# Planar Graphs

## Euler's Formula and the five regular polyhedra

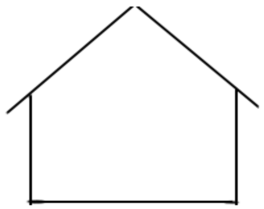
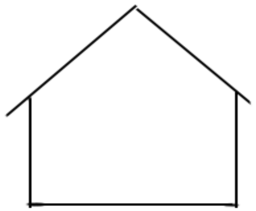
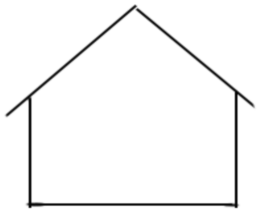
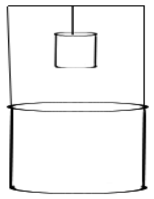
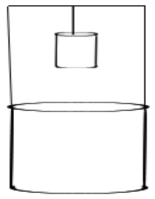
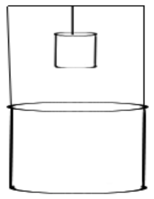
Primes-Switzerland

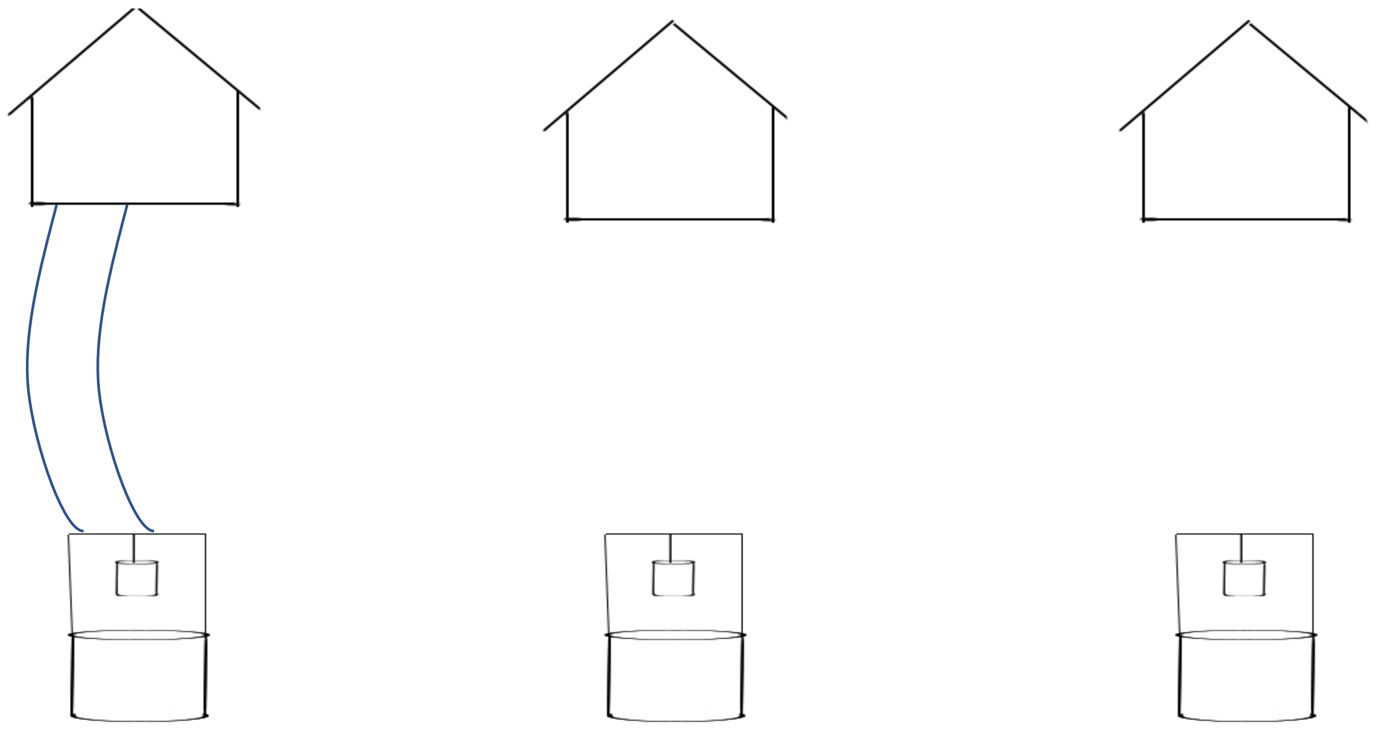
Han-Miru Kim, Susanne Steiner

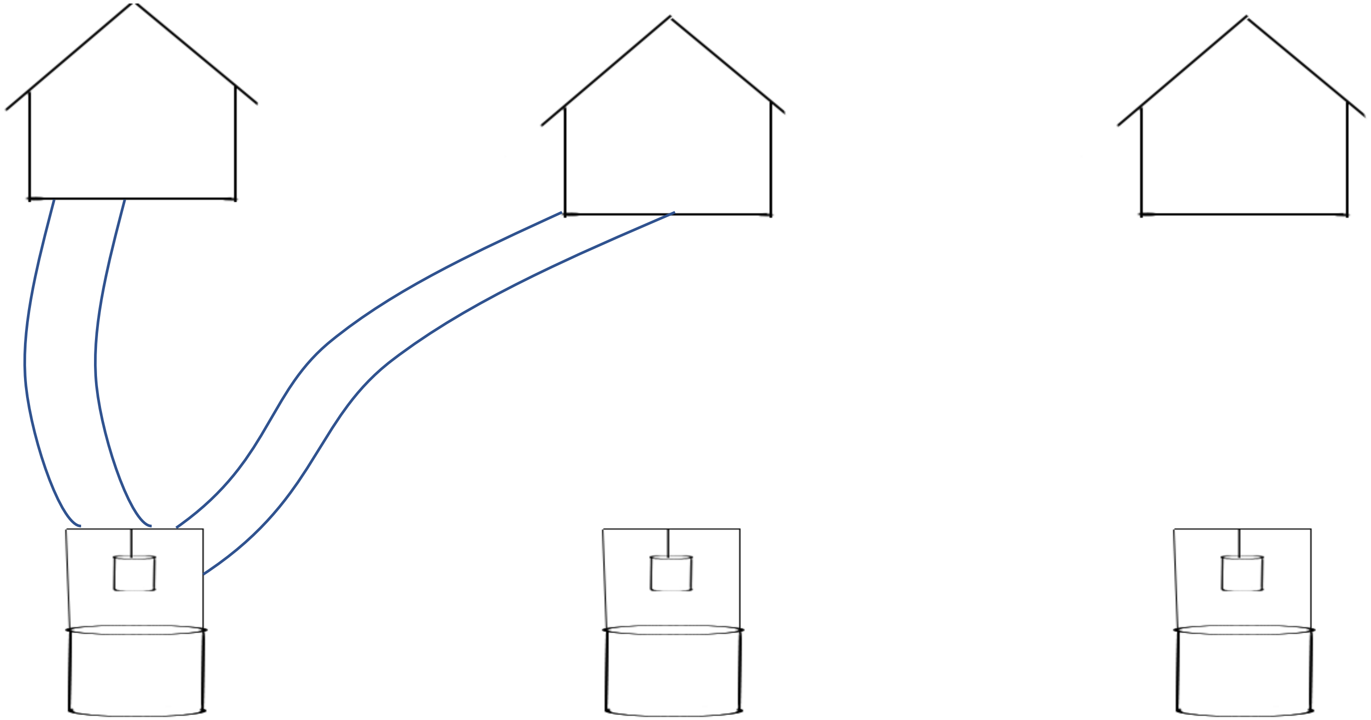
Alte und Neue Kantonsschule Aarau

