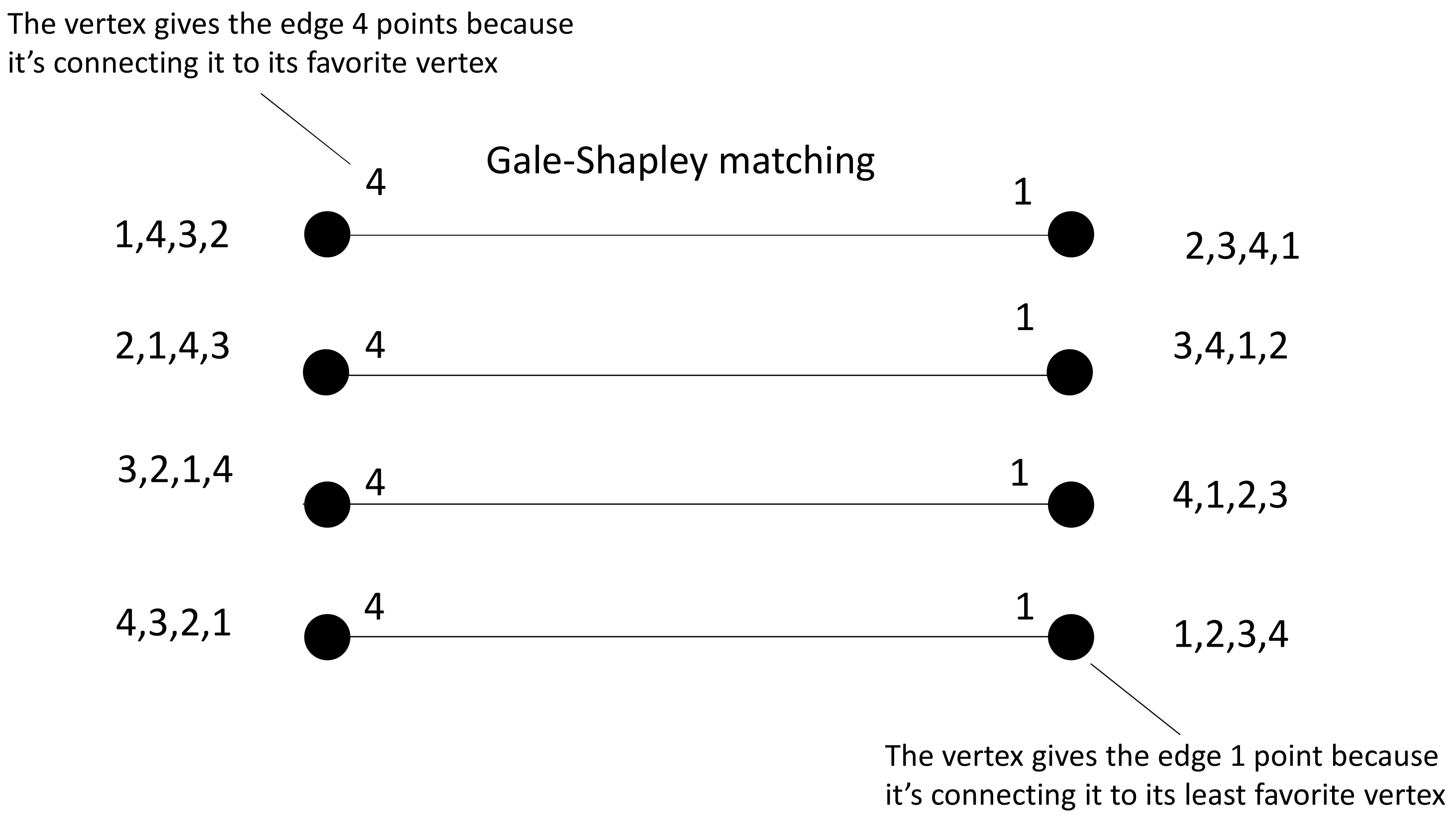