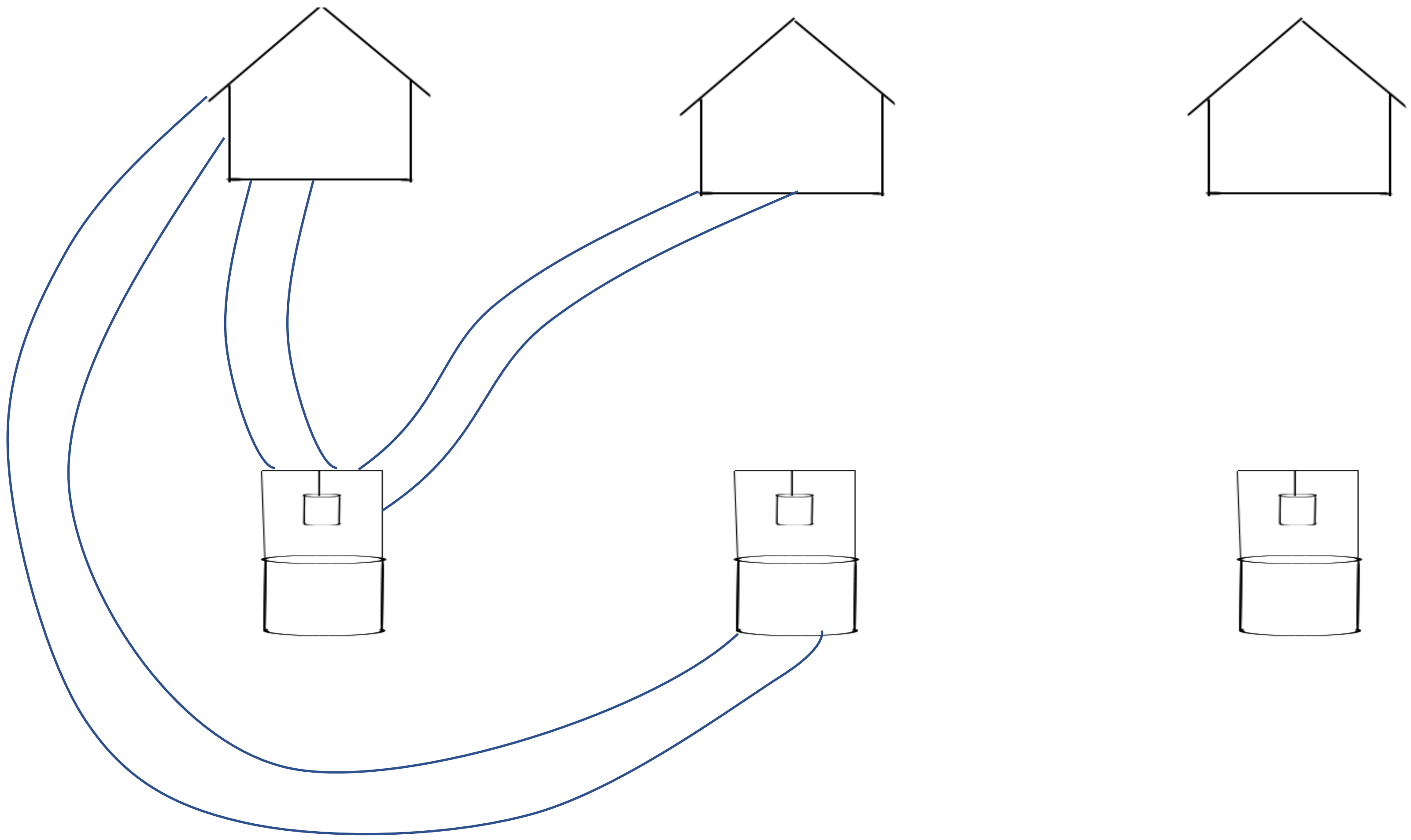
Mentor: Kaloyan Slavov

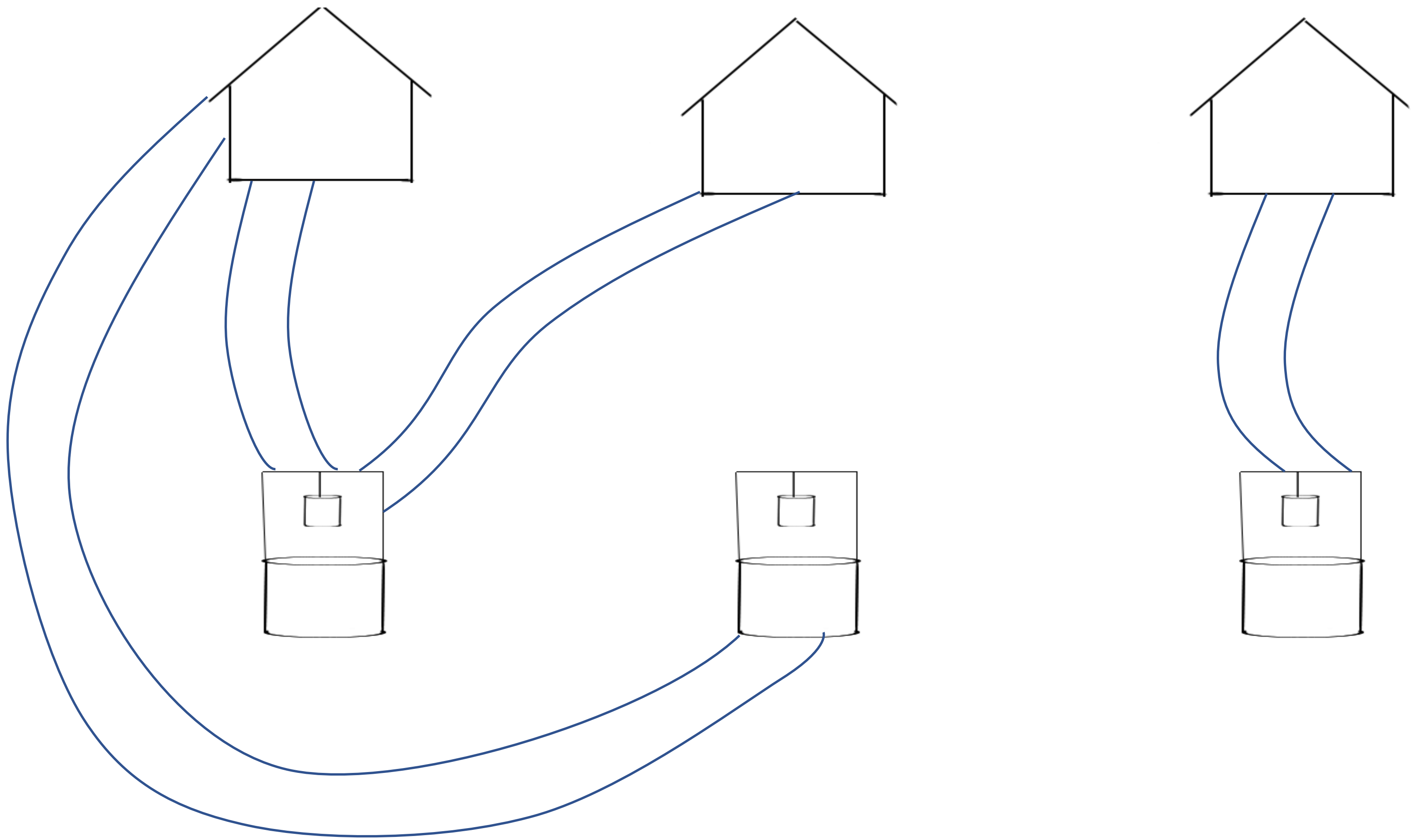
23.06.2018

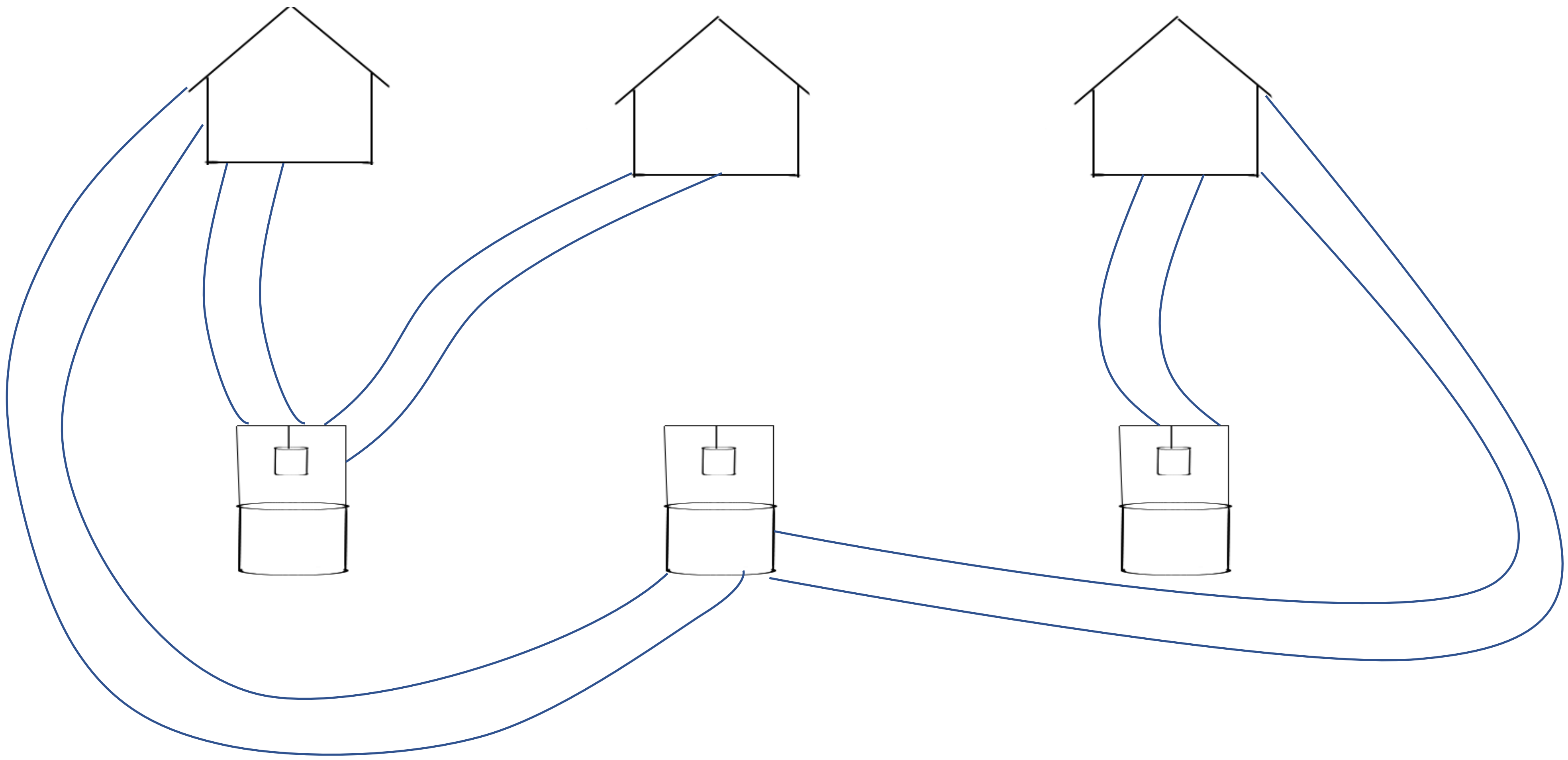







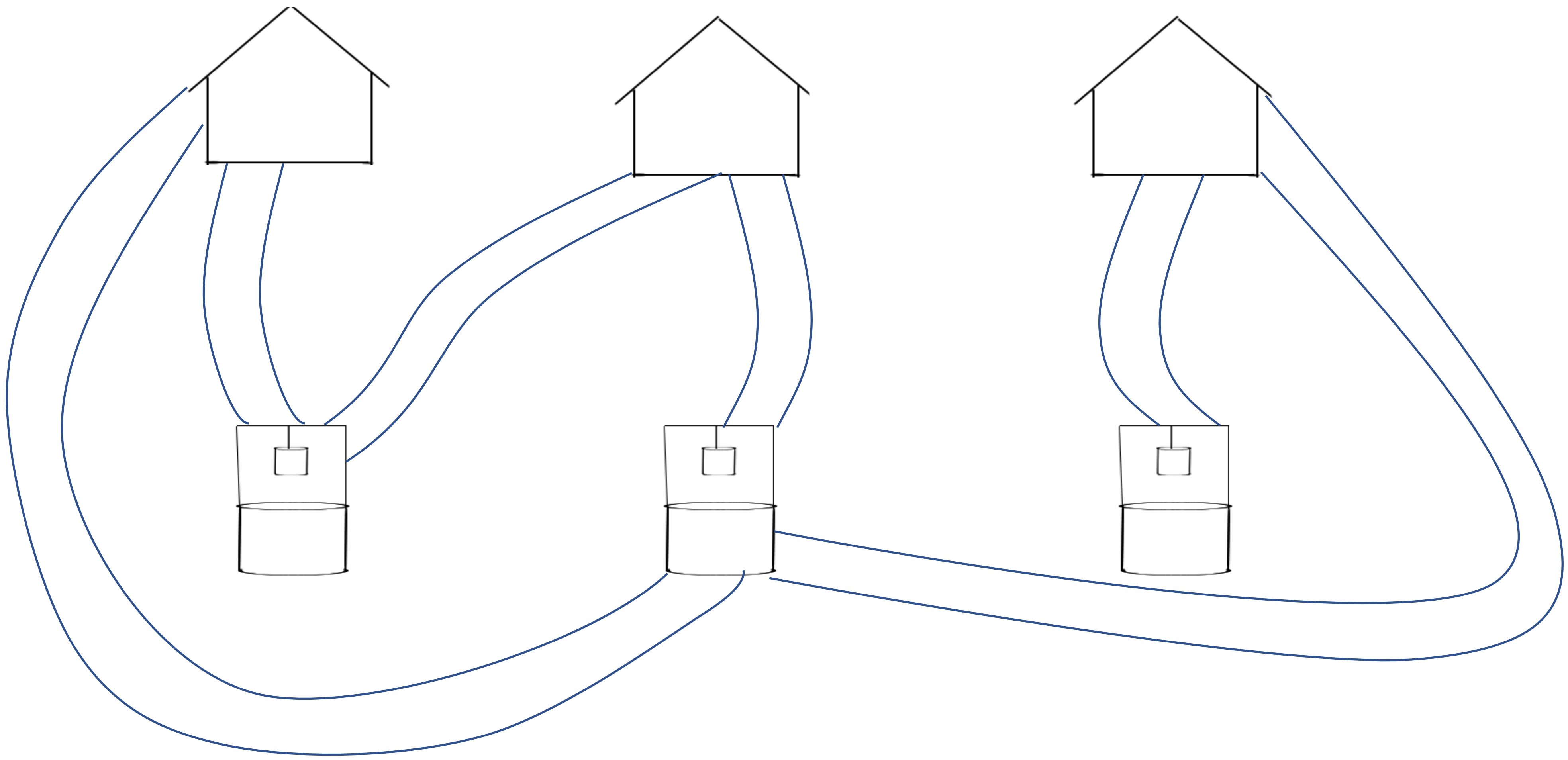


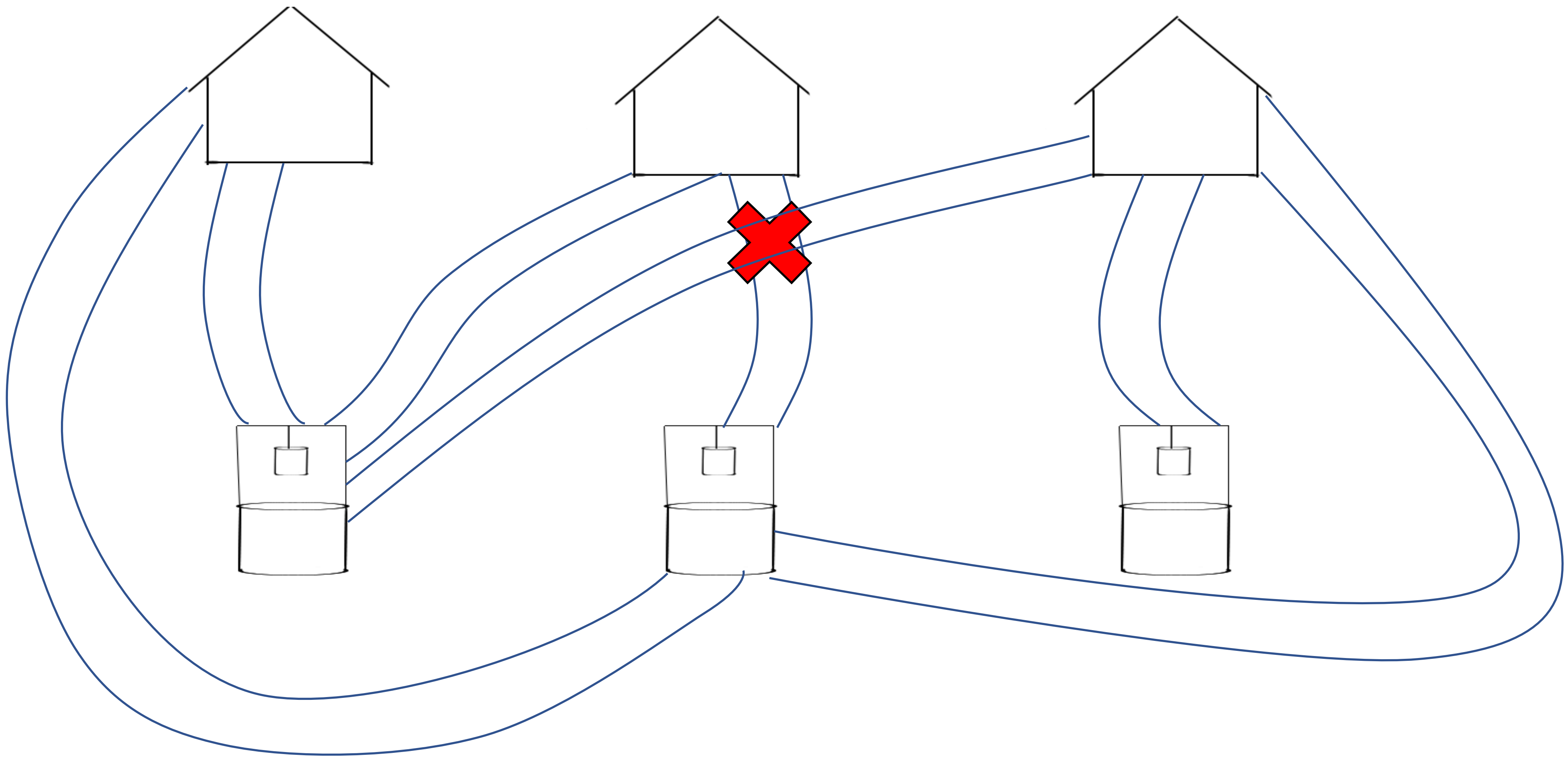


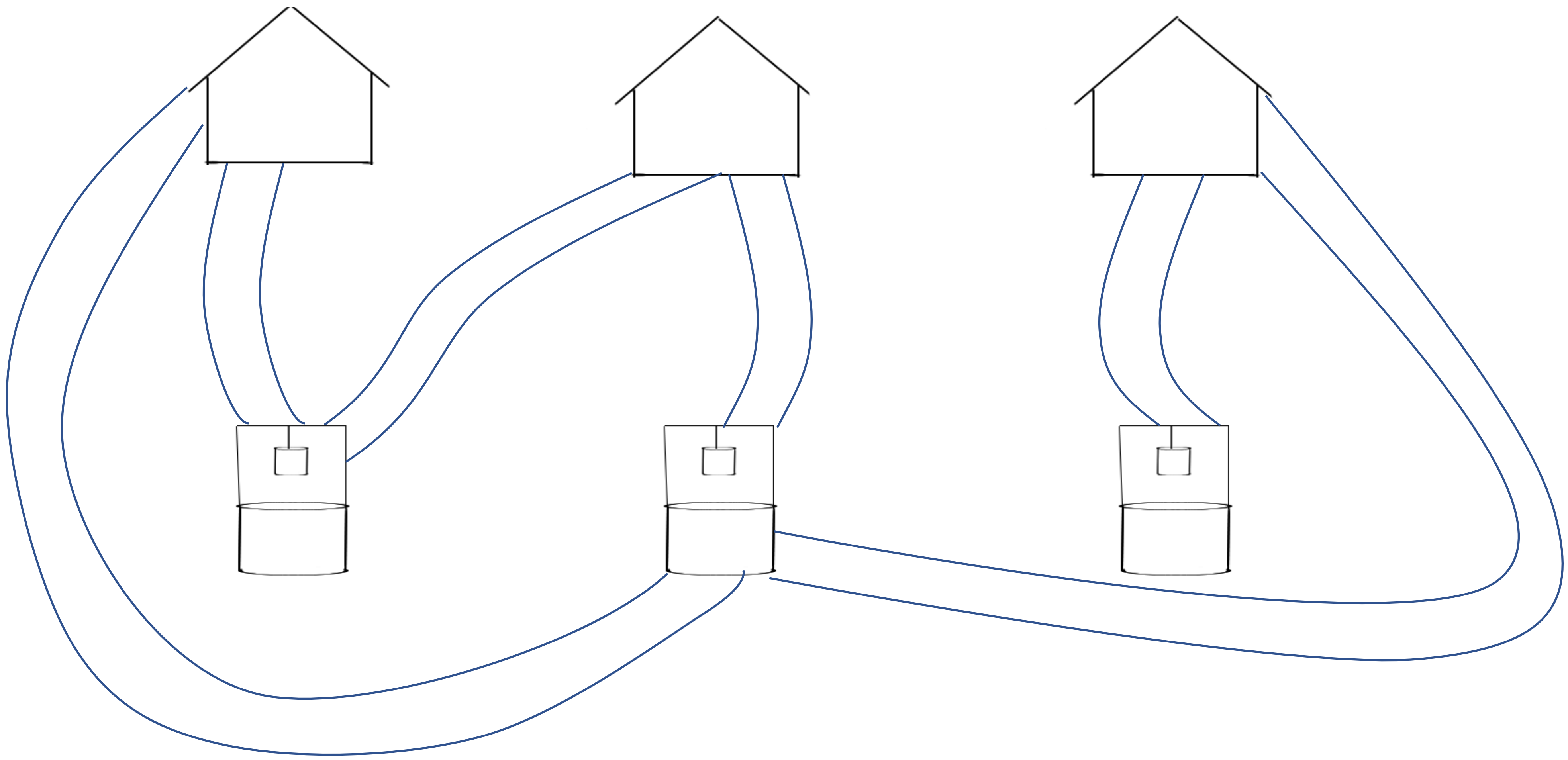


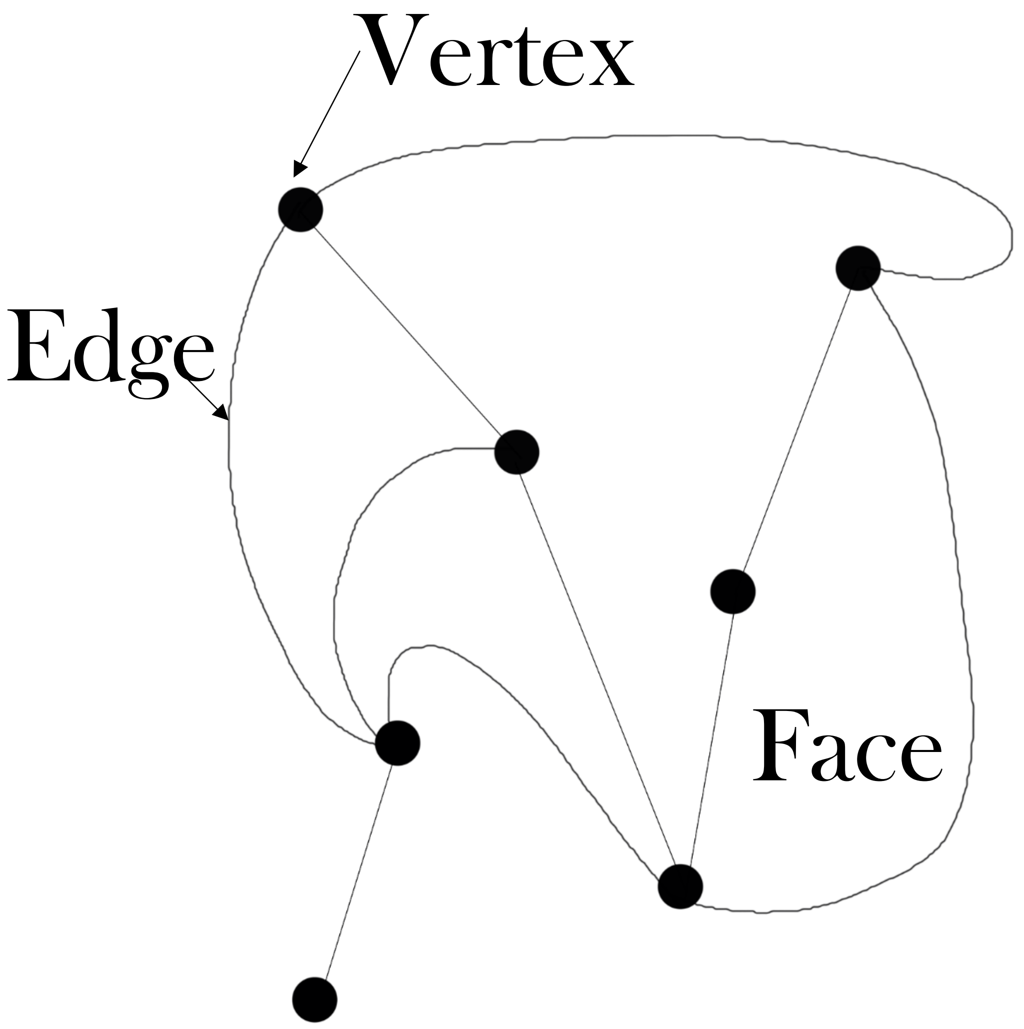


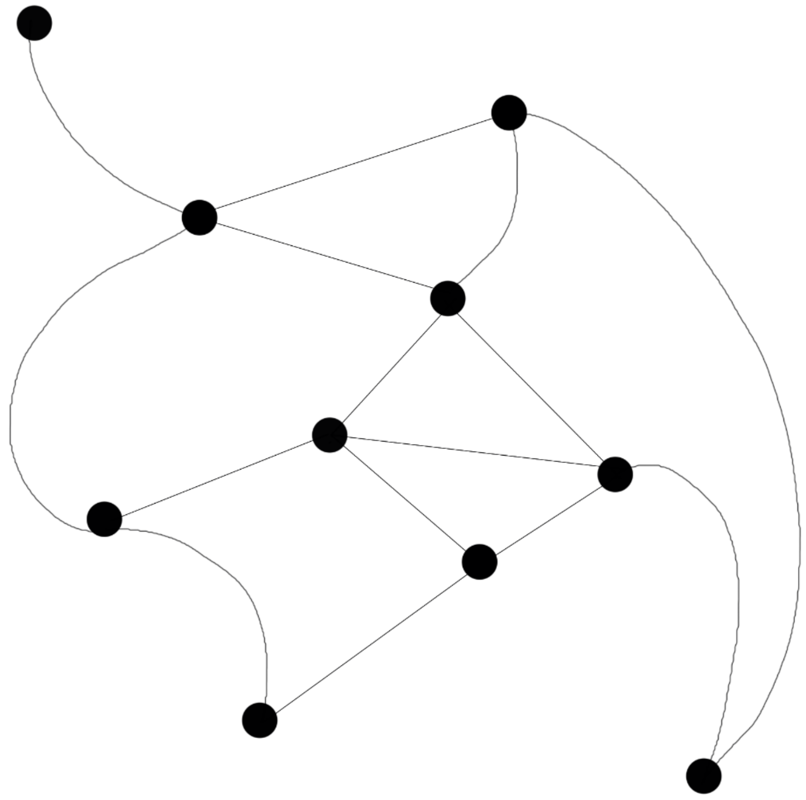


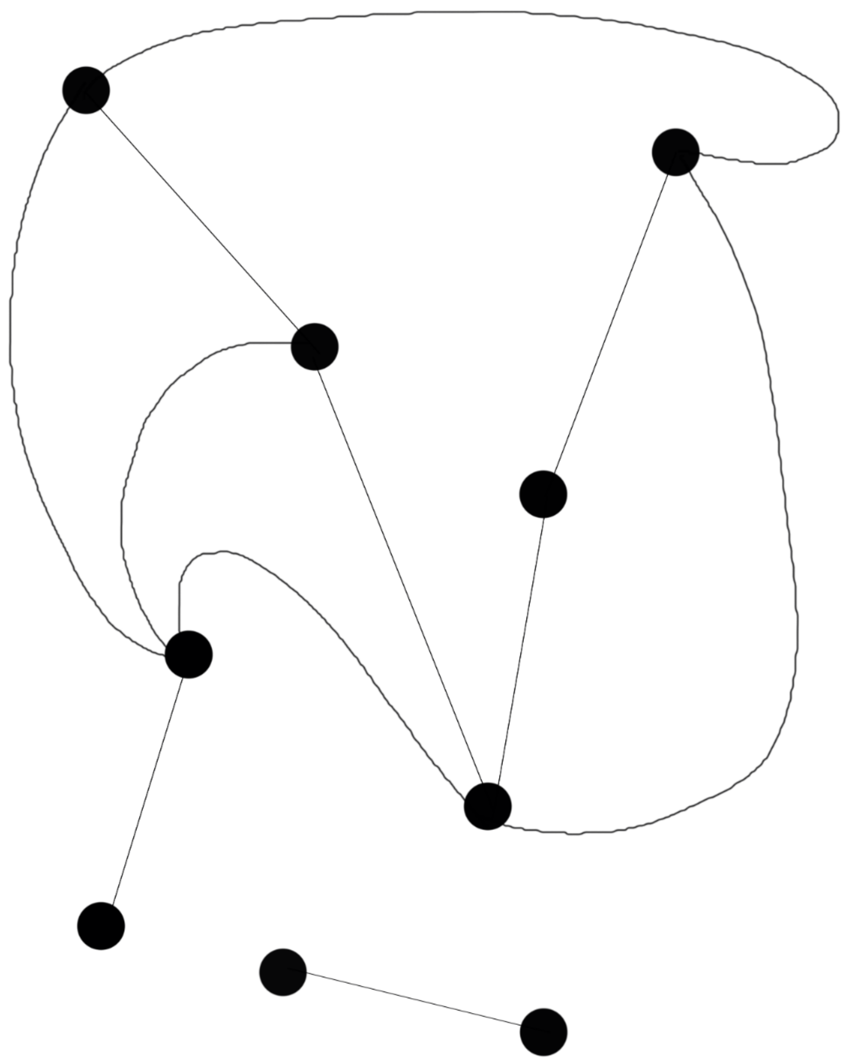


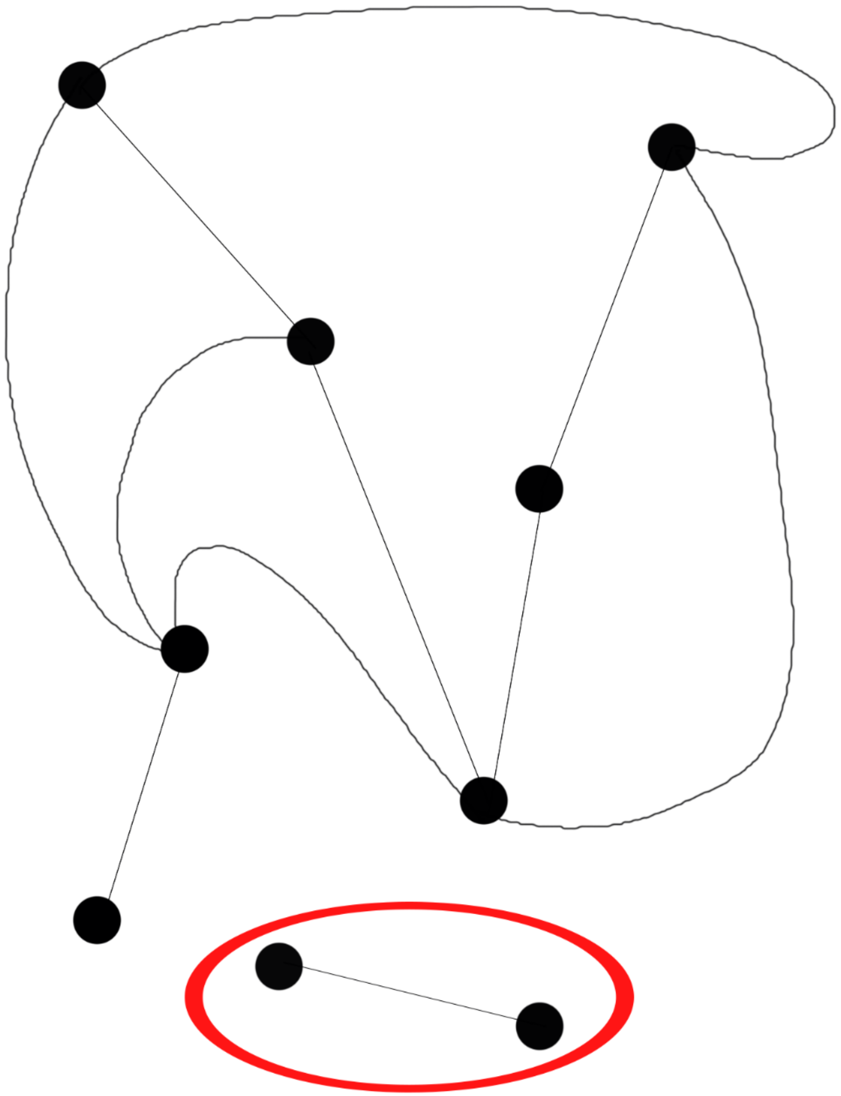


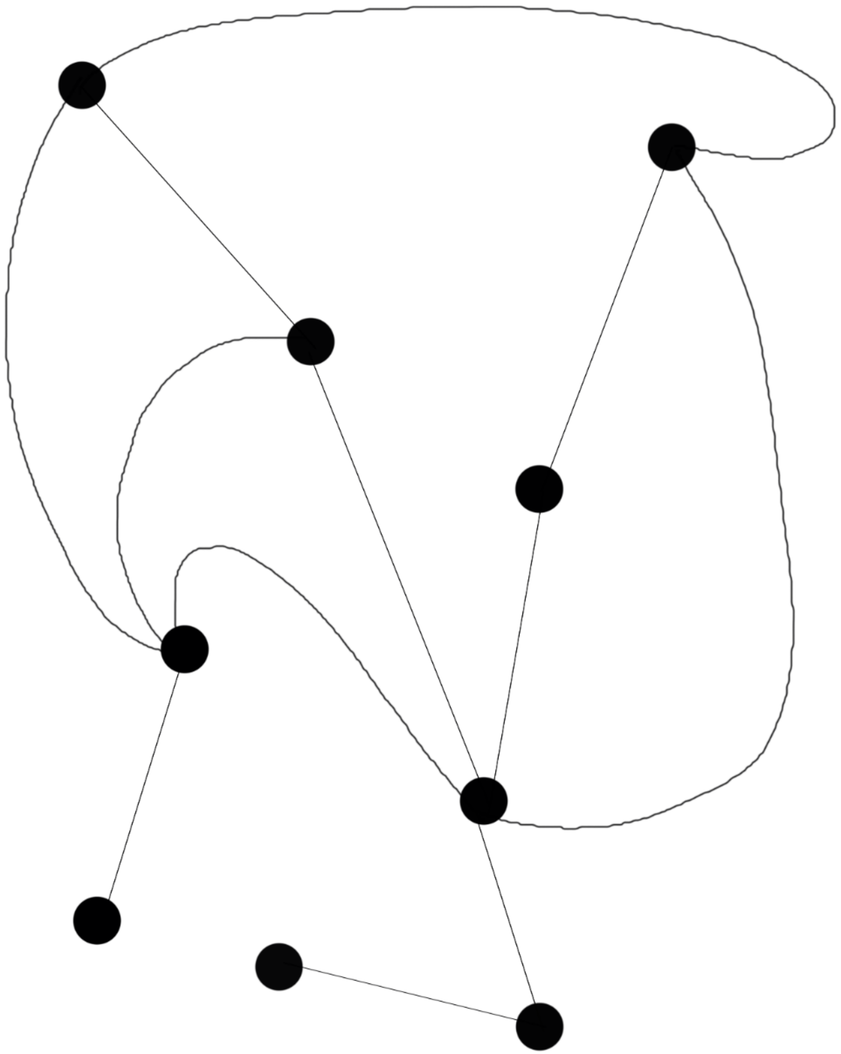


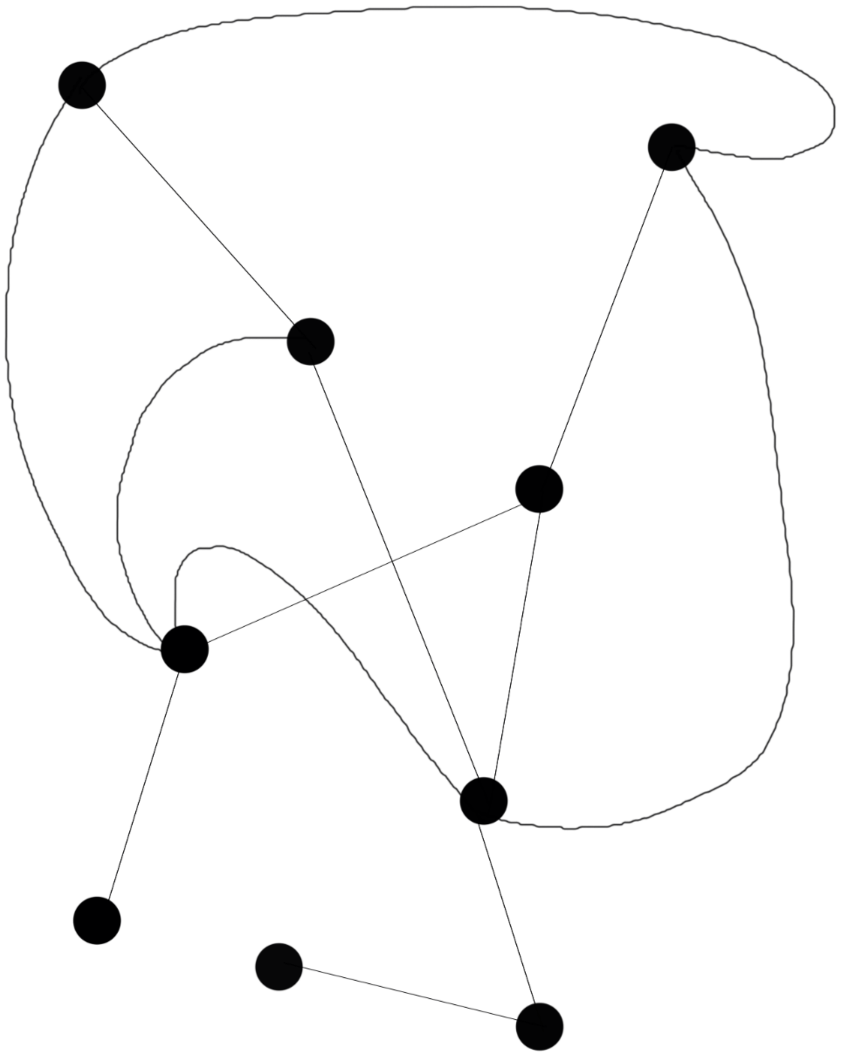

Vertex
Edge
Face

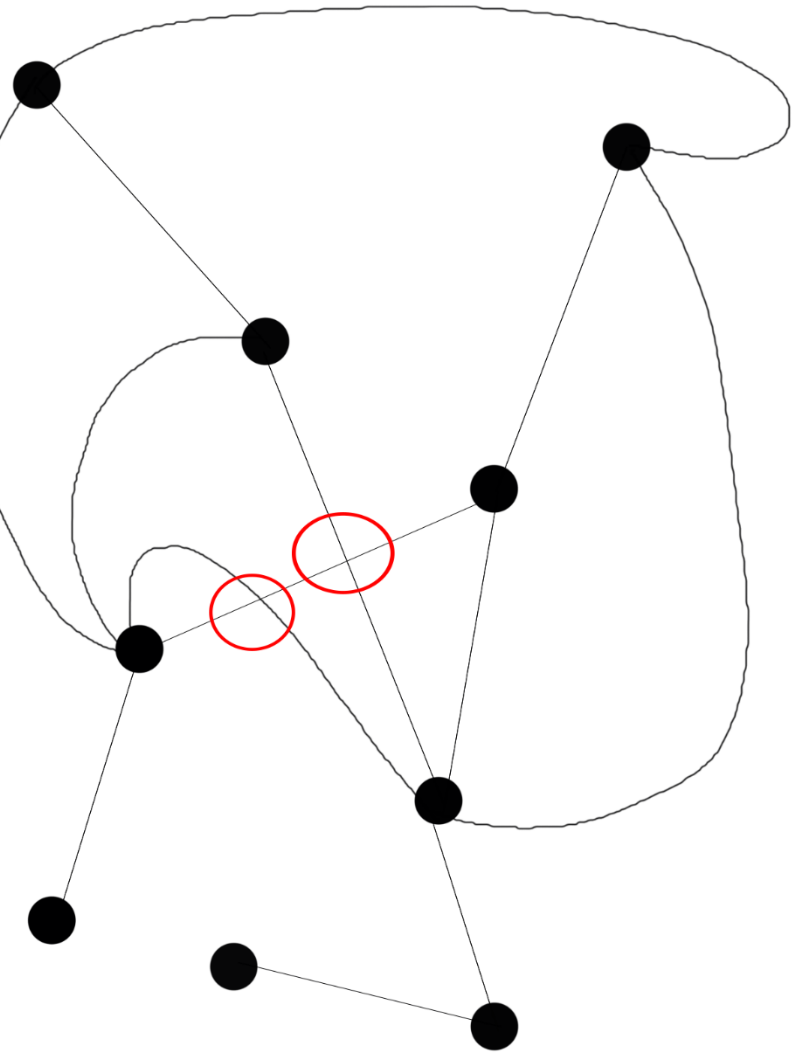


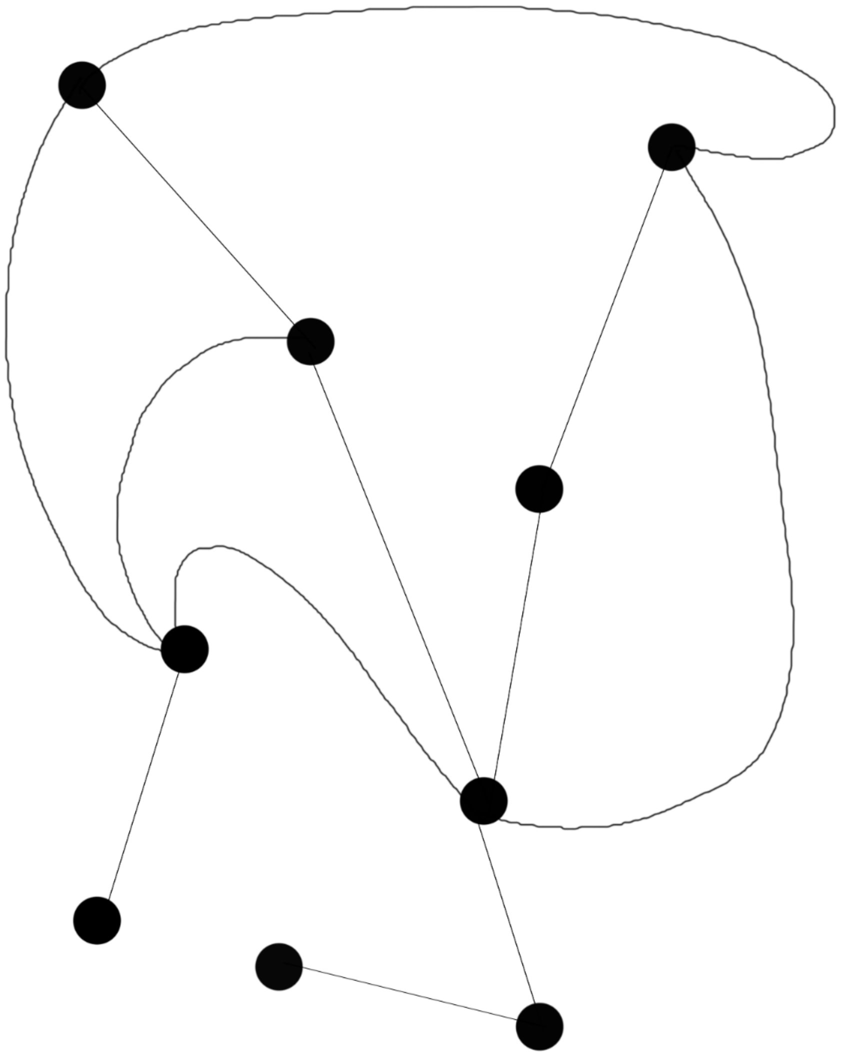


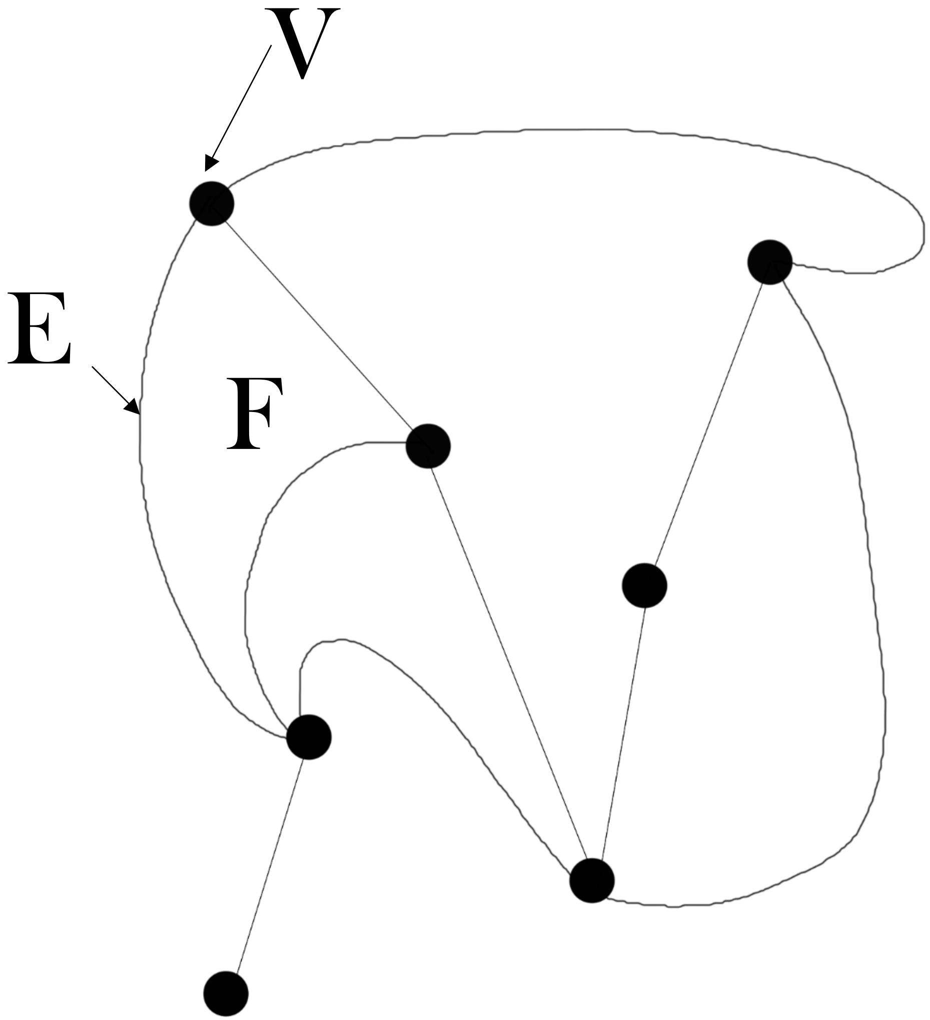


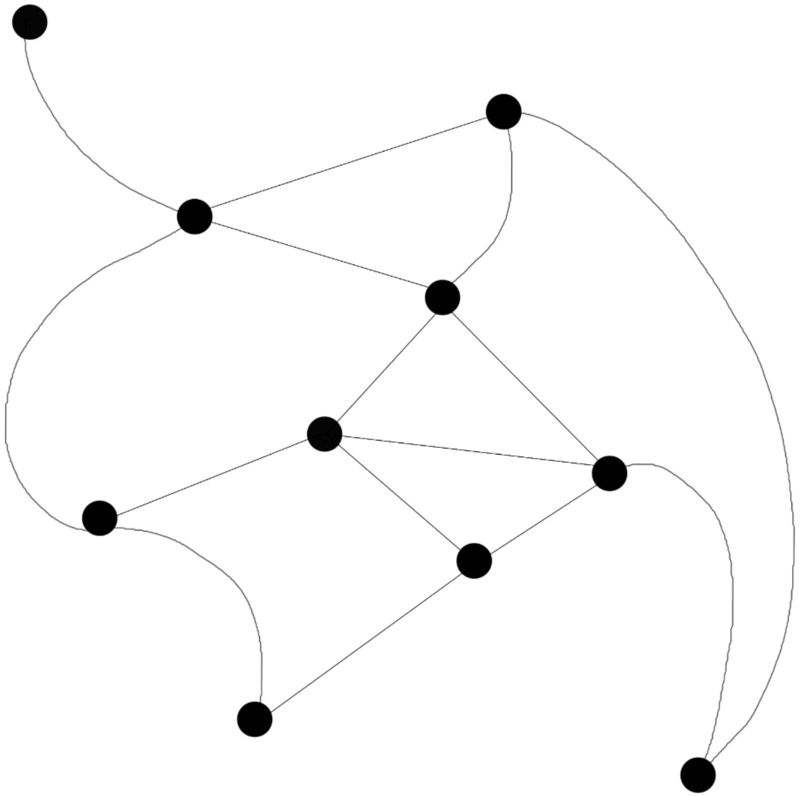




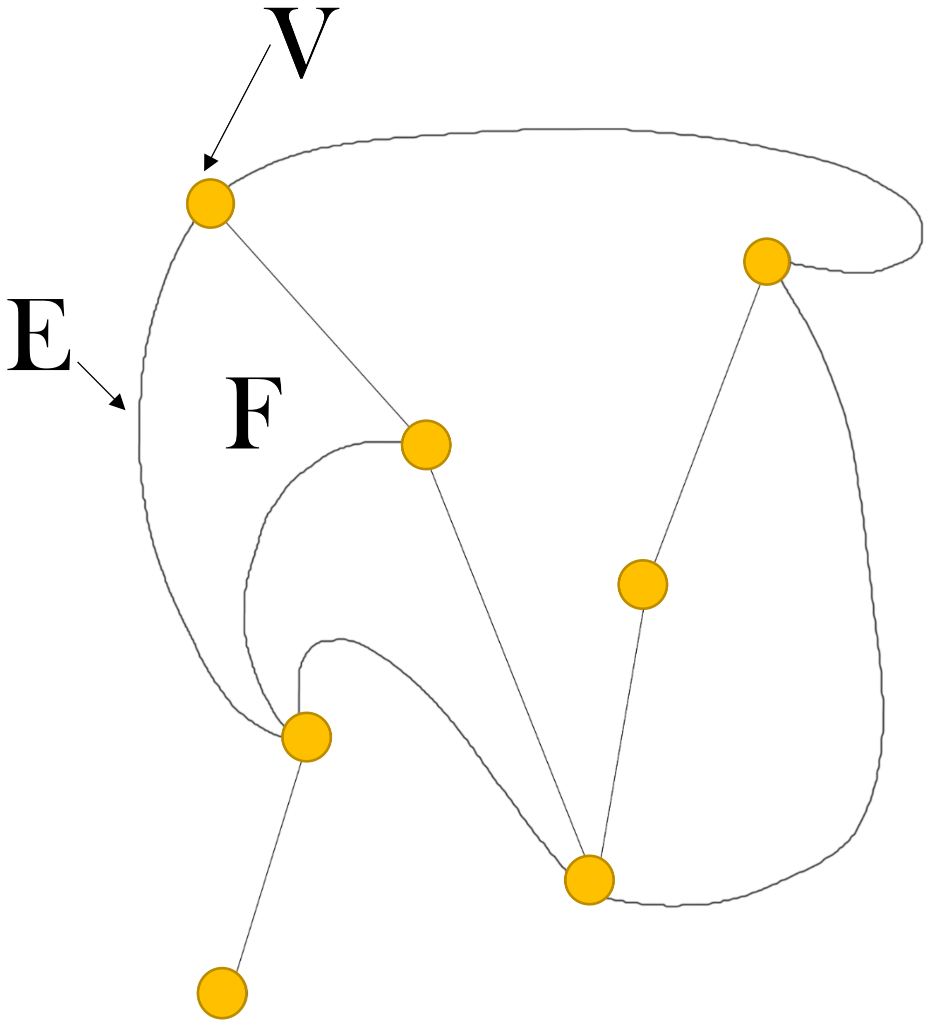





V
E
F

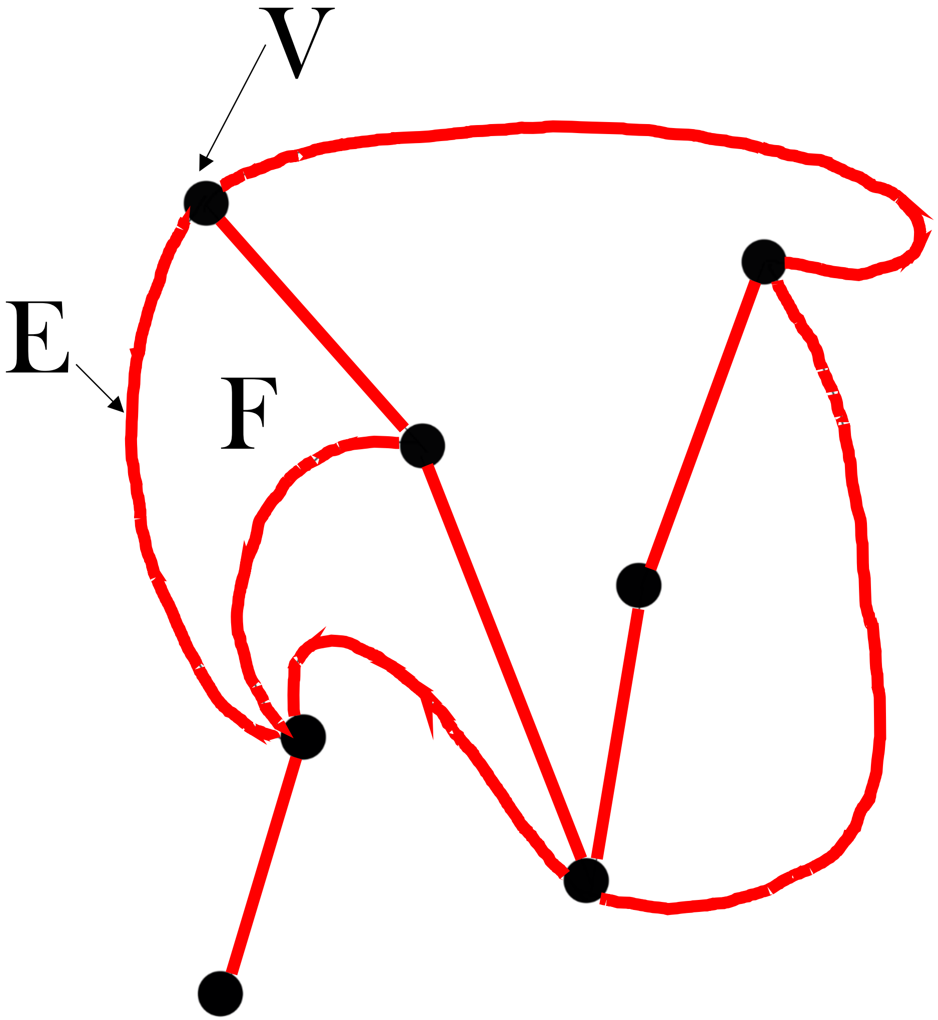


V = [illegible]



$V = 7$

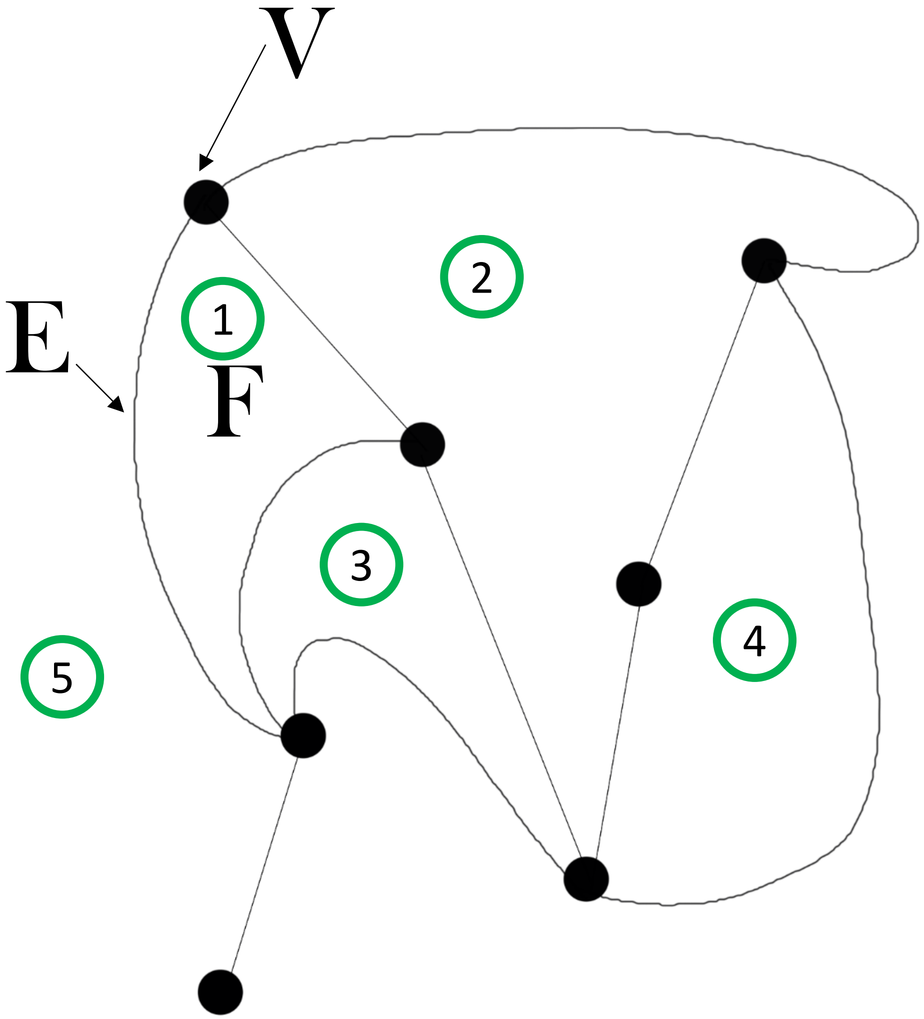
$E = 1$[illegible]