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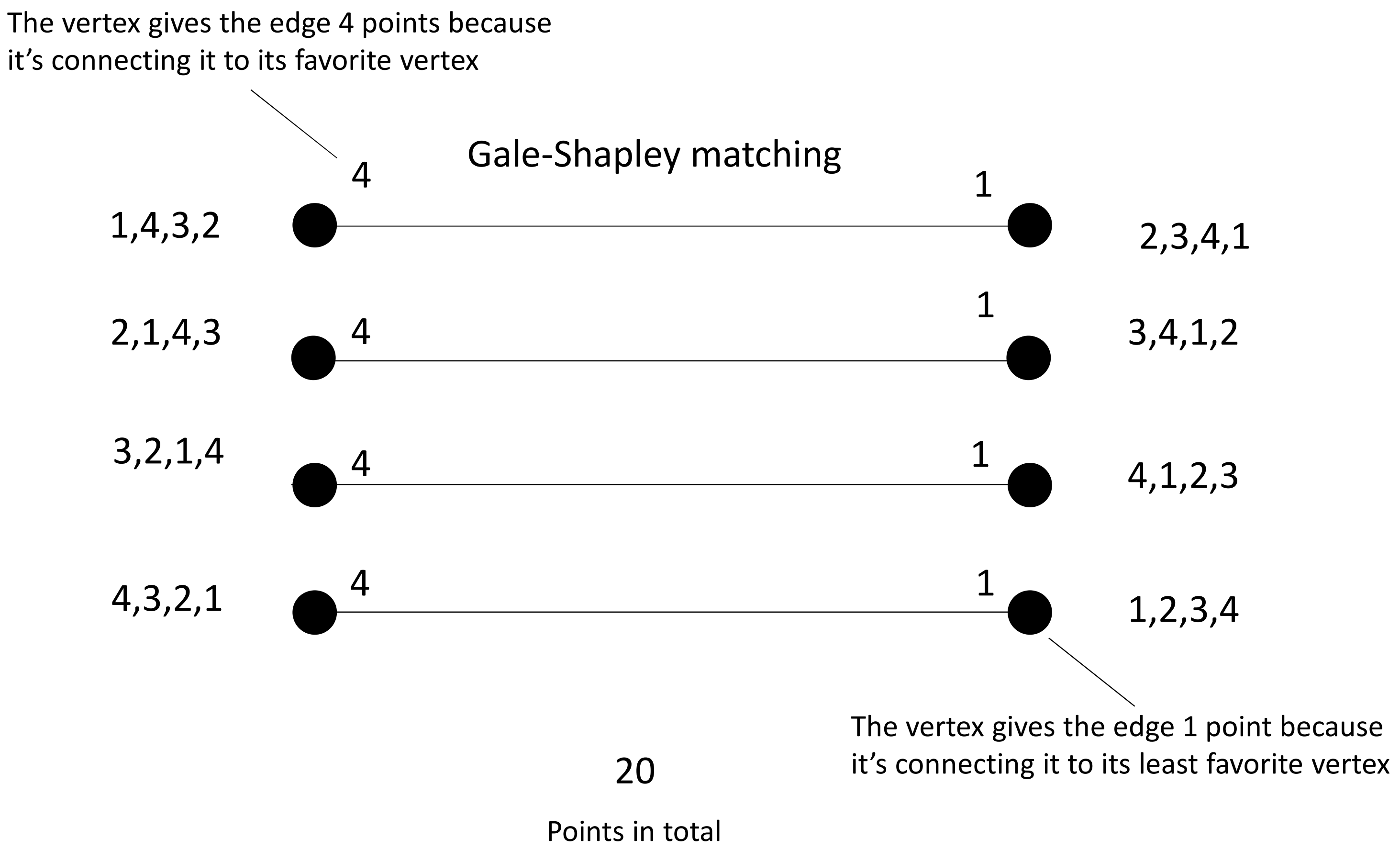