$V = 7$

$E = 10$

$F = 5$

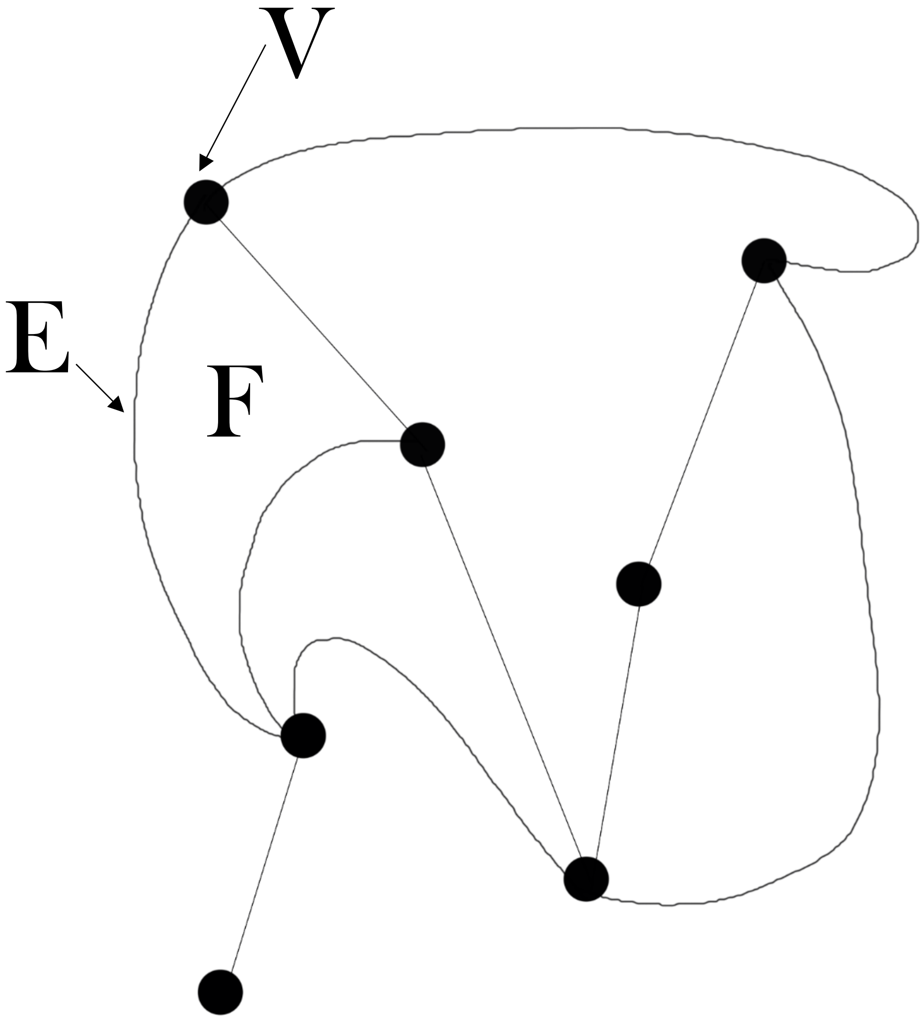


V = 7

E =10

F = 5

7 + 5 = 10 + 2



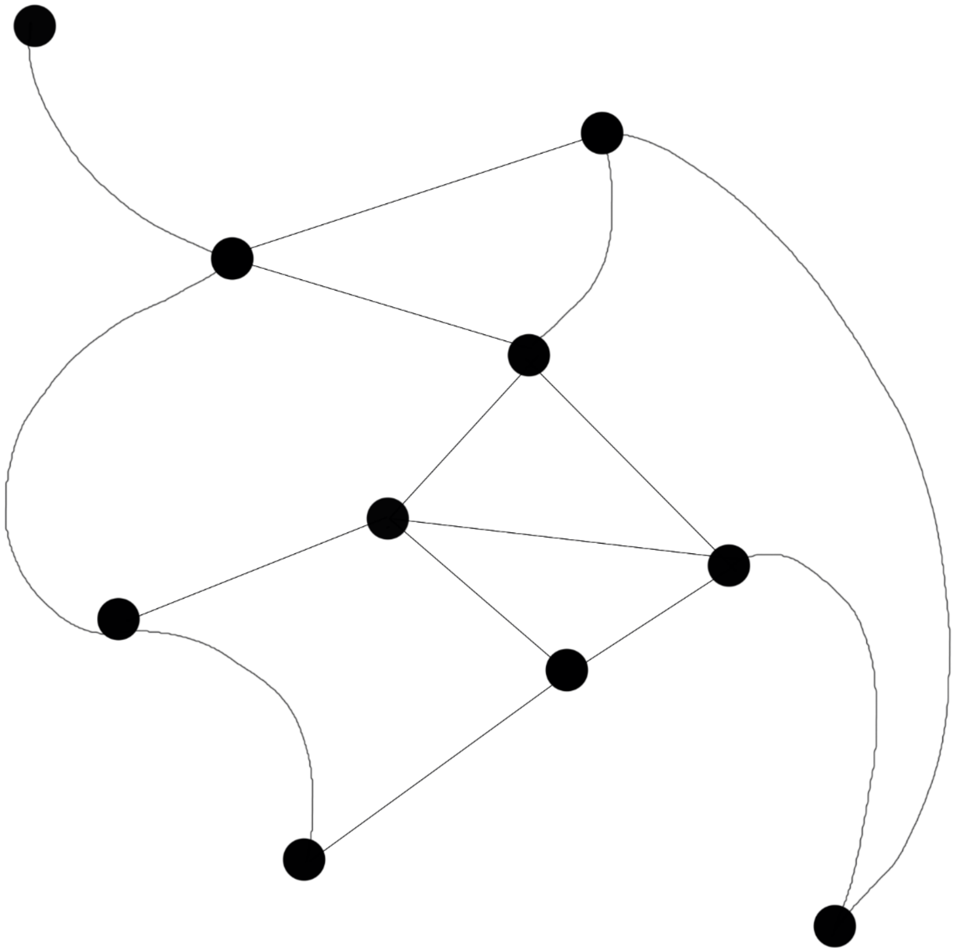

Euler's Forumla

V+F = E+2

$V=10$
$E=15$
$F=7$

$$\boxed{V+F = E+2}$$

$$10+7=15+2$$



# How can we prove the formula?

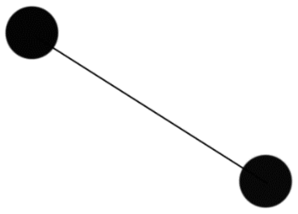
$$\boxed{V+F = E+2}$$

- By induction on E

- E = 0

$$V = 1$$
$$E = 0$$
$$F = 1$$

$$\boxed{V+F = E+2}$$

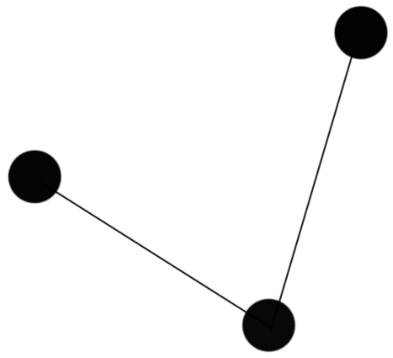


$V = 1 + 1$

$E = 0 + 1$

$F = 1$

$$\boxed{V+F = E+2}$$

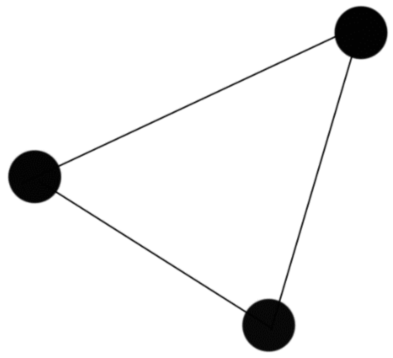


$V = 2 + 1$

$E = 1 + 1$

$F = 1$

$$\boxed{V+F = E+2}$$

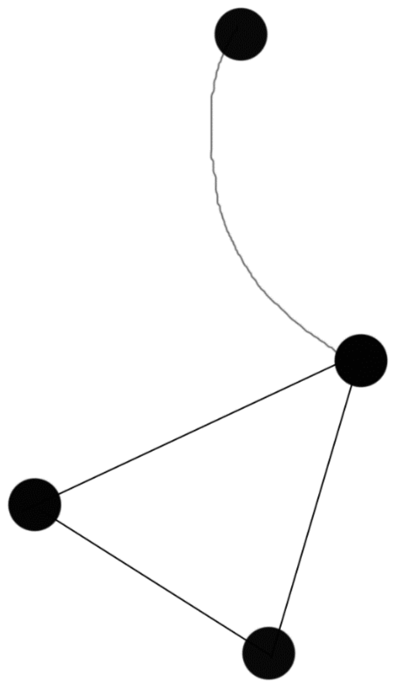


$V = 3$

$E = 2 \; {\color{red}+1}$

$F = 1 \; {\color{red}+1}$

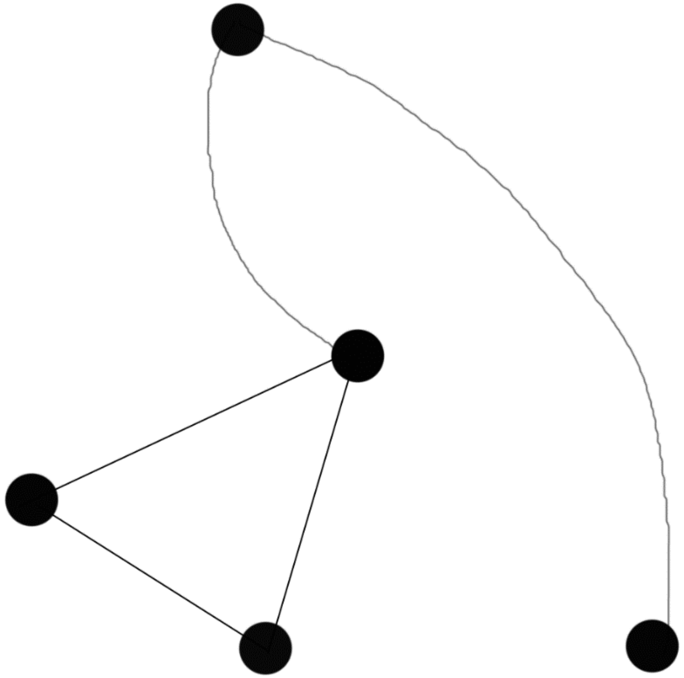
$$\boxed{V+F = E+2}$$



$V = 3 \; {\color{red}+1}$

$E = 3 \; {\color{red}+1}$

$F = 2$



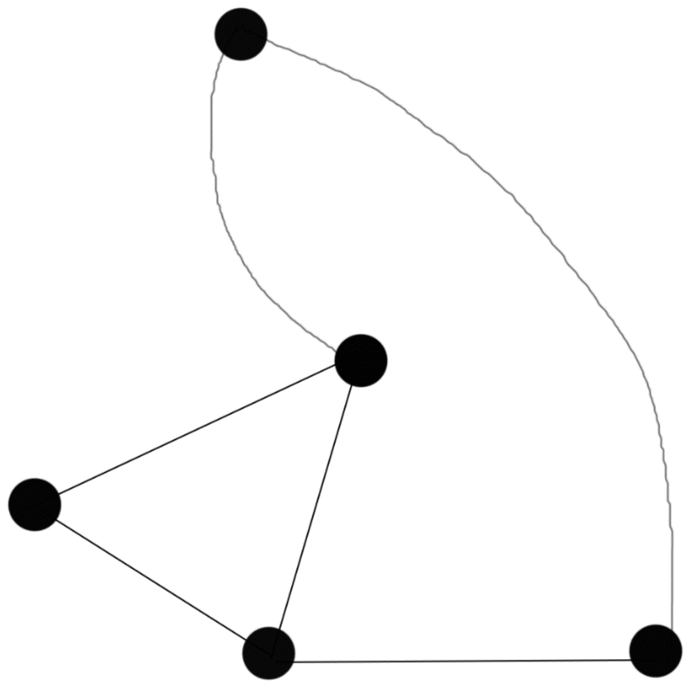
$$\boxed{V+F = E+2}$$

$V = 4$ $+1$

$E = 4$ $+1$

$F = 2$

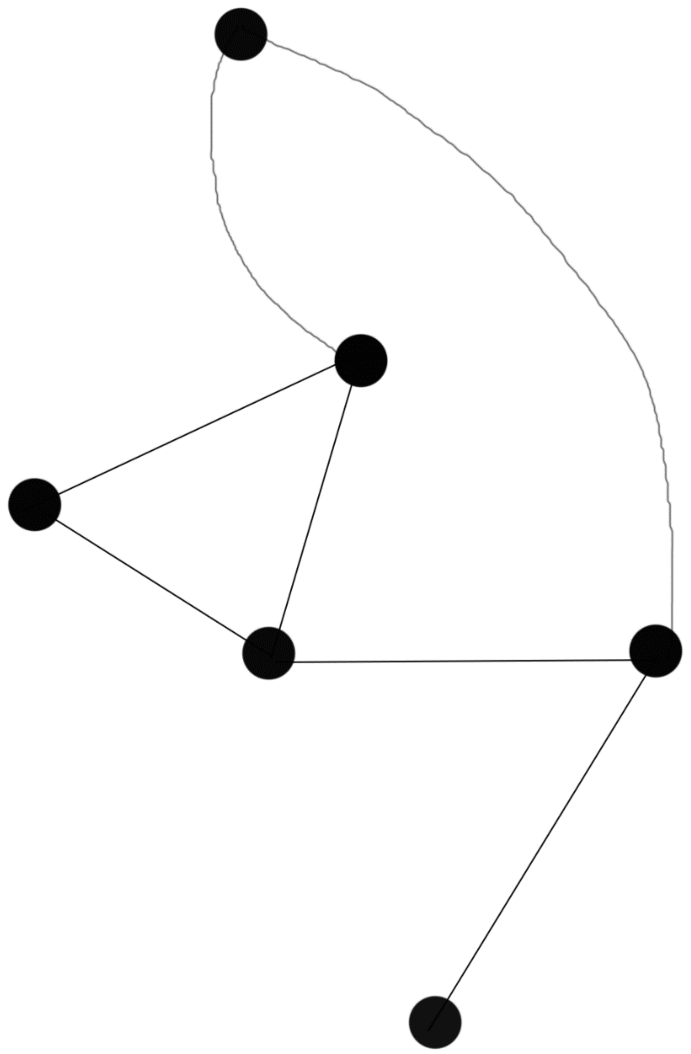
$$V+F = E+2$$



$V = 5$

$E = 5$ $+1$

$F = 2$ $+1$

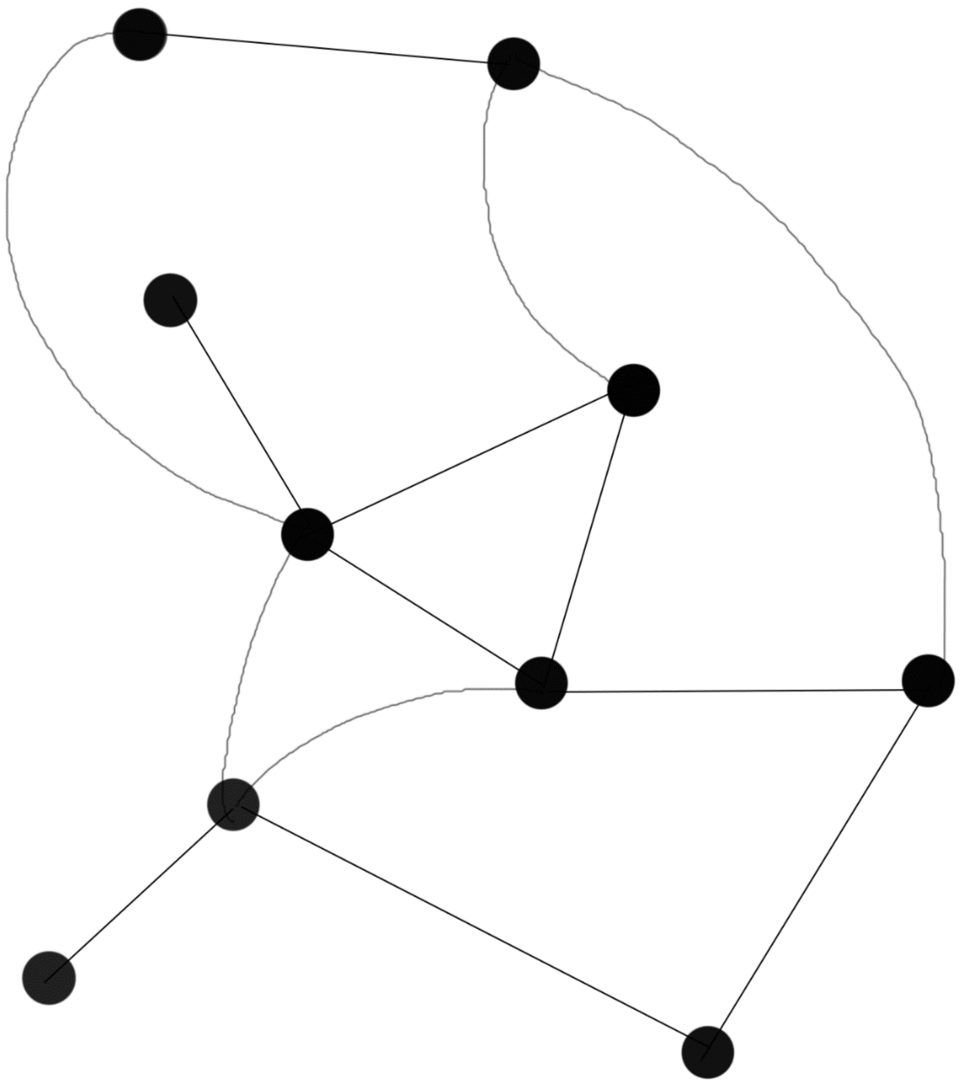


$$\boxed{V+F = E+2}$$

$V = 5$ $+1$

$E = 6$ $+1$

$F = 3$

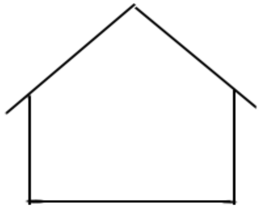
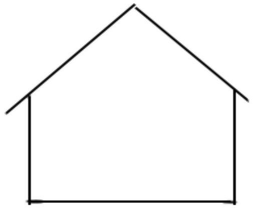
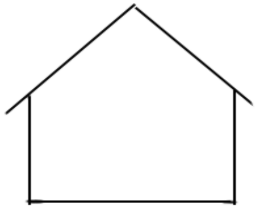
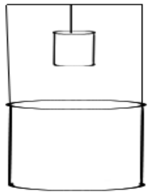
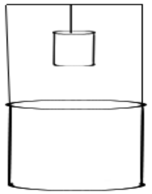
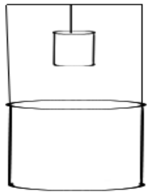


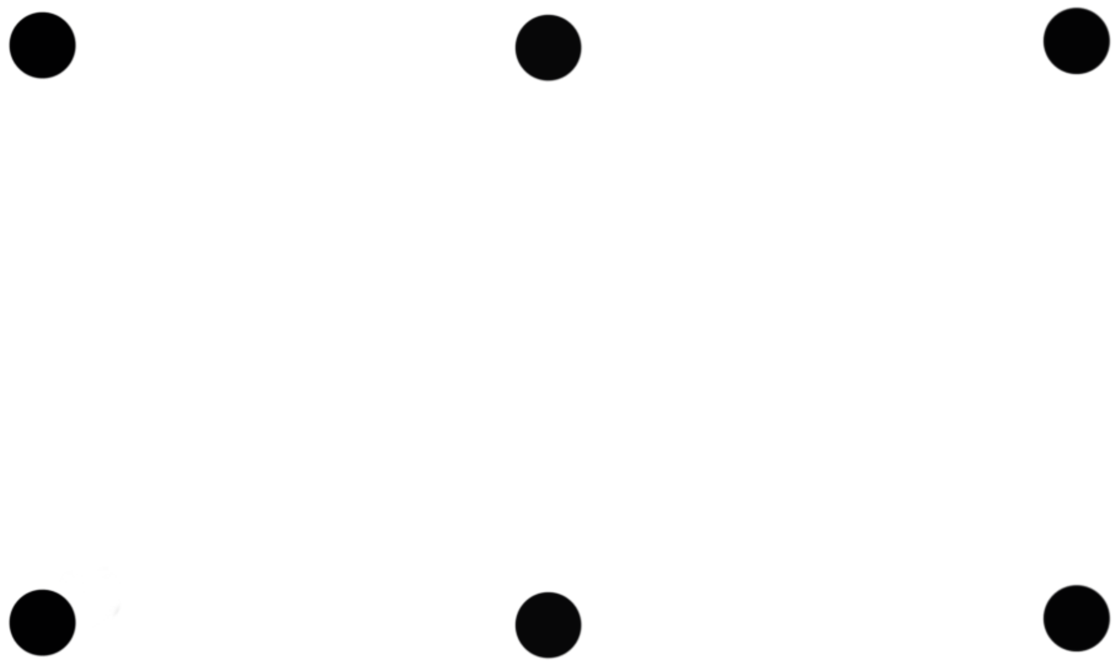
$$V+F = E+2$$

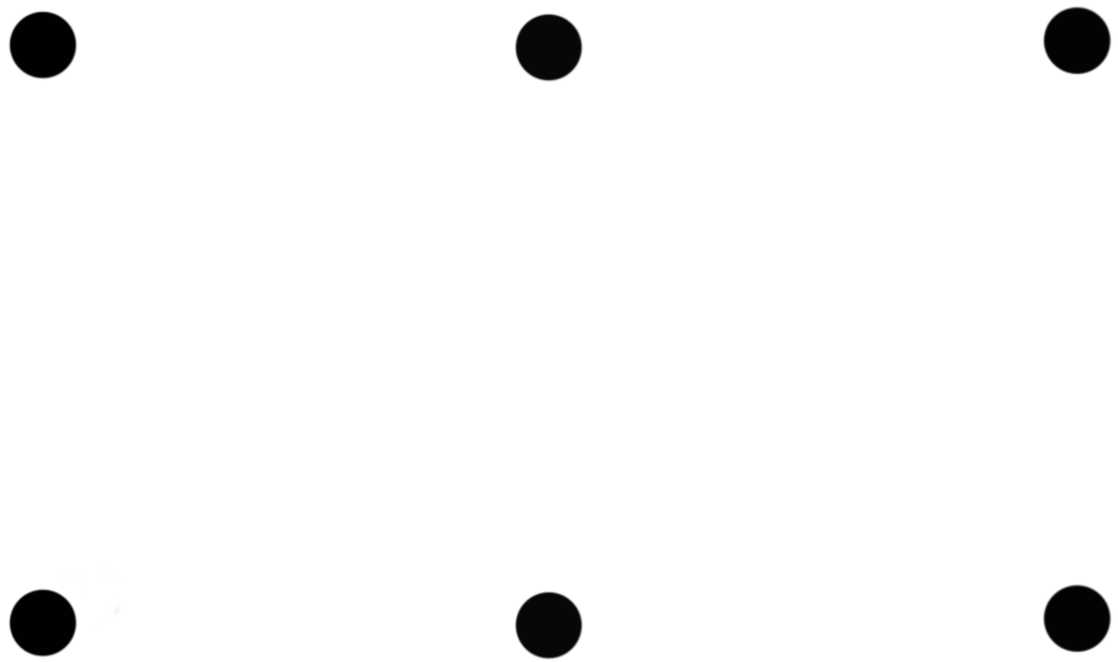
$V = 10$

$E = 14$

$F = 6$

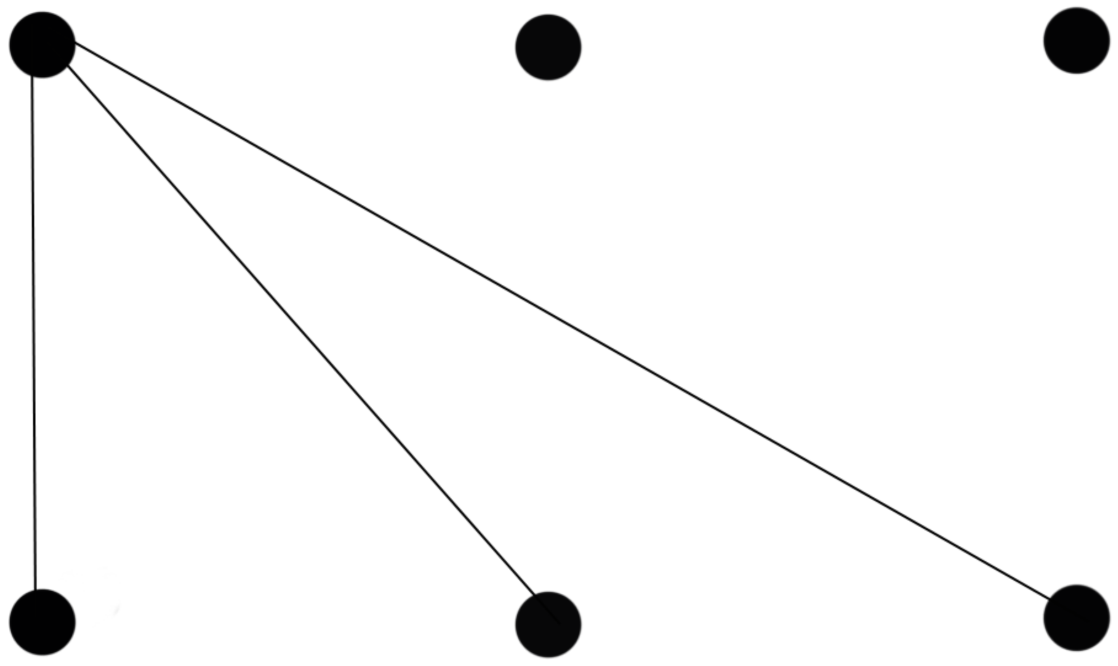




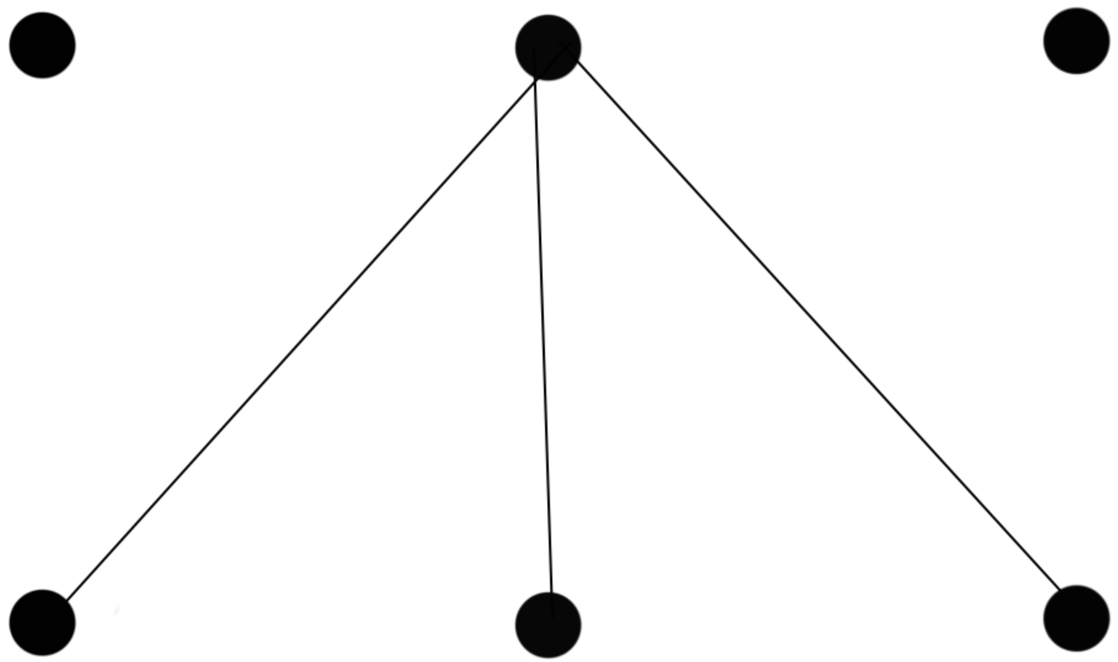
$V = 6$

$E = ?$

$F = ?$



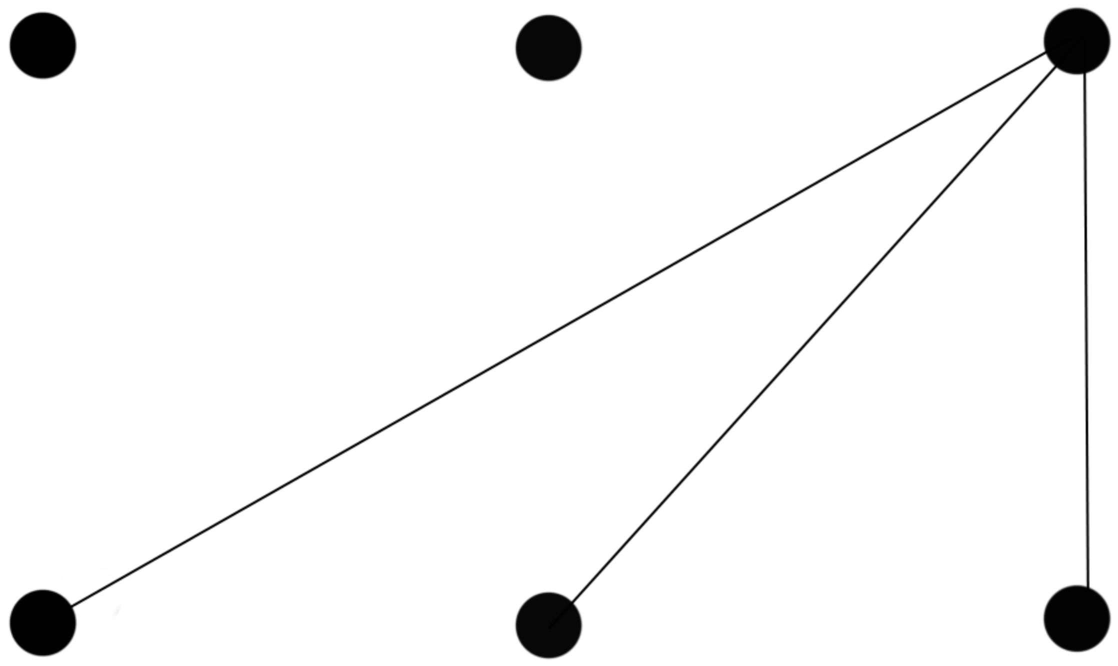
V = 6
E = 3 +
F = ?