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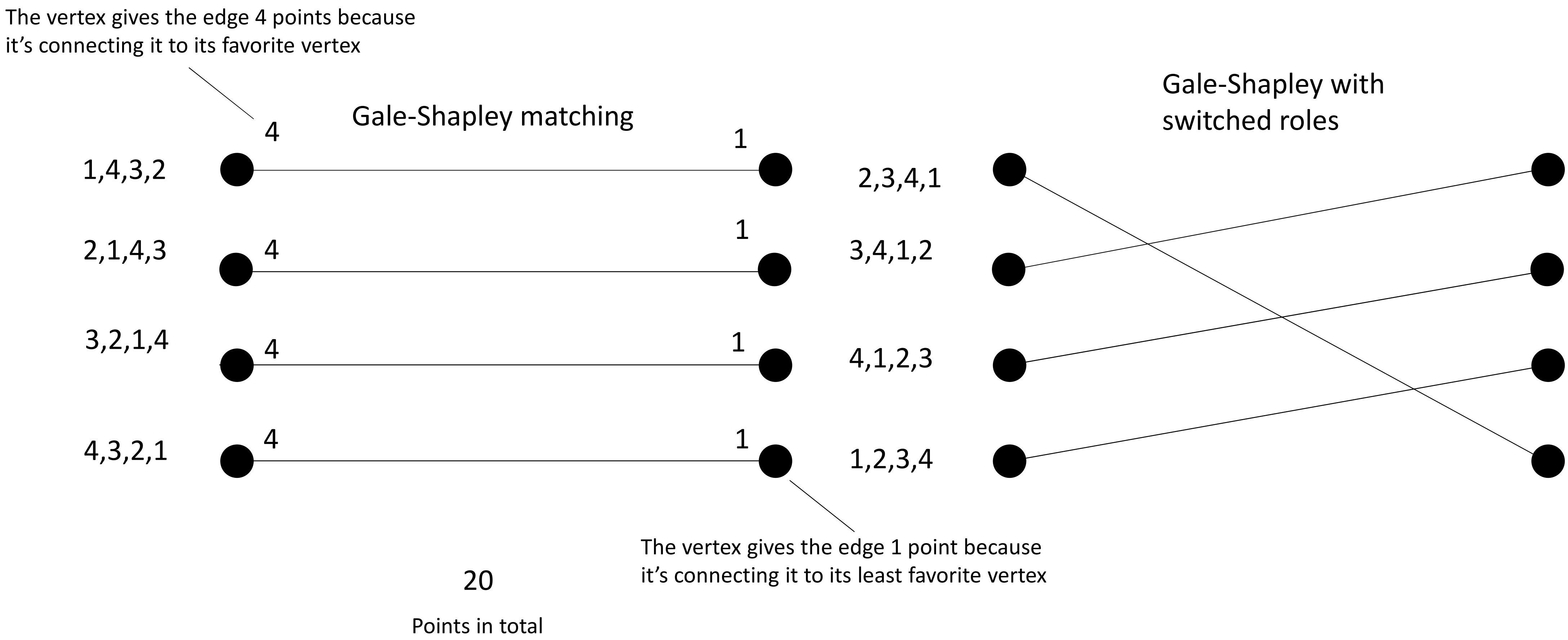

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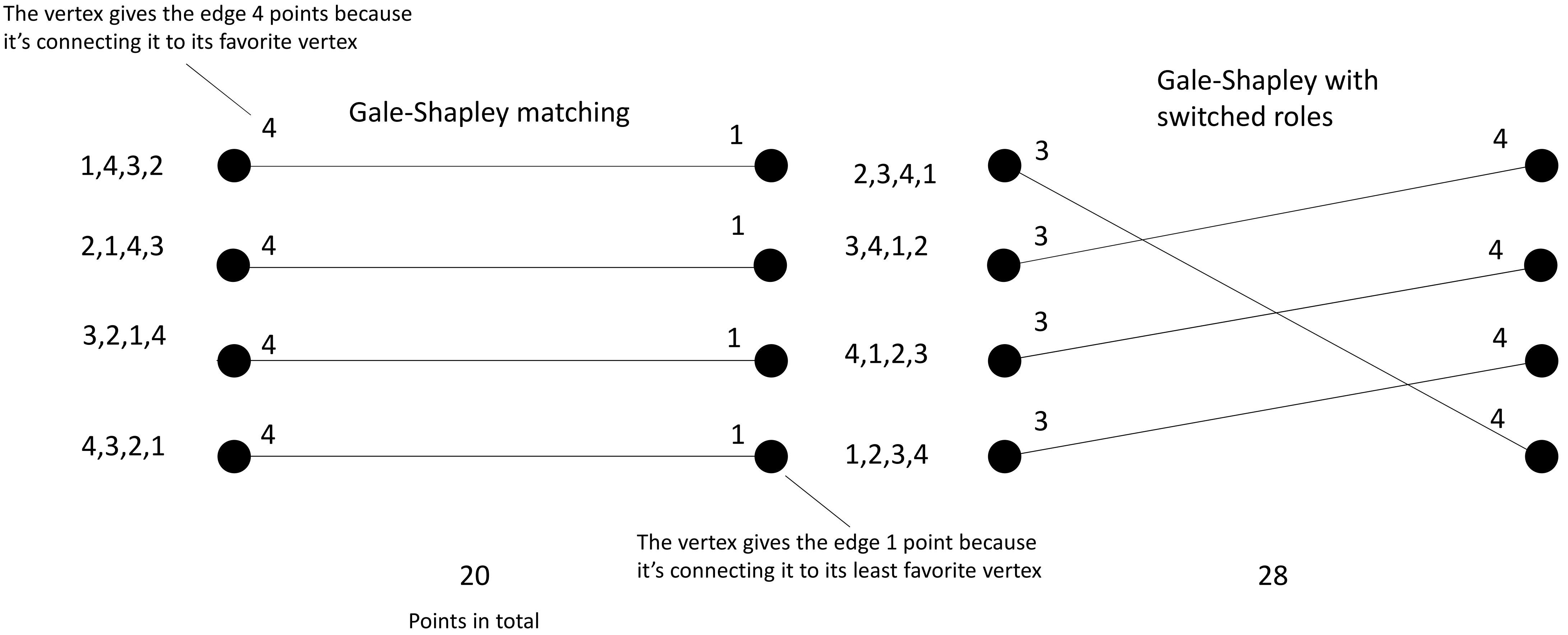