V = 6

E = 3 + 3 +

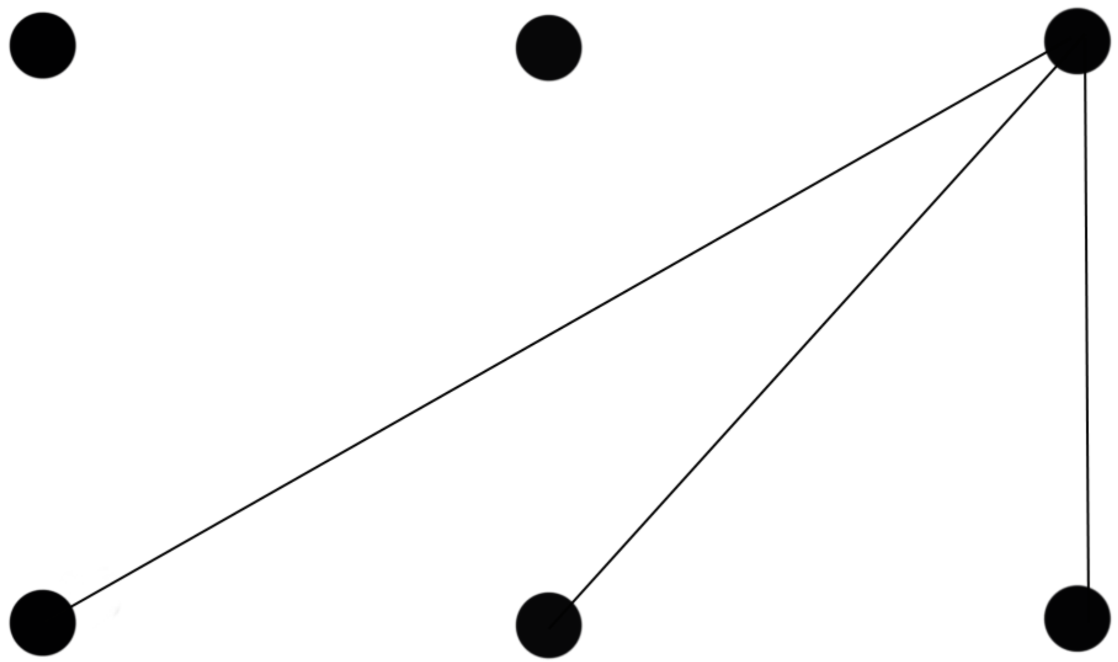
F = ?



$\mathbf{V} = 6$

$\mathbf{E} = 3 + 3 + 3 = 9$

$\mathbf{F} = ?$



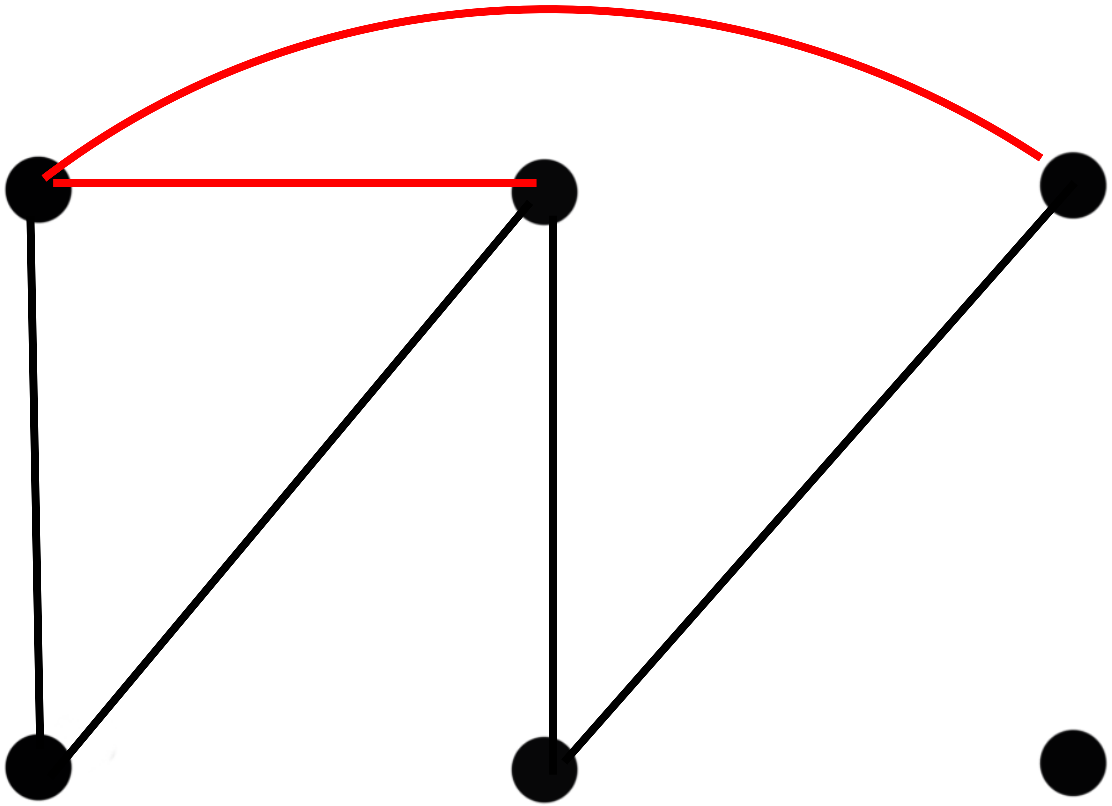
$$\boxed{V+F = E+2}$$

$V = 6$

$E = 3 + 3 + 3 = 9$

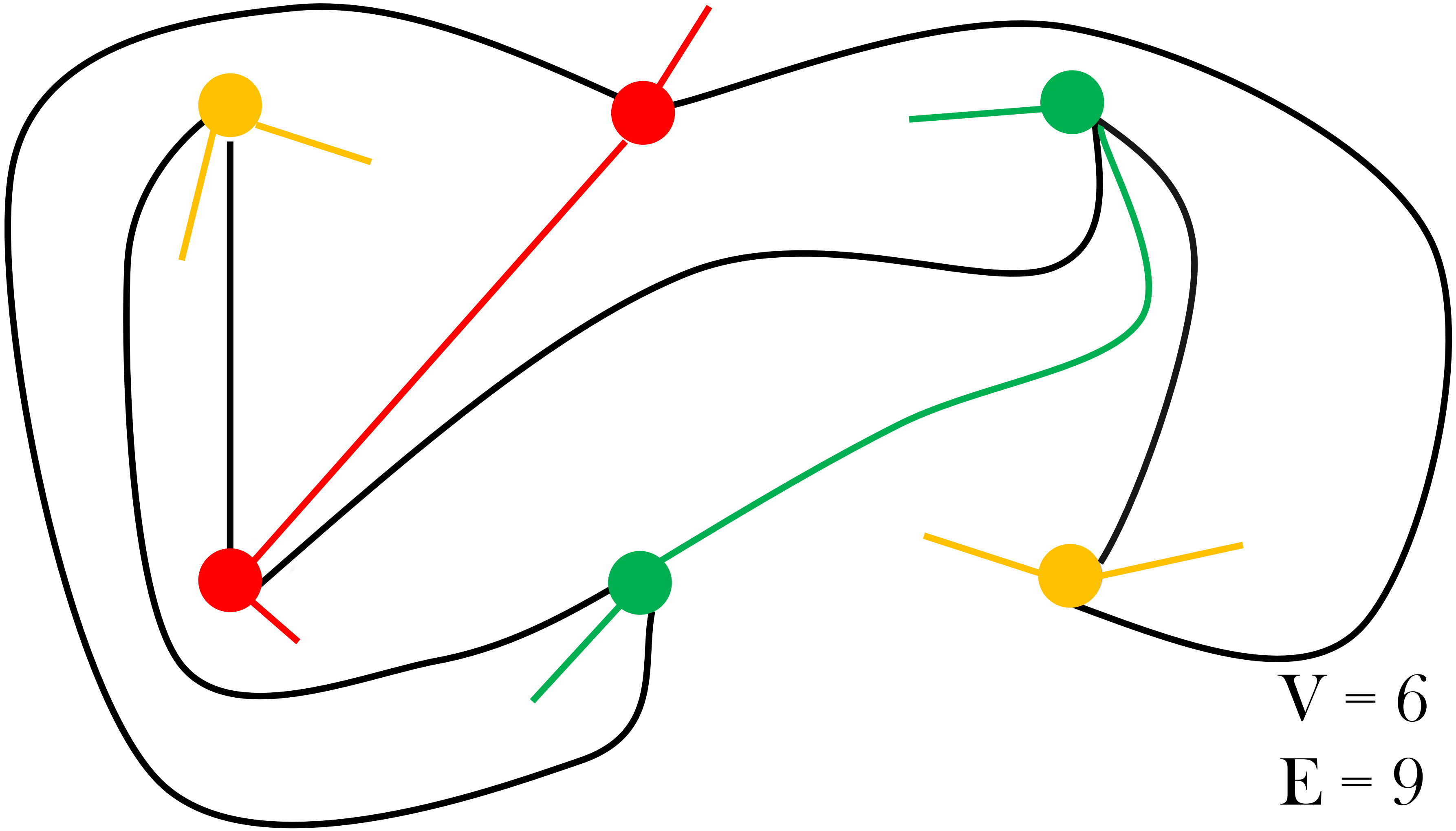
$F = 5$

# But, are there really 5 faces?



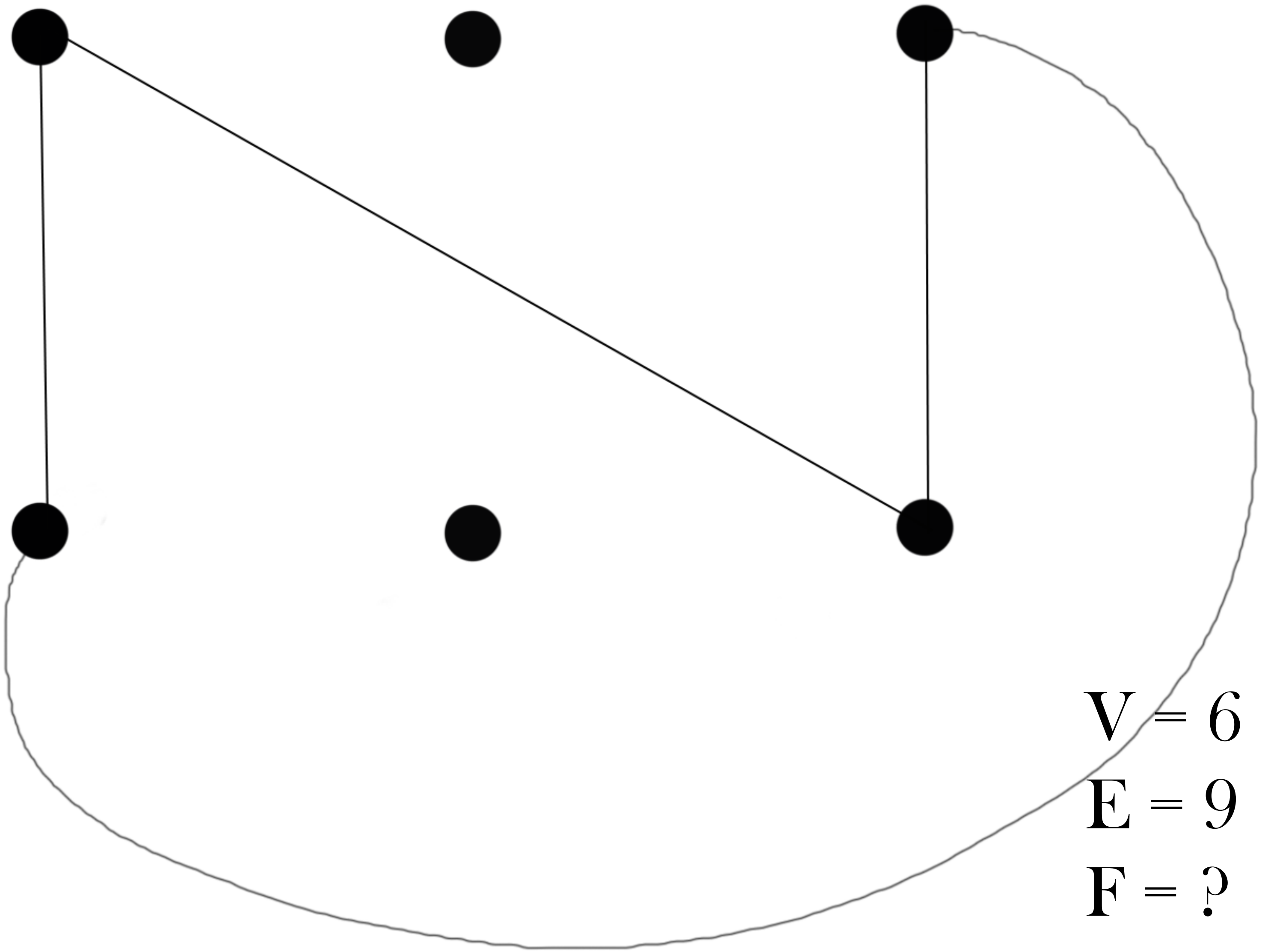
$V = 6$
$E = 9$
$F = ?$

# But, are there really 5 faces?

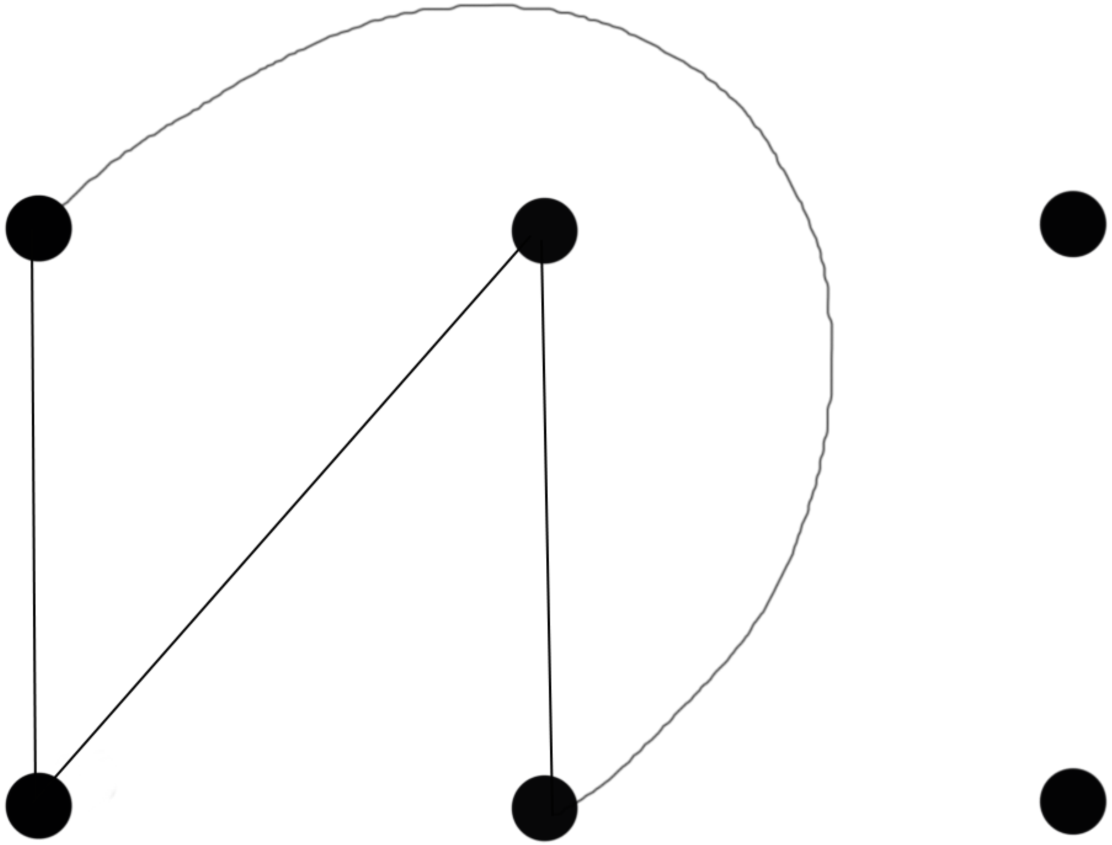


$V = 6$
$E = 9$
$F = ?$

# But, are there really 5 faces?



$V = 6$
$E = 9$
$F = ?$



V = 6
E = 9
F = ?