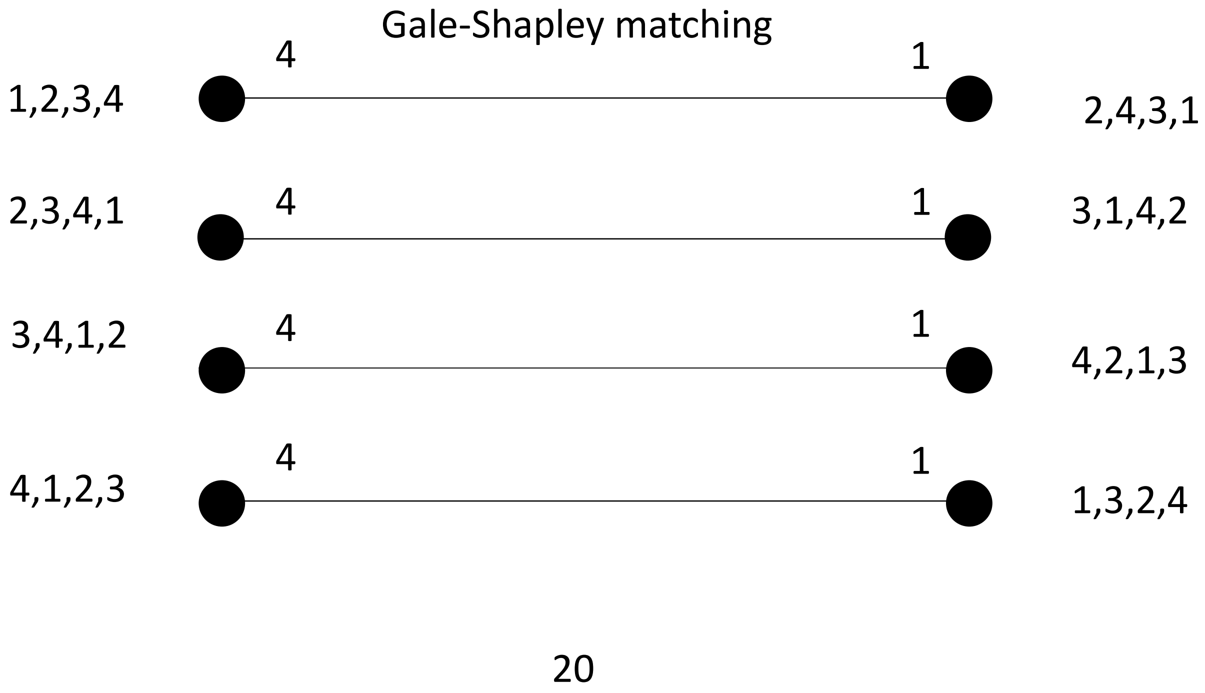
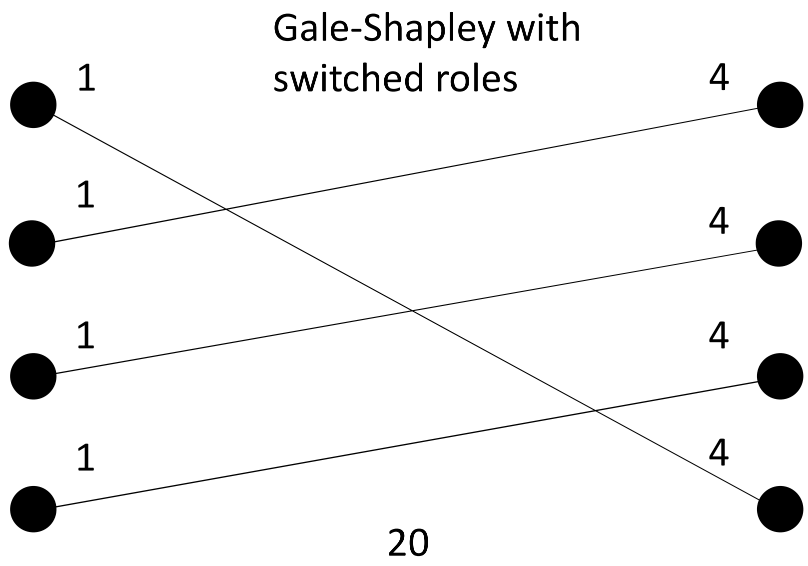


Does running it from both sides produce the best matching?





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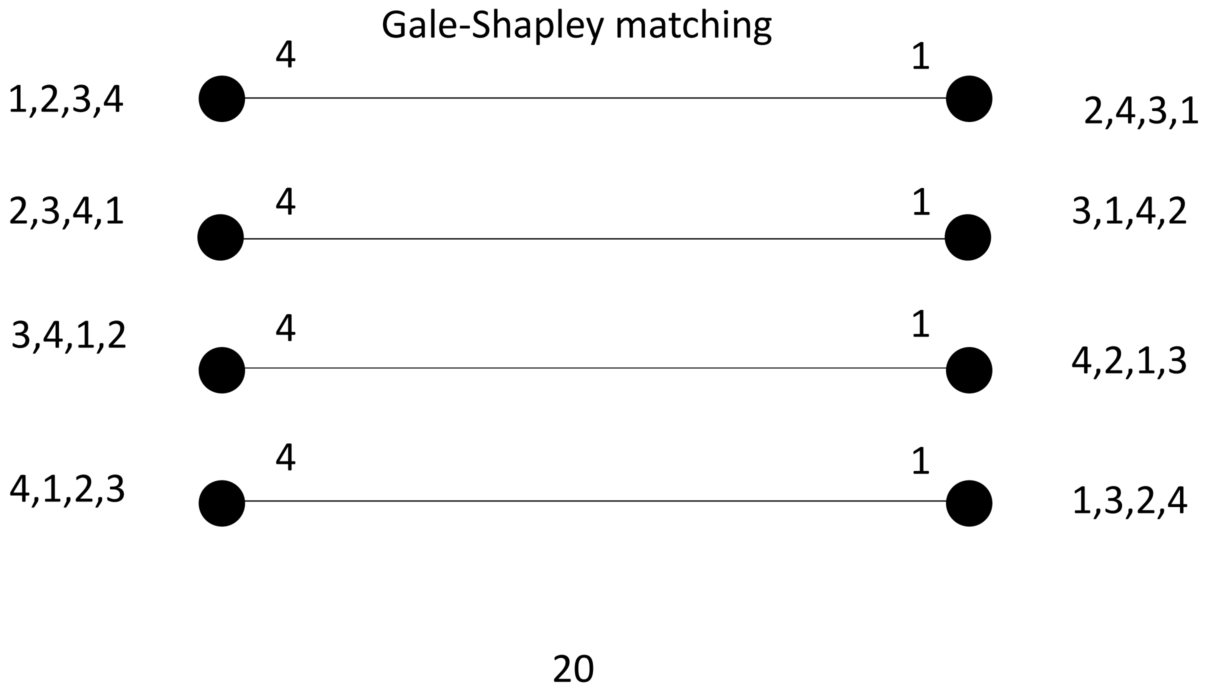
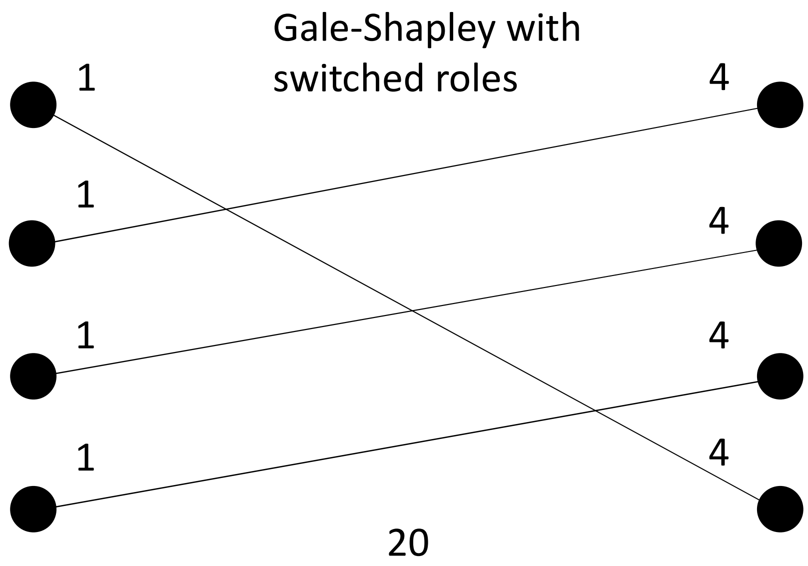
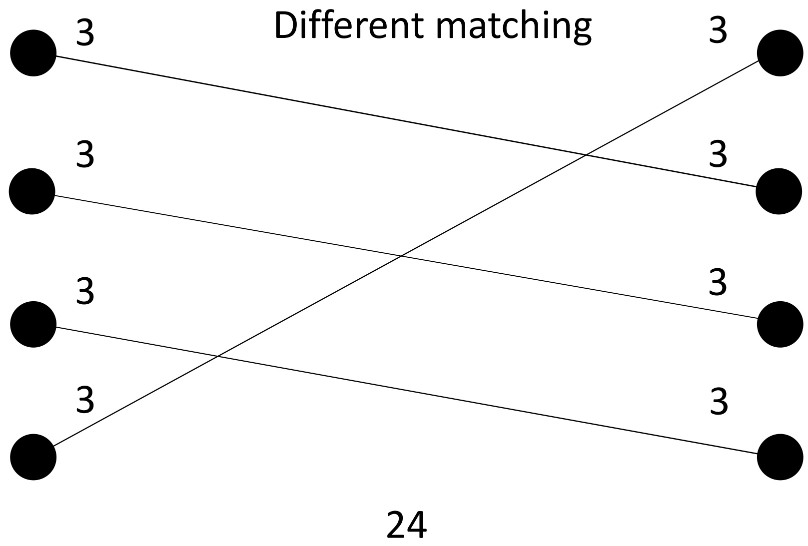