2 Marks per Edge

4 Marks per Face

1 E = 2 F

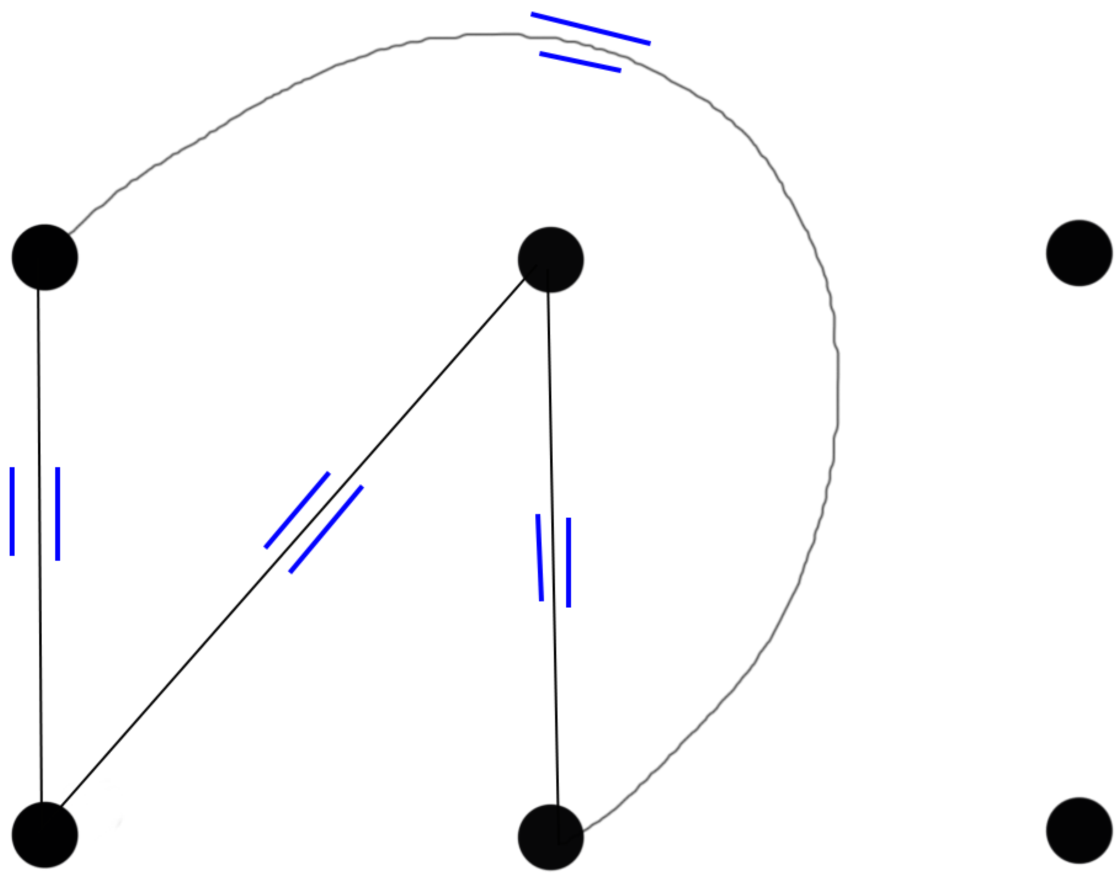
5 Faces => 10 Edges

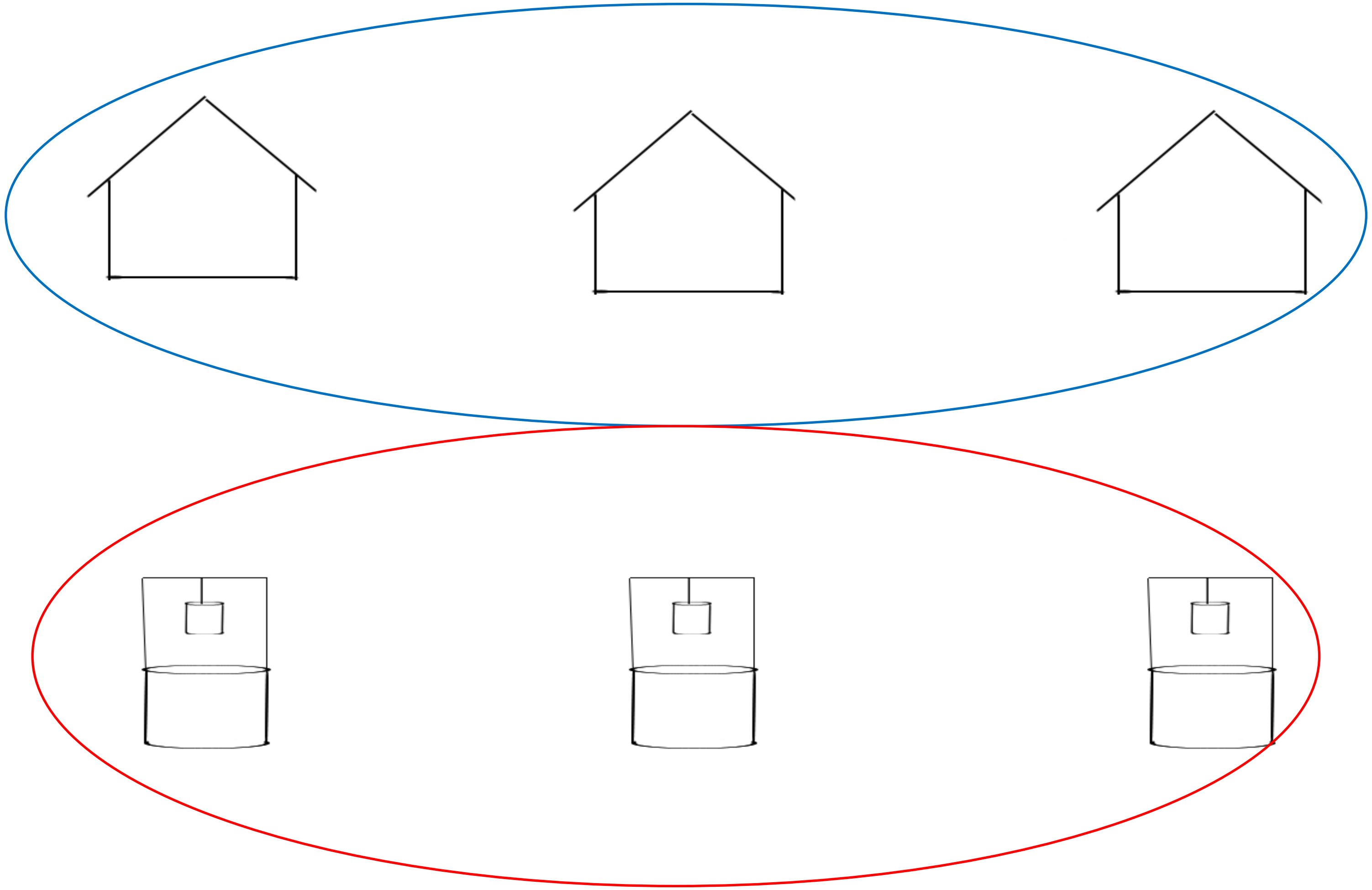
$$\boxed{V+F = E+2}$$

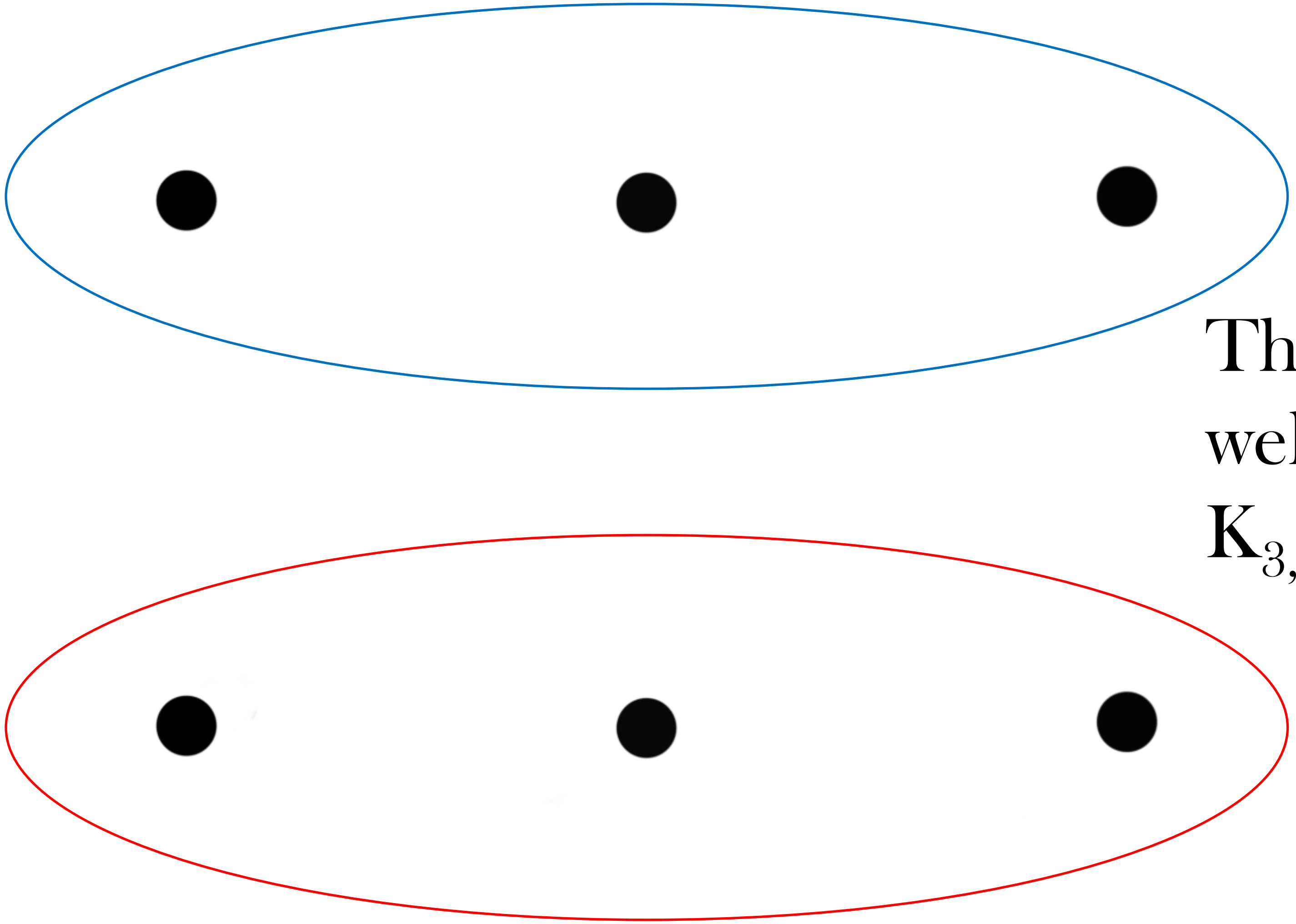
V = 6

E = 9

F = 5



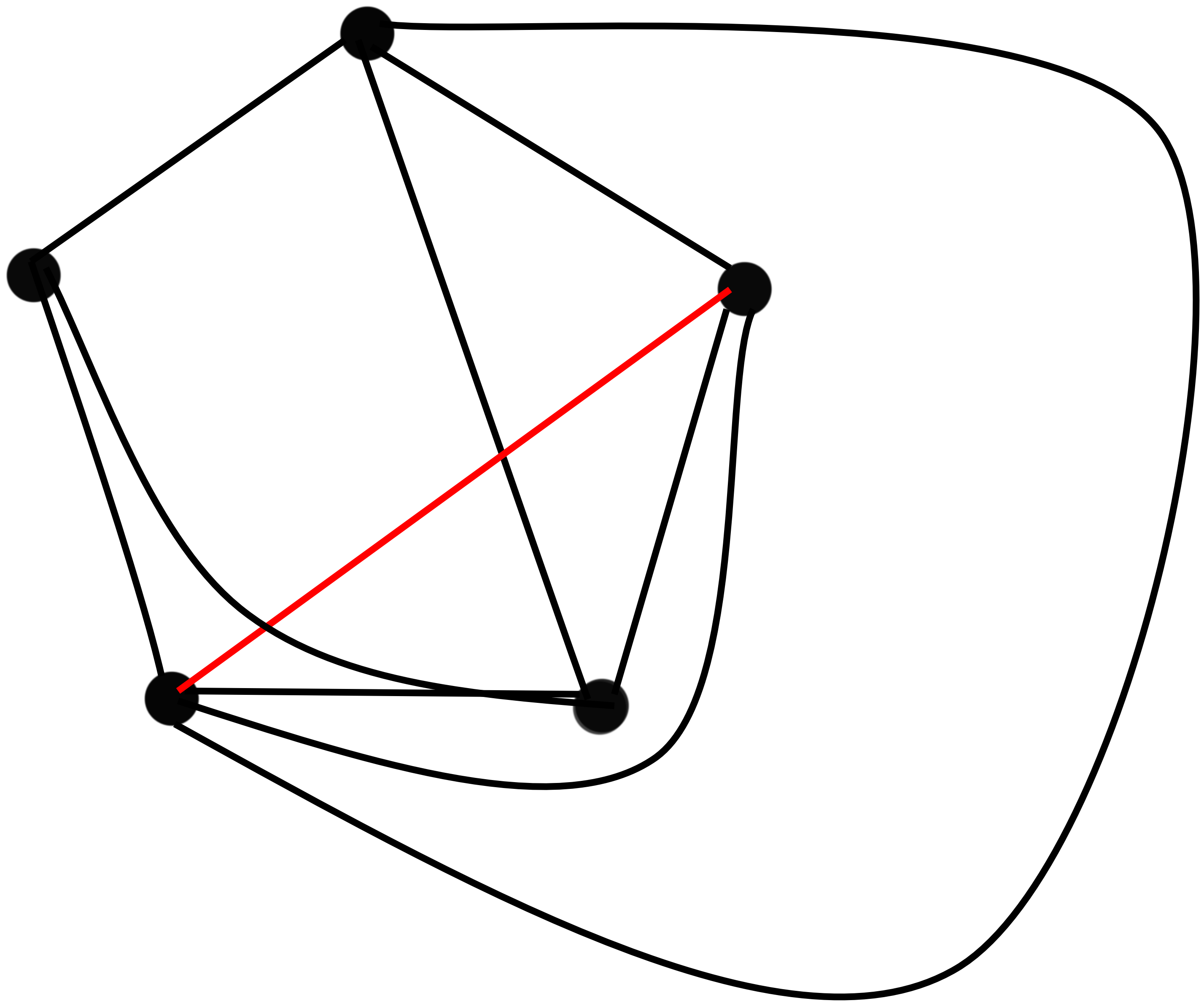




The houses and wells problem is a $K_{3,3}$ - Graph

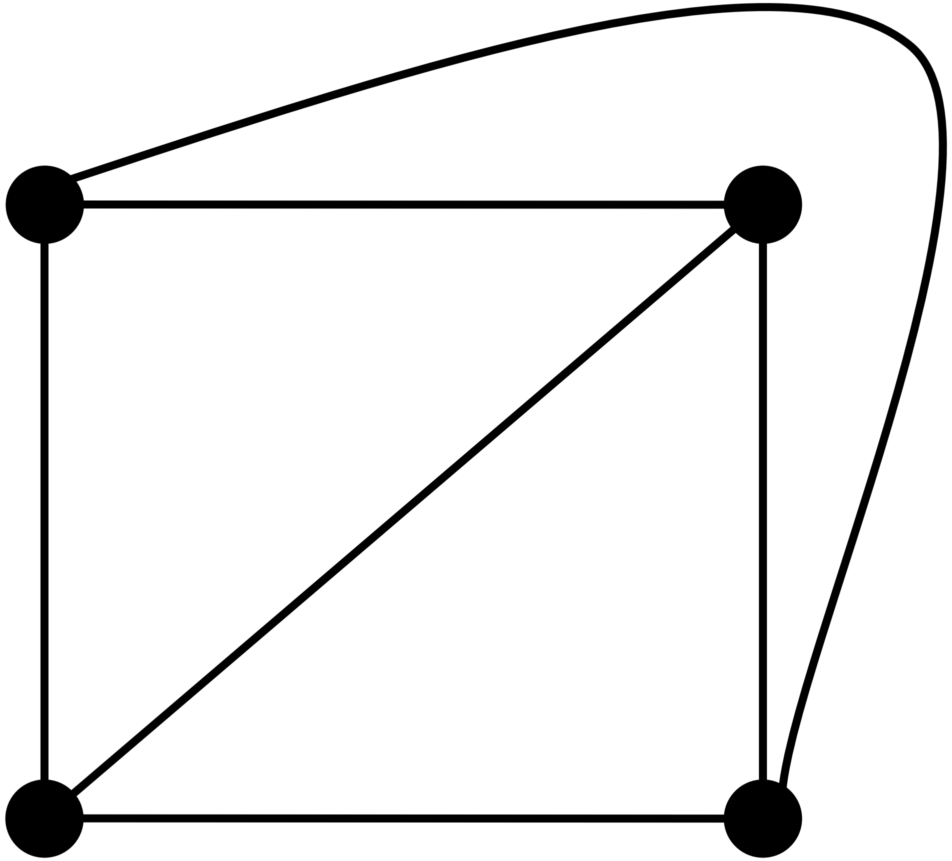
The $K_{3,3}$ - Graph is not planar

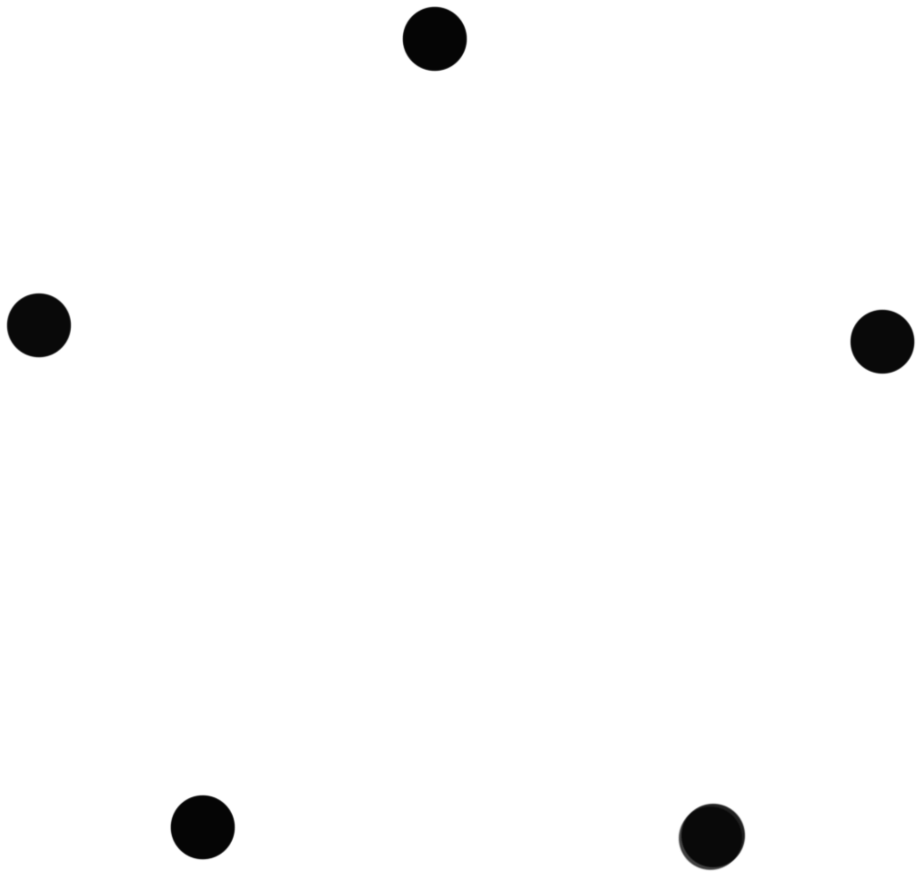
Is the $K_5$ - Graph planar?



Is the $K_4$ - Graph planar?

It is!

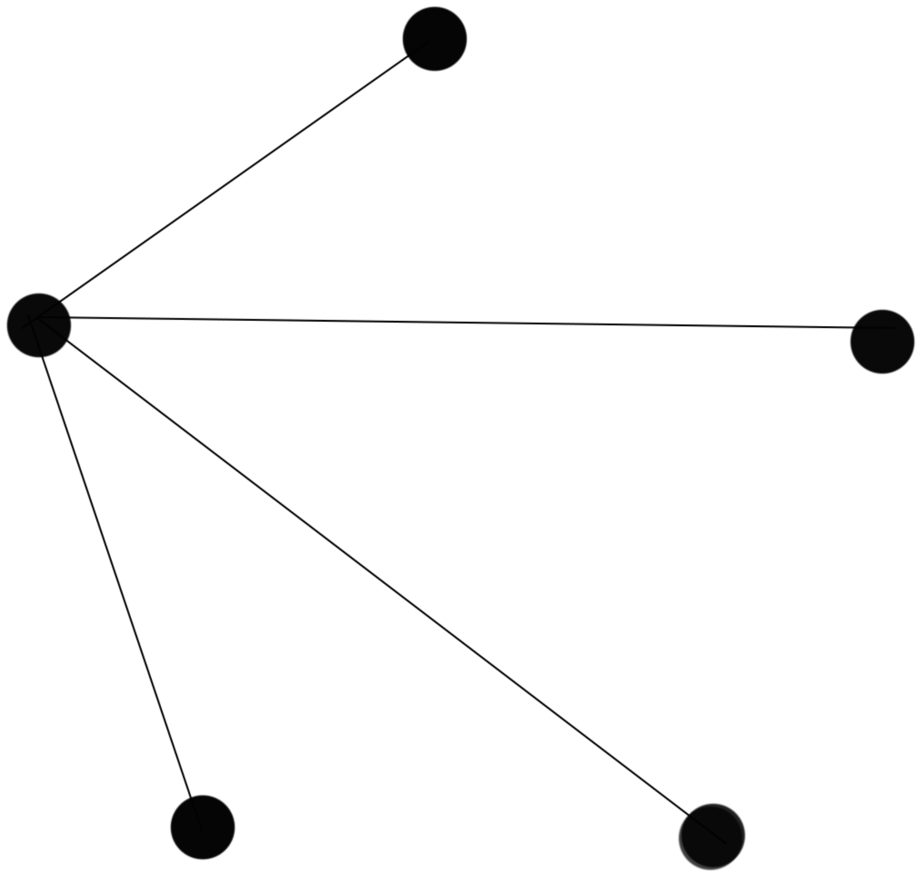




$V = 5$

$E = ?$

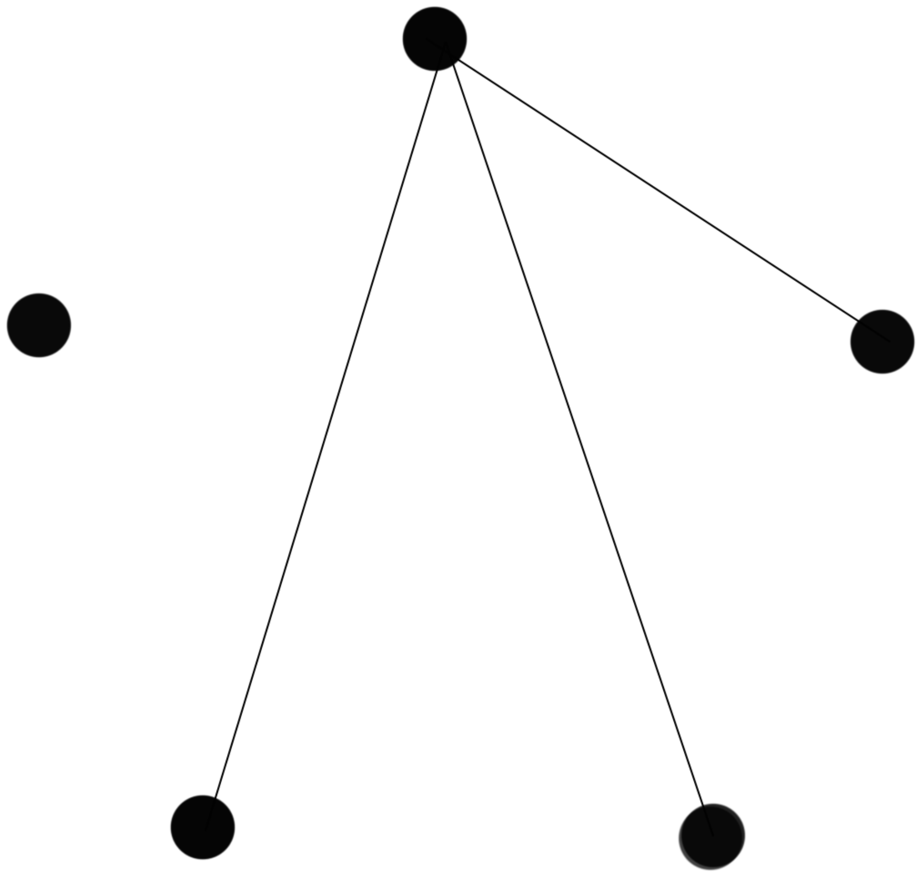
$F = ?$



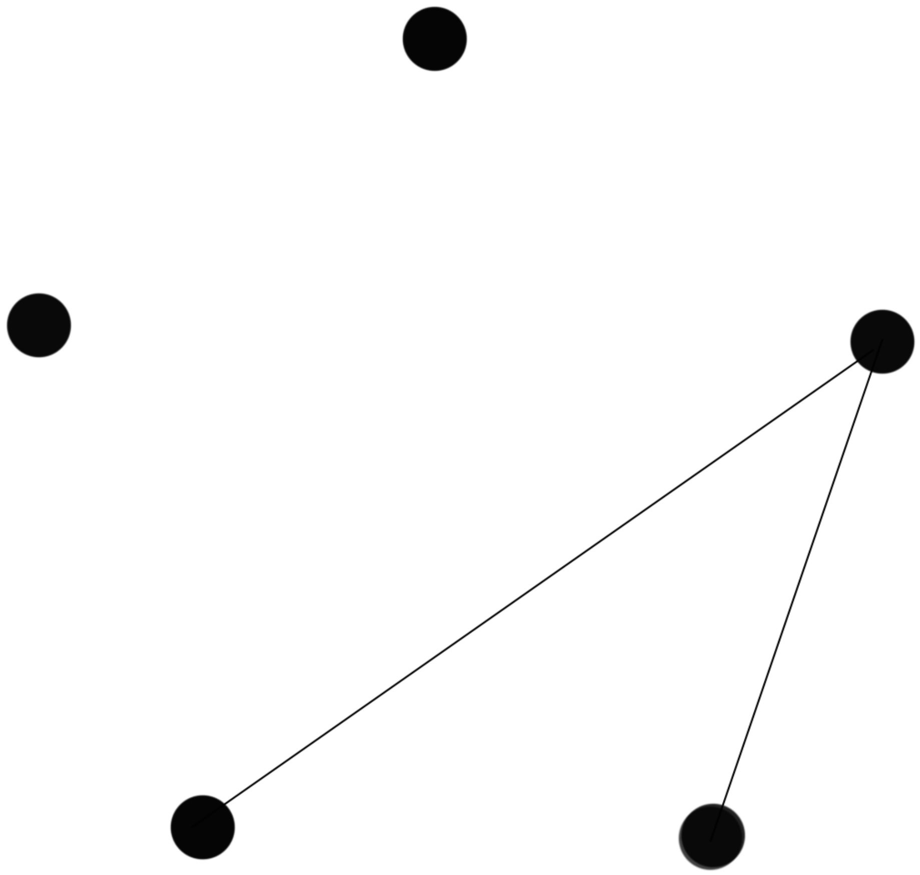
$V = 5$

$E = 4 +$

$F = ?$



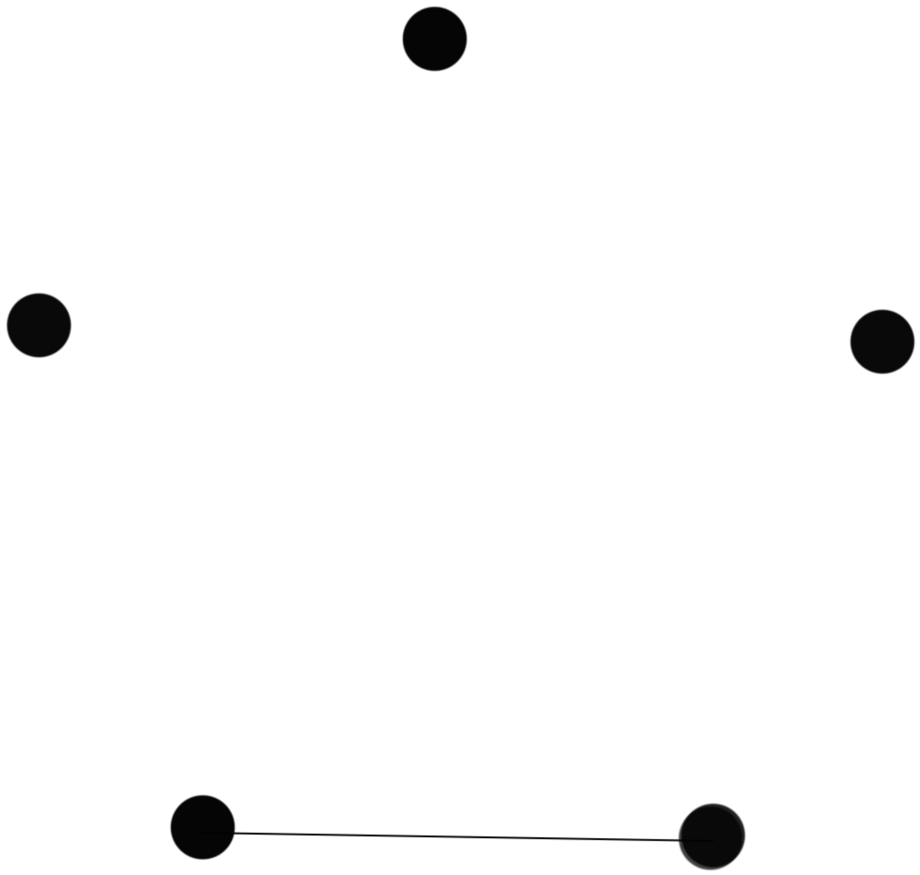
V = 5
E = 4 + 3 +
F = ?