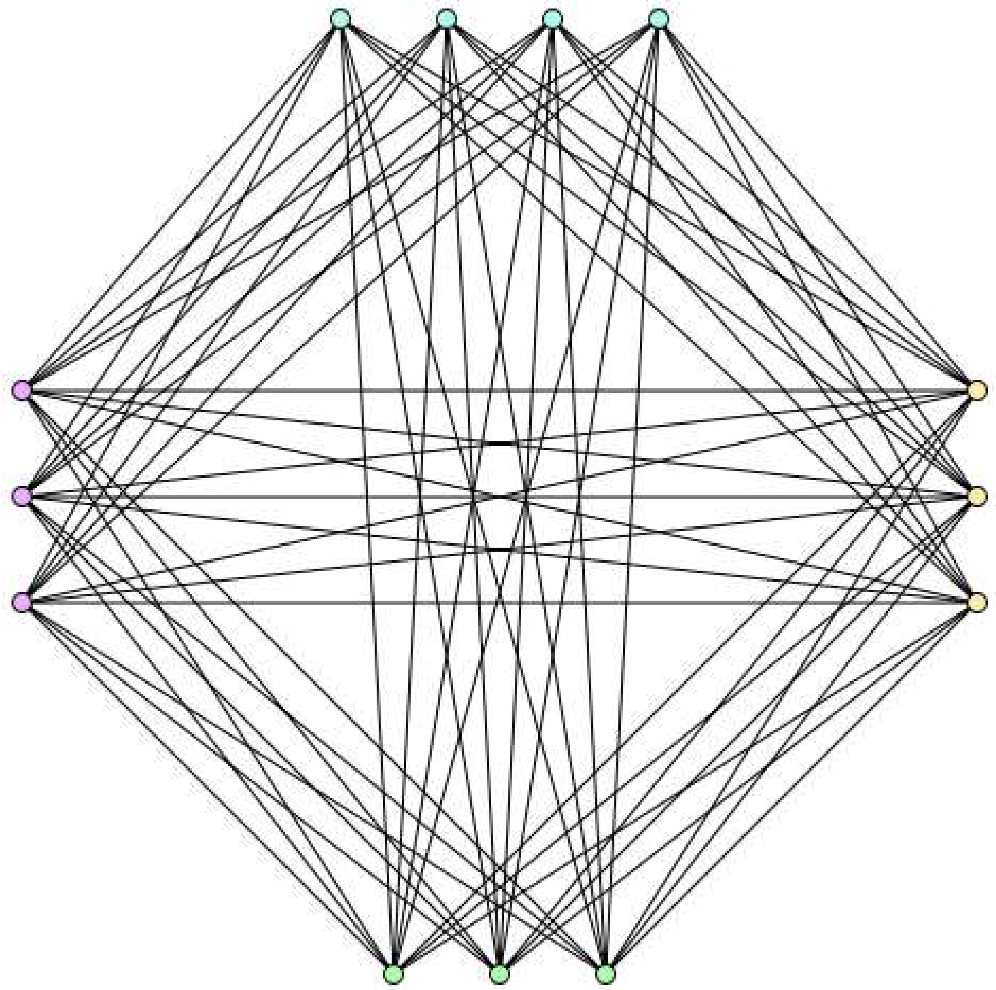




Turan’s Theorem

Karl Robert Kristenprun
Kantonsschule Olten
2ML

Turán Graph



- graph with n-vertices.
- k sets of equal size.
- no two vertices of the same set connected
- doesn’t contain k+1-clique.

- Turán's theorem states that the Turán graph has the largest number of edges among all K_{k+1}-free n-vertex graphs.

Assuming that:

$$\frac{n}{k} \in \mathbb{Z}$$

$$e(T) = \binom{k}{2} \cdot \left(\frac{n}{k}\right)^2 = \left(1 - \frac{1}{k}\right) \cdot \frac{n^2}{2}$$

Let G be maximal K_{k+1}-free graph,

therefore it must contain K_k

If G has $n \leq k$ vertices:

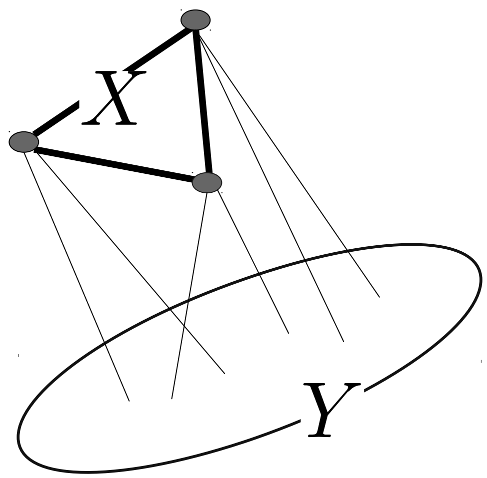
$$\frac{n(n-1)}{2} \leq \left(1 - \frac{1}{k}\right) \frac{n^2}{2}$$

Dividing by n^2 we get:

$$1 - \frac{1}{n} \leq 1 - \frac{1}{k}$$

We divide the graph into two:

- X := K_k
- Y := G \ X



Edges:

$$\binom{k}{2}$$

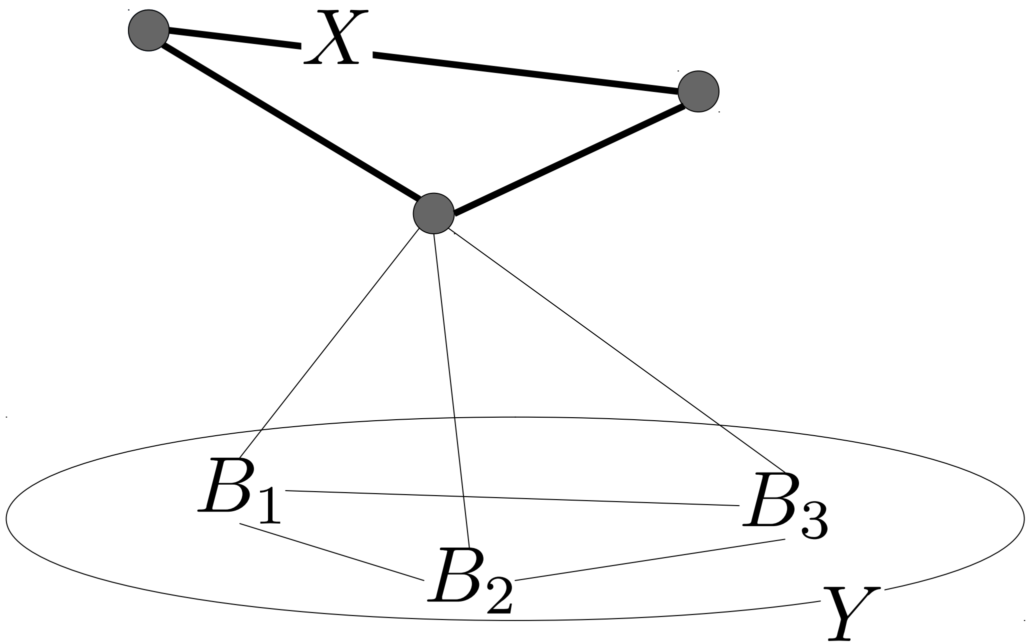
- In X: $\left(1 - \frac{1}{k}\right) \cdot \frac{(n-k)^2}{2}$

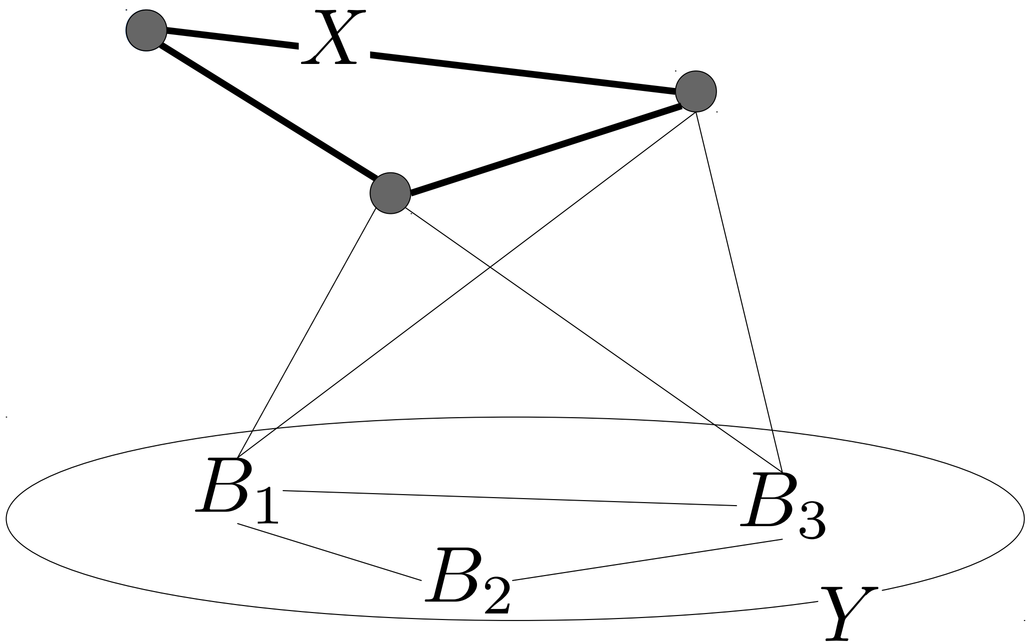
$$(n-k)(k-1)$$

- In Y:
- Between Y and X:

$$\binom{k}{2} + (n-k)(k-1) + \left(1 - \frac{1}{k}\right)\frac{(n-k)^2}{2} =$$

$$= \frac{k-1}{k} \cdot \frac{n^2}{2} = \left(1 - \frac{1}{k}\right)\frac{n^2}{2}$$


X
B_1
B_3
B_2
Y


X
B_1
B_2
B_3
Y

Suppose equality holds:

$$e(G) \leq \left(1 - \frac{1}{k}\right) \frac{n^2}{2}$$