$V = 5$

$E = 4 + 3 + 2$

$F = ?$

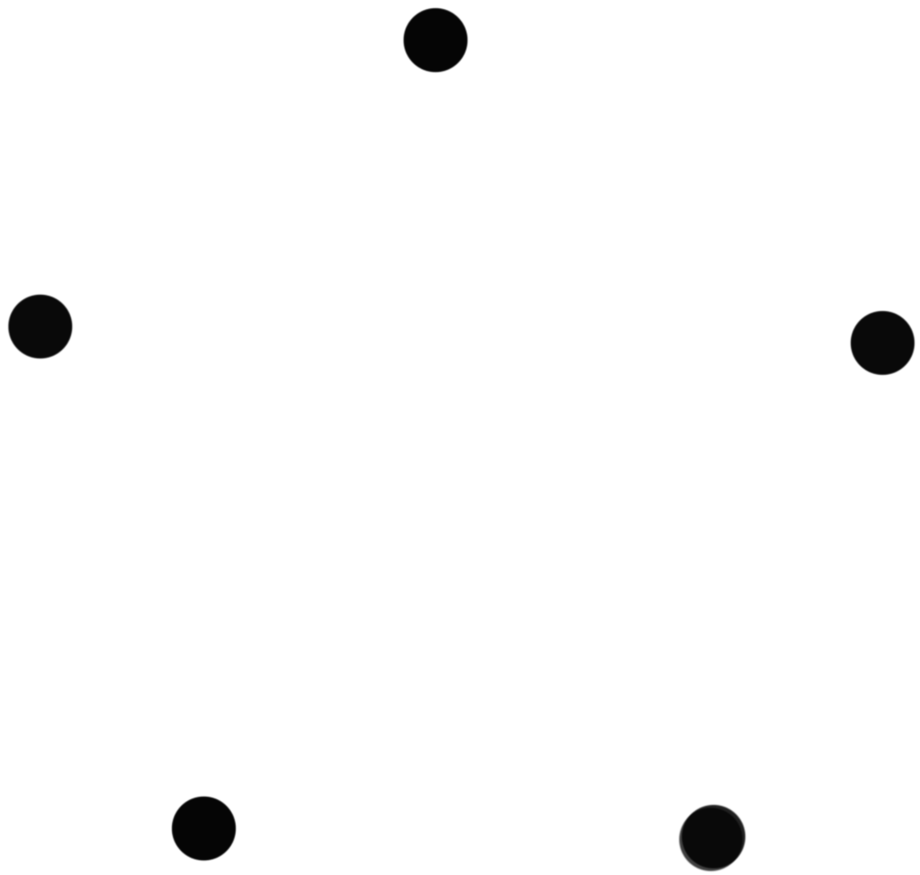


$$\boxed{V+F = E+2}$$

$V = 5$

$E = 4 + 3 + 2 + 1 = 10$

$F = ?$

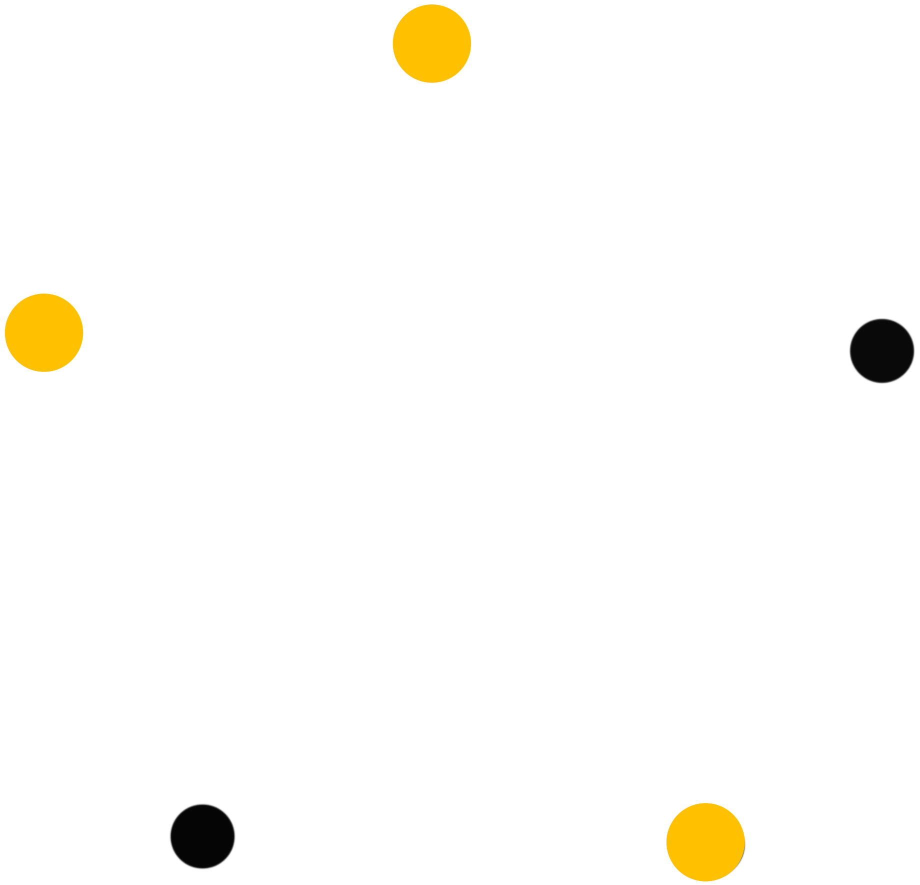


$$\boxed{V+F = E+2}$$

$V = 5$

$E = 10$

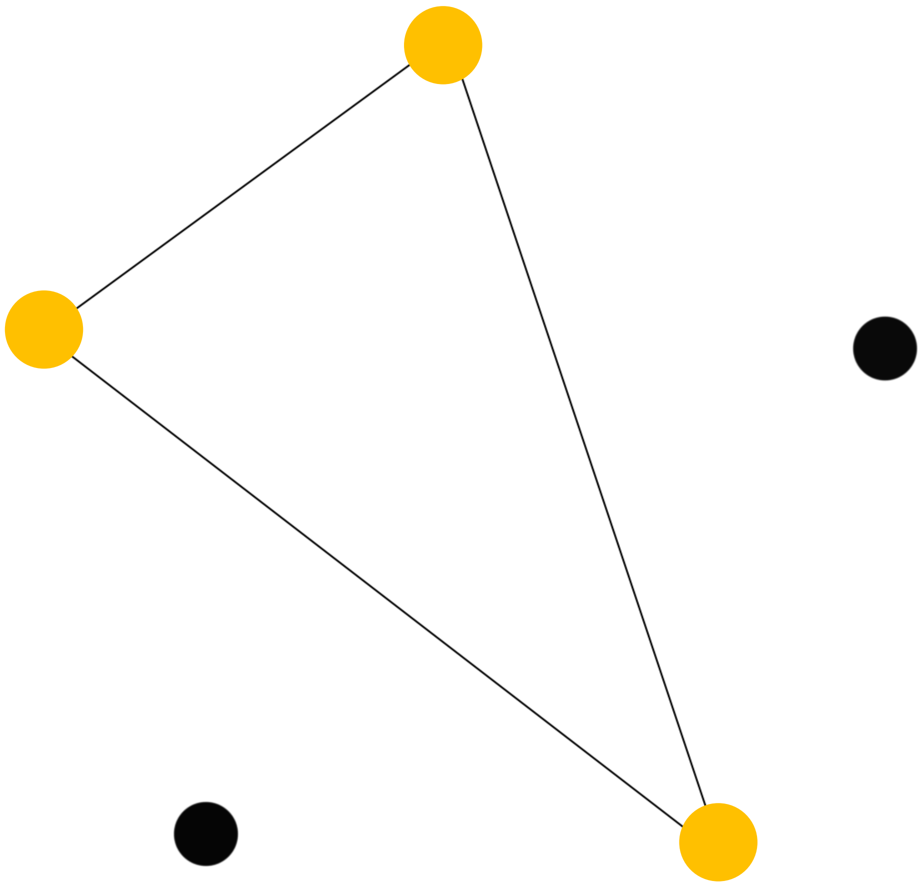
$F = 7$

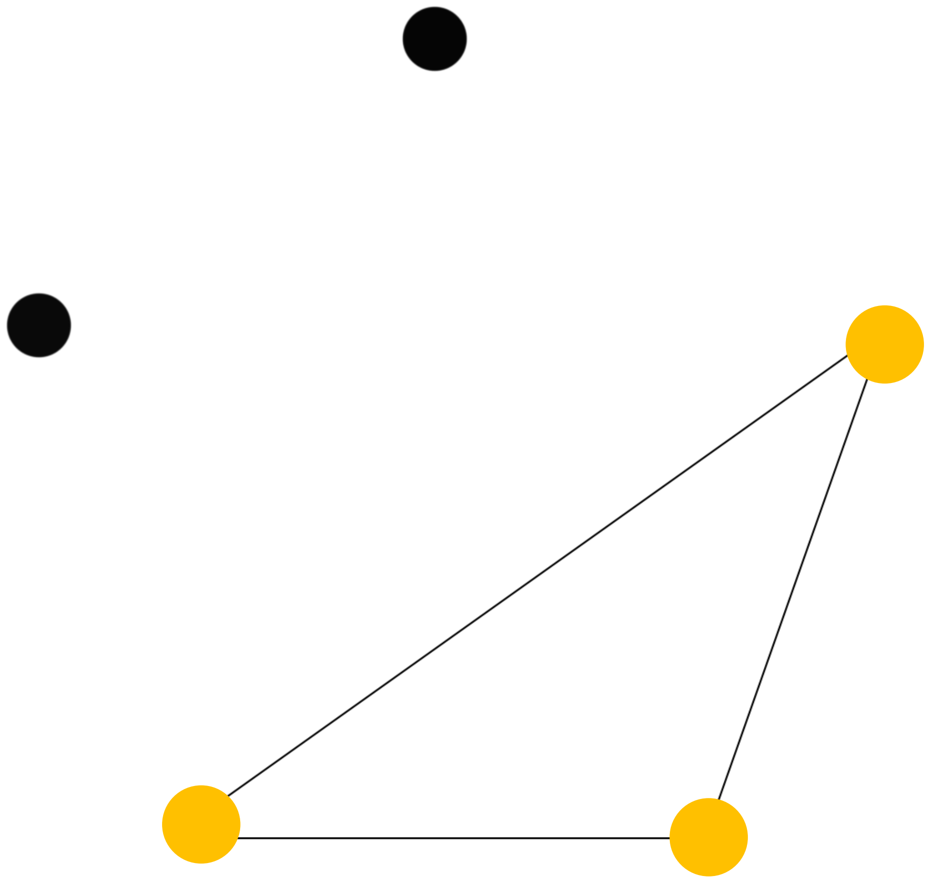


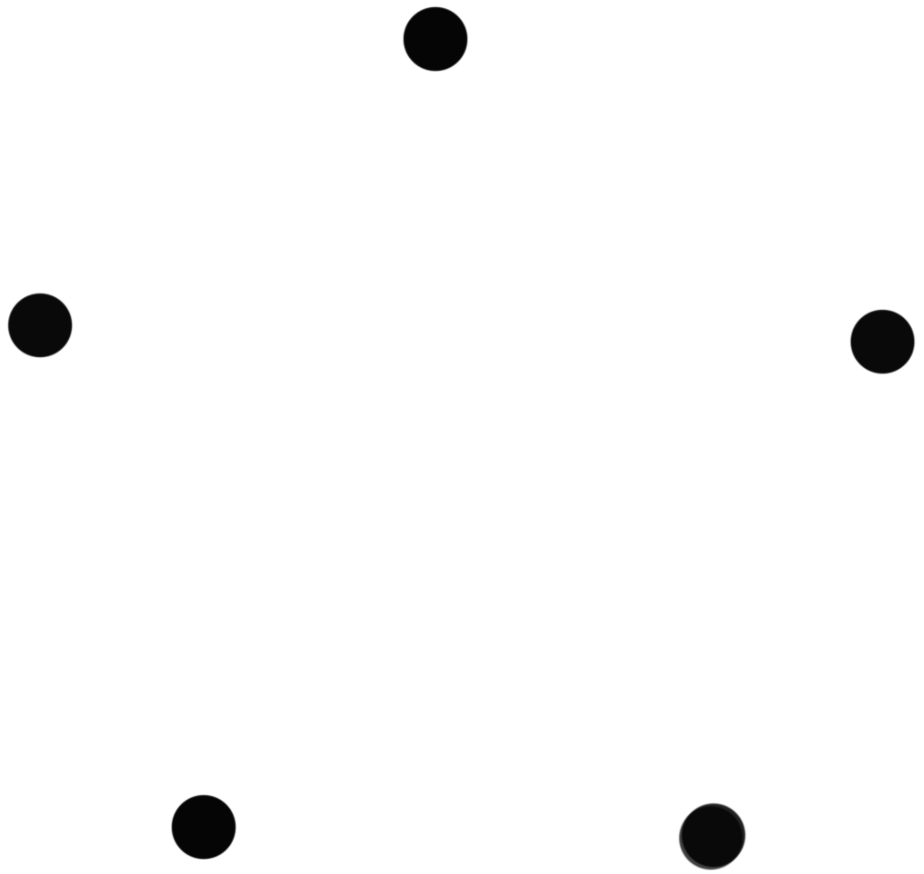
$V = 5$

$E = 10$

$F = ?$





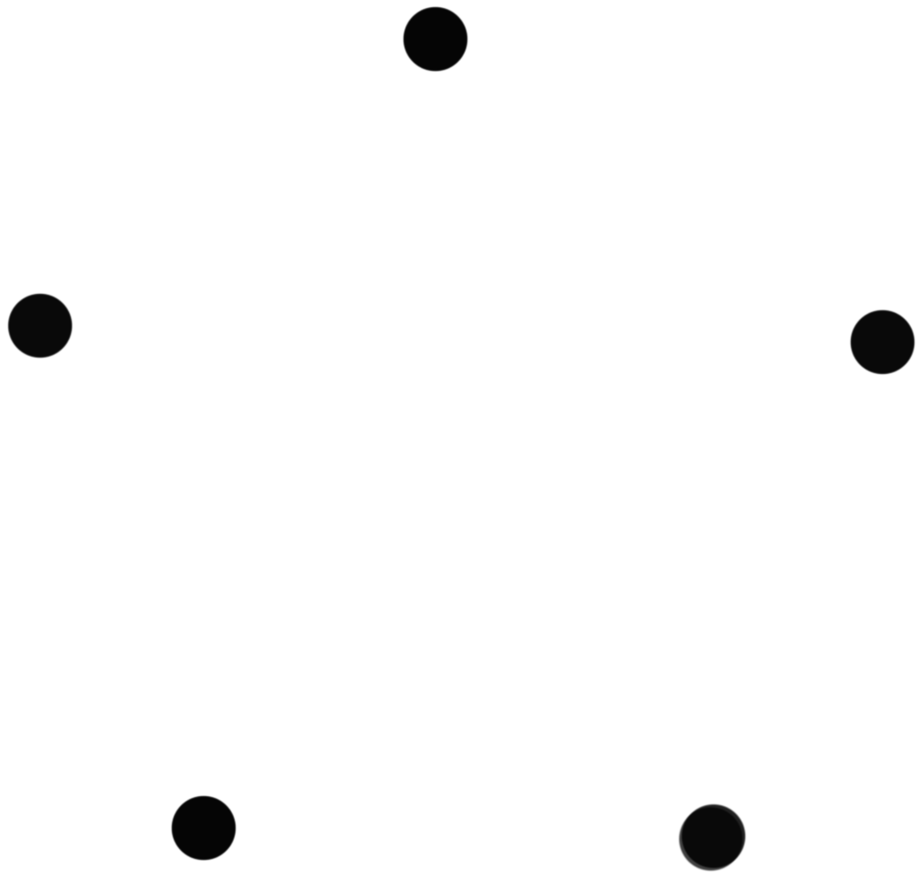


$$\boxed{V+F = E+2}$$

$$\#\Delta = \binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10$$

$V = 5$
$E = 10$
$F = 7$

F is at least 10



The $K_5$ – Graph is not planar

# Planar Graphs

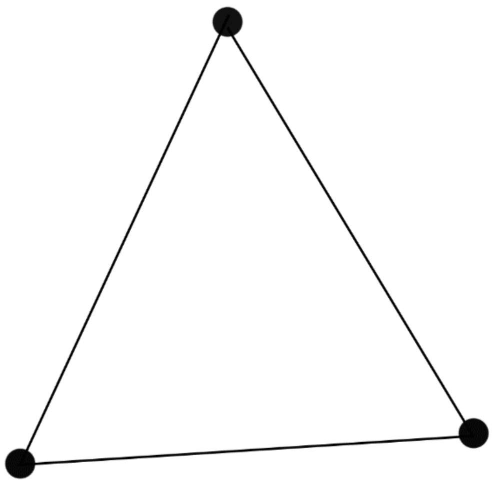
## Euler's Formula and the five regular polyhedra

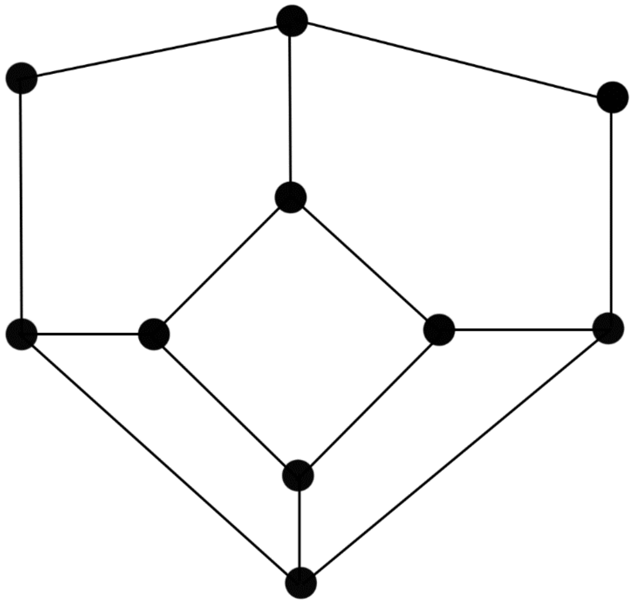
Primes-Switzerland

Han-Miru Kim, Susanne Steiner
Alte und Neue Kantonsschule Aarau

Mentor: Kaloyan Slavov

23.06.2018





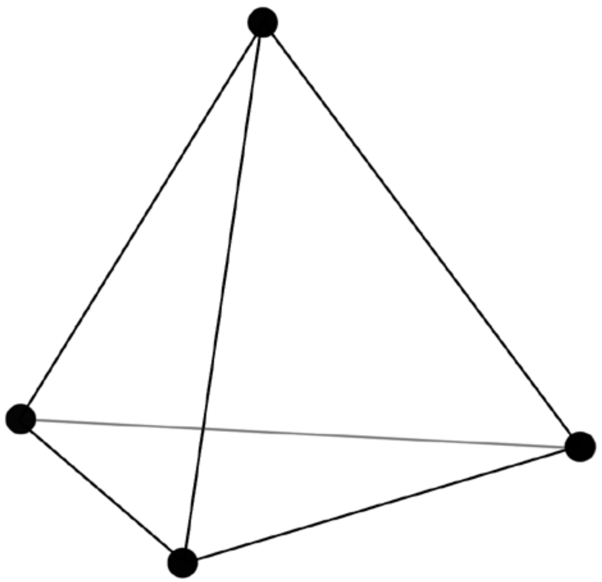
$$V + F = E + 2\,?$$

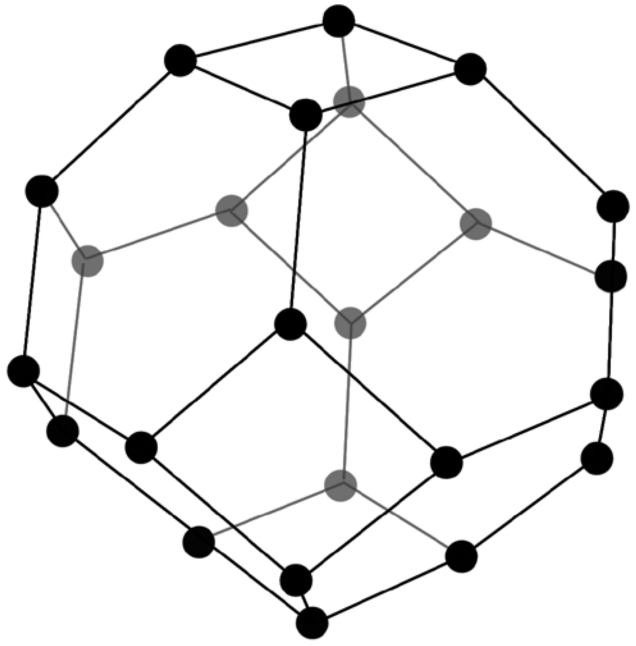
$V = 4$

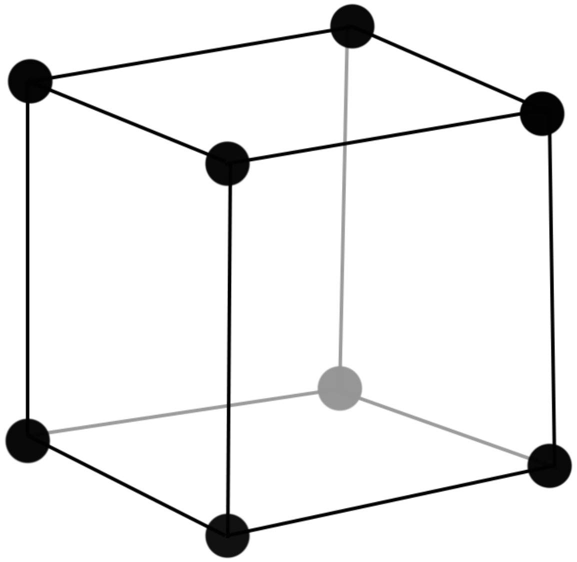
$F = 4$

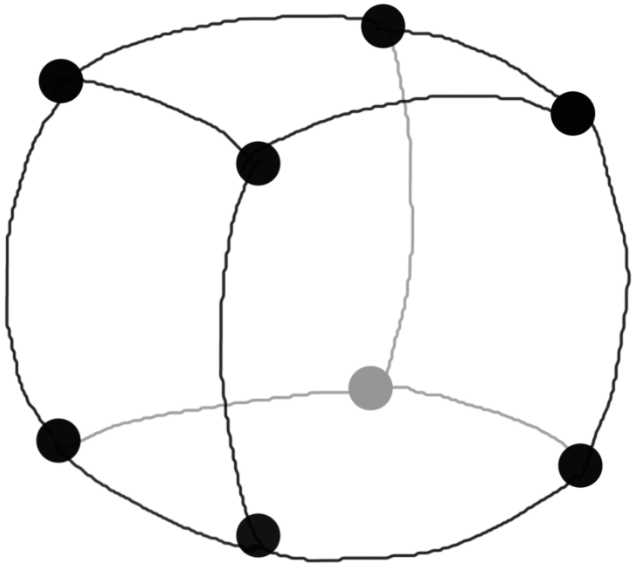
$E = 6$

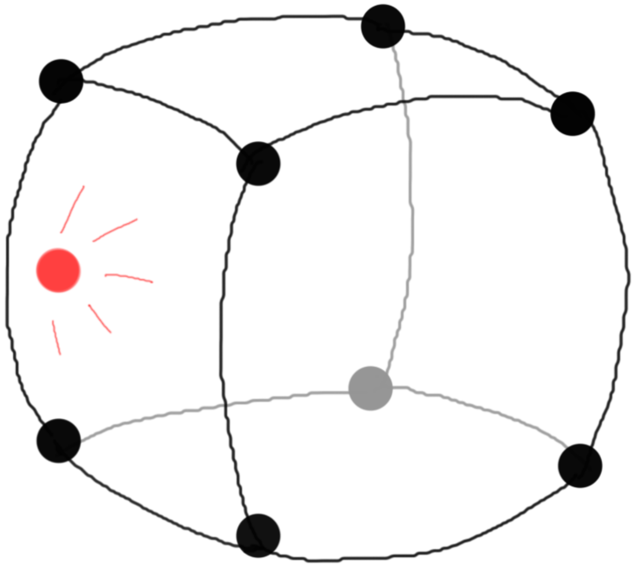
$$4 + 4 = 6 + 2$$

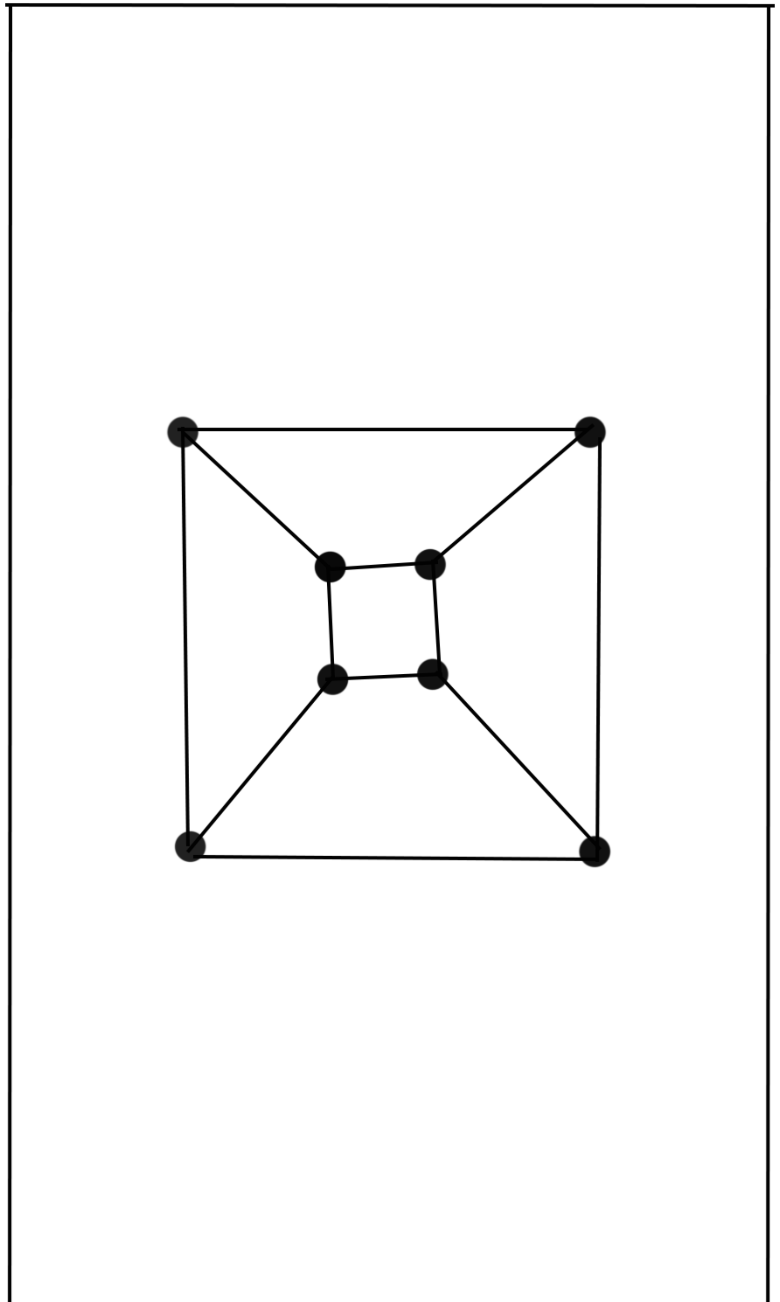




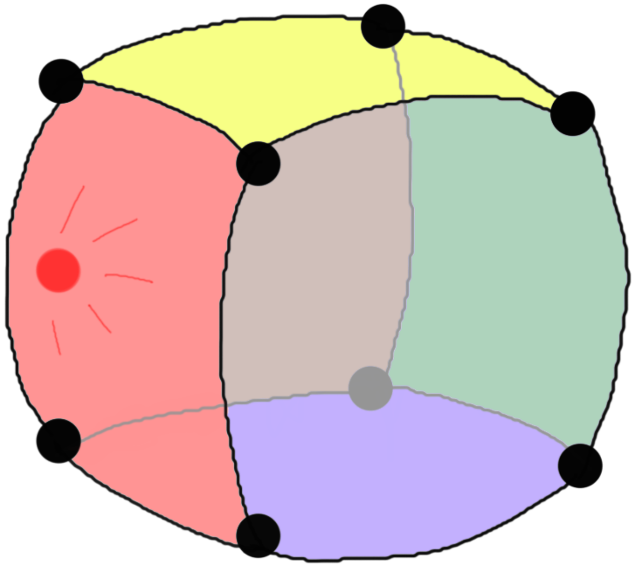


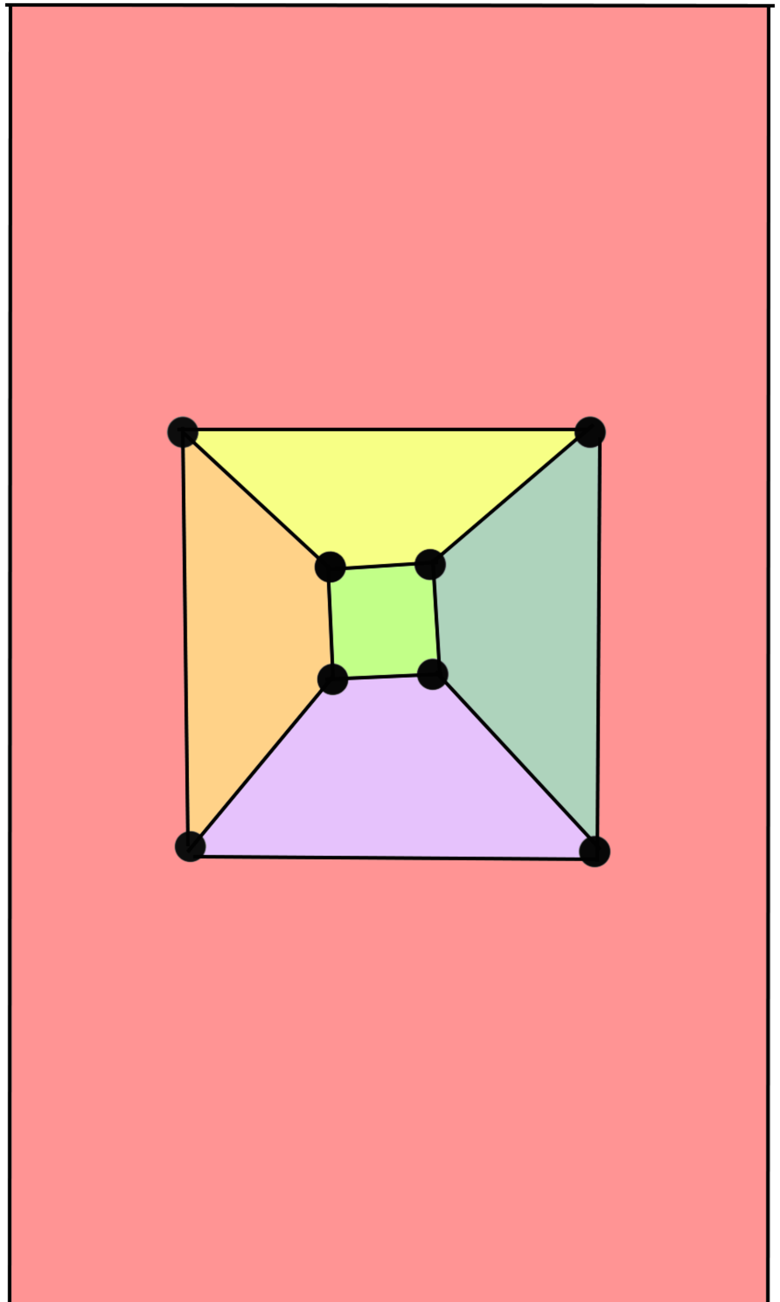


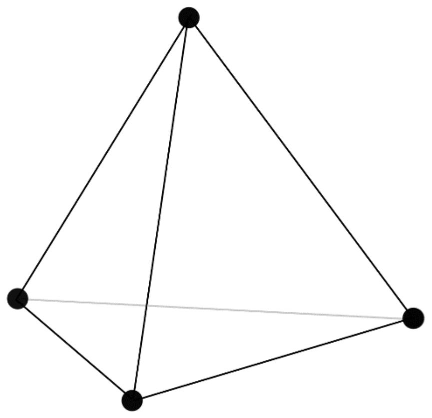




$$V + F = E + 2$$

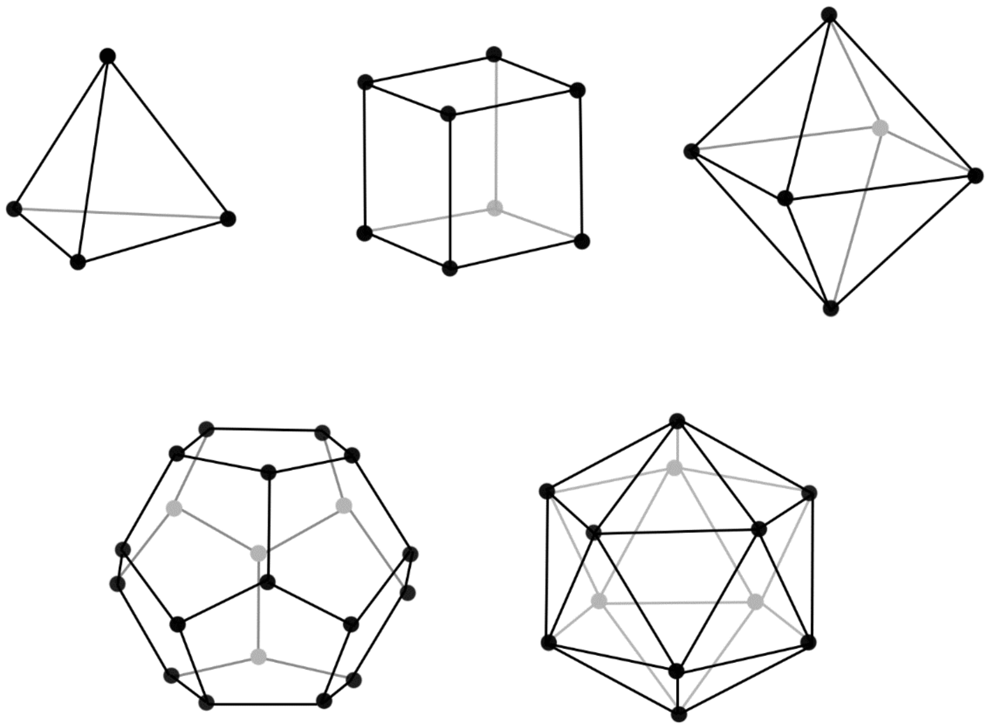


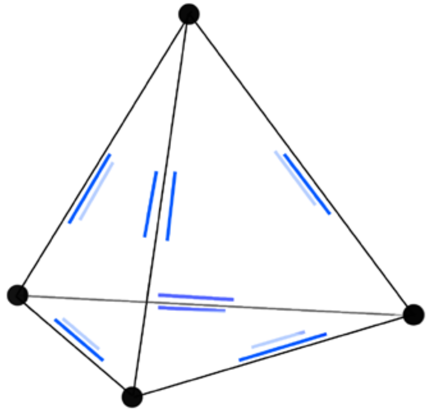




$d \geq 3$

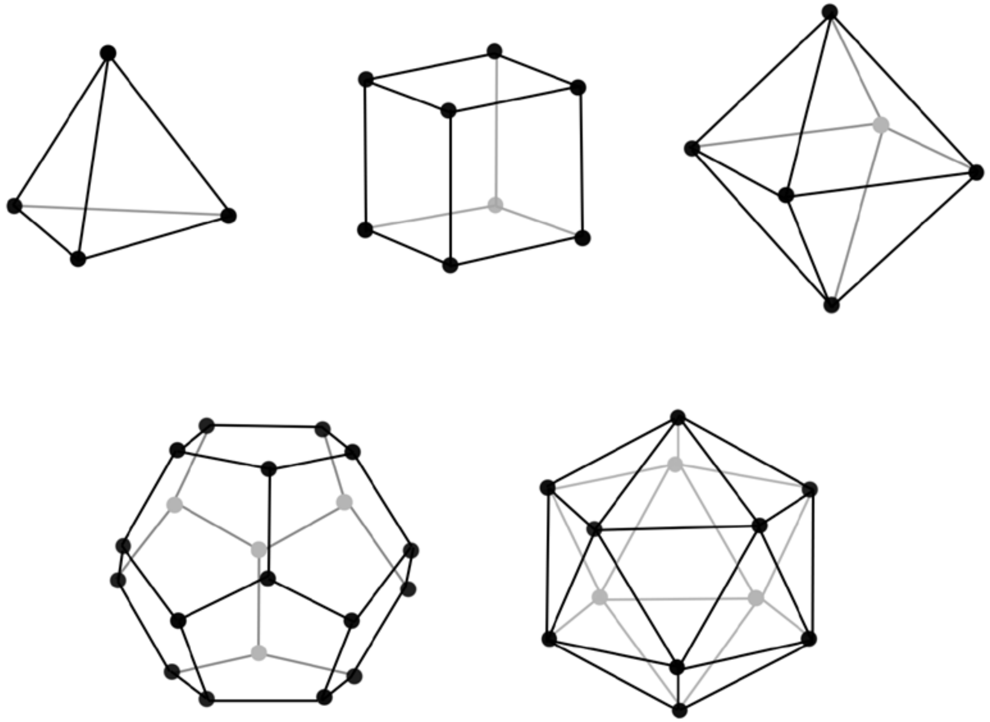
$l \geq 3$





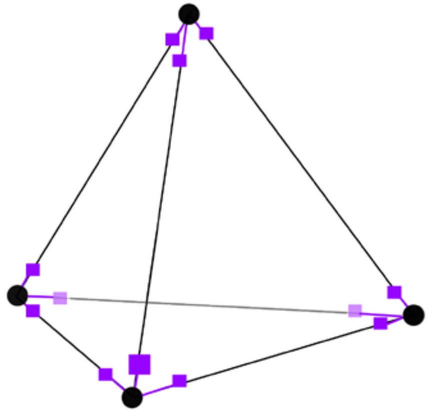
$$d \geq 3$$

$$l \geq 3$$



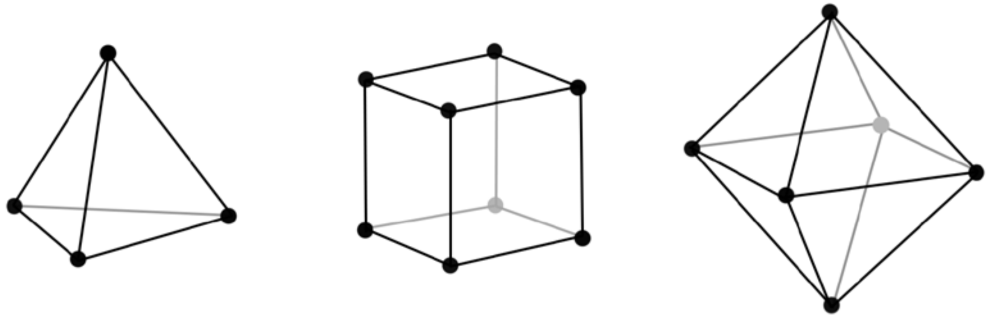
$$F \cdot l = 2E$$

$$3F \leq 2E$$



$d \geq 3$

$l \geq 3$

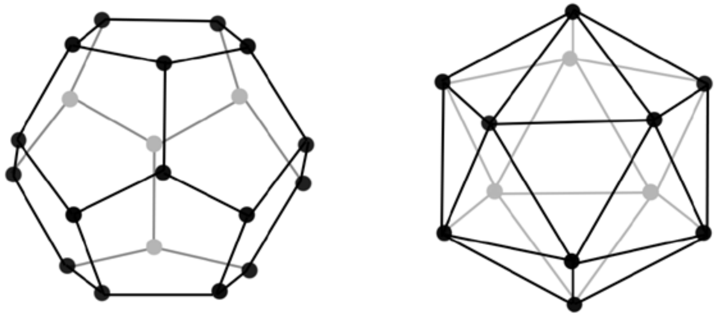


$F \cdot l = 2E$

$3F \leq 2E$

$V \cdot d = 2E$

$3V \leq 2E$



$$F \cdot l = 2E \qquad V \cdot d = 2E$$

$$\boldsymbol{E + 2 = V + F}$$

$$3F \leq 2E \qquad 3V \leq 2E$$

$$\boldsymbol{F \cdot l = 2E} \qquad \boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{E + 2 = V + F}$$

$$\boldsymbol{3F \leq 2E} \qquad \boldsymbol{3V \leq 2E}$$

$$F \leq \frac{2E}{3} \qquad V \leq \frac{2E}{3}$$

$$\boldsymbol{E + 2 = V + F} \qquad \boldsymbol{E + 2 = V + F}$$

$$E + 2 \leq \frac{2E}{3} + V \qquad E + 2 \leq \frac{2E}{3} + F$$

$$3E + 6 \leq 2E + 3V \qquad 3E + 6 \leq 2E + 3F$$

$$\boldsymbol{E \leq 3V - 6} \qquad \boldsymbol{E \leq 3F - 6}$$

$$\boldsymbol{F \cdot l = 2E} \qquad \boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{E \leq 3F - 6} \qquad \boldsymbol{E \leq 3V - 6}$$

$$E = \frac{F \cdot l}{2} \leq 3F - 6$$

$$\boldsymbol{F \cdot l = 2E} \qquad \boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{E \leq 3F - 6} \qquad \boldsymbol{E \leq 3V - 6}$$

$$E = \frac{F \cdot 6}{2} \leq 3F - 6 \qquad E = \frac{V \cdot d}{2} \leq 3V - 6$$

$$\mathrm{E} = 3F \leq 3F - 6$$

$$\longrightarrow \boldsymbol{l \leq 5}$$

$$\boldsymbol{F \cdot l = 2E} \qquad \boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{E \leq 3F - 6} \qquad \boldsymbol{E \leq 3V - 6}$$

$$E = \frac{F \cdot 6}{2} \leq 3F - 6 \qquad E = \frac{V \cdot 6}{2} \leq 3V - 6$$

$$l = 3{,}4{,}5$$

$$\mathrm{E} = 3F \leq 3F - 6 \qquad E = 3V \leq 3V - 6$$

$$d = 3{,}4{,}5$$

$$\longrightarrow \boldsymbol{l \leq 5} \qquad \longrightarrow \boldsymbol{d \leq 5}$$

Case 1

$$d = 3$$

$$2E = 3V$$

$$V = \frac{2E}{3}$$

$$\boldsymbol{V + F = E - 2}$$

$$3F - 6 = E$$

$$6F - 12 = F \cdot \mathrm{l}$$

$$\boldsymbol{F(6 - l) = 12}$$

$$V + F = E + 2$$

$$\boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{F \cdot l = 2E}$$

Case 1

$$d = 3$$

$$V + F = E + 2$$

$$\boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{F \cdot l = 2E}$$

$$3F - 6 = E$$

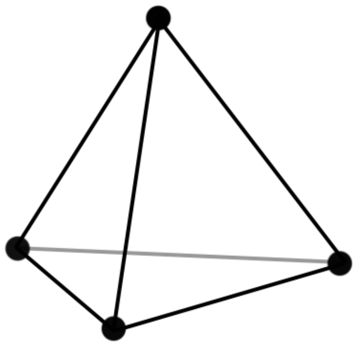
$$\boldsymbol{F(6 - l) = 12}$$

for l = 3

$F(6-3) = 12 \rightarrow \boldsymbol{F = 4}$

$E = 3F - 6 \rightarrow \boldsymbol{E = 6}$

$V = E + 2 - F \rightarrow \boldsymbol{V = 4}$

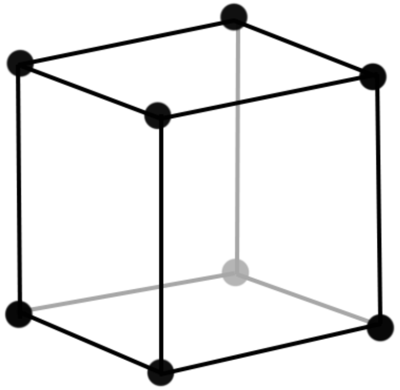


for l = 4

$F(6-4) = 12 \rightarrow \boldsymbol{F = 6}$

$E = 3F - 6 \rightarrow \boldsymbol{E = 12}$

$V = E + 2 - F \rightarrow \boldsymbol{V = 8}$

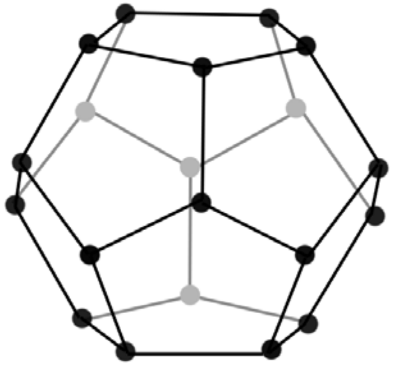


for l = 5

$F(6-5) = 12 \rightarrow \boldsymbol{F = 12}$

$E = 3F - 6 \rightarrow \boldsymbol{E = 30}$

$V = E + 2 - F \rightarrow \boldsymbol{V = 20}$



Case 2

$$d = 4$$

$$2E = 4V$$

$$V = \frac{E}{2}$$

$$\boldsymbol{V + F = E + 2}$$

$$2F - 4 = E$$

$$4F - 8 = F \cdot \mathrm{l}$$

$$\boldsymbol{F(4 - l) = 8}$$

$$V + F = E + 2$$

$$\boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{F \cdot l = 2E}$$

Case 2

$$d = 4$$

$$V + F = E + 2$$

$$\boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{F \cdot l = 2E}$$

$$2F - 4 = E$$

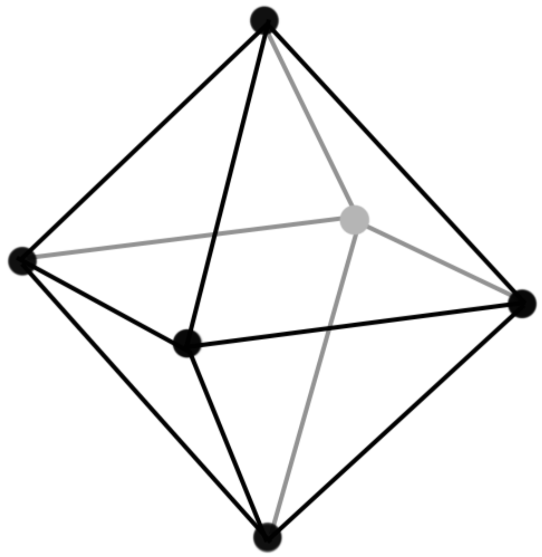
$$\boldsymbol{F(4 - l) = 8}$$

for $\mathrm{l} = 3$

$$\boldsymbol{F = 8}$$

$$\boldsymbol{E = 12}$$

$$\boldsymbol{V = 6}$$



Case 3

$$d = 5$$

$$5F - 10 = 3E$$

$$\boldsymbol{F(10 - 3l) = 20}$$

$$V + F = E + 2$$

$$\boldsymbol{V \cdot d = 2E}$$

$$\boldsymbol{F \cdot l = 2E}$$

for $\mathrm{l} = 3$

$$\boldsymbol{F = 20}$$
$$\boldsymbol{E = 30}$$
$$\boldsymbol{V = 12}$$